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Direct Geolocation of Wideband Emitters Based on Delay and Doppler

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***Abstract*—The localization of a stationary transmitter using receivers mounted on fast moving platforms is considered. It is assumed that the transmitted radio signal is random with known statistics. The conventional approach is based on two steps. In the first step the time difference of arrival and the differential Doppler shift are measured and in the second step these measurements are used for geolocation. We advocate a direct position determination approach that proves to be more computationally efficient and more precise for weak signals than the conventional approach. A secondary but important result is a derivation of closed-form and compact expressions of the Cramér-Rao Lower Bound . All results are verified by Monte-Carlo computer simulations.**

***Index Terms*—Emitter location, Maximum likelihood estimation, Differential Doppler.**

1. Introduction

Passive geolocation of a stationary transmitter based on the delayed and Doppler shifted signal, observed by at least a single moving platform, is a well known technique as can be concluded from [1]-[8]. Since the receivers location and velocity along their trajectory are known, the emitter location can be estimated.

Common methods use two steps for localization. The system first estimates the differential delay and differential Doppler frequency shift along the receivers trajectories. In the second step the system estimates the transmitter location based on the results obtained in the first step. The two step methods are not guaranteed to yield optimal location results since in the first step the differential delay and differential Doppler estimates are obtained by ignoring the constraint that all measurements must be consistent with a geolocation of a single point emitter. Thus, the lines of position (LOP) obtained from the delay/Doppler measurements are not guaranteed to intersect in a single geographical location. However, it can be shown that *asymptotically* (for large number of observations), the two-step methods are equivalent to the single step approach. The theory of the asymptotic equivalence, also known as the Extended Invariance Principle (EXIP) is proved in [14] and [15]. We therefore conclude that the single step approach outperforms the two step methods for low Signal to Noise Ratio (SNR) and/or short data records. This is demonstrated in this paper. Other advantages of the single step approach is the reduced computational complexity. In the conventional methods the Differential-Doppler and the differential-delay are estimated along the receivers track and finally the location of the transmitter is deduced. In the single-step method, only the transmitter location is estimated, directly from the observed signals. Therefore, our algorithm is not only more precise it is also easier to implement.

In a previous publication we proposed a single-step solution for narrowband signals [11] based on Doppler shift only and assuming known or unknown non-random signals. Herein, we propose a maximum likelihood location estimation by using a single step approach for wideband random signals using both the Doppler Effect and the relative delay.

2. Problem Formulation

Consider a stationary radio emitter and L receivers, synchronized in frequency and time, who are moving at a considerable speed. The receivers intercept the transmitted signal at K short intervals along their track. Let $\mathbf{p}$ stand for the coordinates vector of the emitter and $\mathbf{p}_{\ell,k}$ and $\mathbf{v}_{\ell,k}$ denote the coordinates vector and the velocity vector of the ℓ-th receiver at the k-th interception interval. The frequency down-converted complex signal observed by the ℓ-th receiver at the k-th interception interval at time t is given by

$$r_{\ell,k}(t) = s_k(t - \tau_{\ell,k}) e^{\imath 2\pi f_{\ell,k} t} + w_{\ell,k}(t), \qquad (1)$$

where $-T/2 \leq t \leq T/2$ is the observation time interval, $s_k(t)$ is the observed signal envelope during the k-th interception interval, $\tau_{\ell,k}$ is the signal propagation time from the emitter to the receiver, $w_{\ell,k}(t)$ is a wide-sense stationary, white, zero mean, complex, Gaussian noise and $f_{\ell,k}$ is the observed Doppler frequency shift given by

$$f_{\ell,k} \triangleq f_c \mu_{\ell,k}(\mathbf{p}), \qquad (2)$$

$$\mu_{\ell,k}(\mathbf{p}) \triangleq \frac{1}{c} \mathbf{v}_{\ell,k}^T (\mathbf{p} - \mathbf{p}_{\ell,k}) / \|\mathbf{p} - \mathbf{p}_{\ell,k}\|, \qquad (3)$$

where f_c is the known nominal carrier frequency of the transmitted signal, c is the signal propagation speed and $\|\cdot\|$ is the Euclidean norm. Similarly to [7] we assume that $\tau_{\ell,k} \ll T$.

Assume now that the signals $s_k(t)$ are realizations of a zero-mean, Gaussian process. We shall consider the waveform received by each sensor to be represented by its Fourier coefficients defined by

$$\tilde{r}_{\ell,k}(f_n) \triangleq \frac{1}{T} \int_{-T/2}^{T/2} r_{\ell,k}(t) e^{-\imath 2\pi f_n t} \, dt, \qquad (4)$$

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where $f_n = n/T,\ n = 0, \pm 1, \pm 2, \cdots$ is the frequency associated with the n-th coefficient. The Fourier coefficients of $s_k(t)$ and of $w_{\ell,k}(t)$ are denoted by $\tilde{s}_k(f_n)$ and by $\tilde{w}_{\ell,k}(f_n)$, respectively. In the frequency domain (1) becomes

$$\tilde{r}_{\ell,k}(f_n) = \tilde{s}_k(f_n - f_{\ell,k}) e^{-\imath 2\pi f_n \tau_{\ell,k}} + \tilde{w}_{\ell,k}(f_n). \tag{5}$$

Define the vectors

$$\begin{aligned} \tilde{\mathbf{r}}_{\ell,k} &\triangleq [\tilde{r}_{\ell,k}(f_{-N}), \ldots, \tilde{r}_{\ell,k}(f_N)]^T, \\ \tilde{\mathbf{w}}_{\ell,k} &\triangleq [\tilde{w}_{\ell,k}(f_{-N}), \ldots, \tilde{w}_{\ell,k}(f_N)]^T, \\ \tilde{\mathbf{s}}_k &\triangleq [\tilde{s}_k(f_{-N}) \ldots, \tilde{s}_k(f_N)]^T, \\ \tilde{\mathbf{A}}_{\ell,k} &\triangleq \text{diag}\{e^{-\imath 2\pi \tau_{\ell,k} f_{-N}}, \ldots, e^{-\imath 2\pi \tau_{\ell,k} f_N}\}. \end{aligned} \tag{6}$$

We get from (5) and (6) the relation

$$\tilde{\mathbf{r}}_{\ell,k} = \tilde{\mathbf{A}}_{\ell,k} \tilde{\mathbf{F}}_{\ell,k} \tilde{\mathbf{s}}_k + \tilde{\mathbf{w}}_{\ell,k}, \tag{7}$$

where $\tilde{\mathbf{F}}_{\ell,k}$ is a down shift operator. The product $\tilde{\mathbf{F}}_{\ell,k}\tilde{\mathbf{s}}_k$ shifts the vector $\tilde{\mathbf{s}}_k$ by $\lfloor T f_{\ell,k} \rfloor$ indices. The first and second order moments of the noise vectors and the signal vectors are summarized by

$$\begin{aligned} E\{\tilde{\mathbf{s}}_{\ell,k}\} &= E\{\tilde{\mathbf{w}}_{\ell,k}\} = 0,\ \forall \ell, k, \\ E\{\tilde{\mathbf{s}}_k \tilde{\mathbf{w}}_{\ell,j}^H\} &= 0,\ \forall \ell, k, j, \\ E\{\tilde{\mathbf{w}}_{\ell,k} \tilde{\mathbf{w}}_{i,j}^H\} &= \sigma^2 \mathbf{I} \delta_{\ell,i} \delta_{k,j}, \\ E\{\tilde{\mathbf{s}}_k \tilde{\mathbf{s}}_j^H\} &= \mathbf{\Lambda} \delta_{k,j}, \end{aligned} \tag{8}$$

where $\mathbf{\Lambda}$ is a diagonal matrix whose main diagonal is the signal spectrum which is independent of the observations index $k \in \{1, \cdots K\}$. Obviously, $\mathbf{\Lambda}$ is diagonal only if the observation time, T, is much larger than the correlation time of the signal and as a result the Fourier coefficients are uncorrelated. Thus, (7) and (8) yield

$$\begin{aligned} \tilde{\mathbf{R}}_{\ell,k,i,j} &\triangleq E\{\tilde{\mathbf{r}}_{\ell,k} \tilde{\mathbf{r}}_{i,j}^H\} \\ &= \tilde{\mathbf{A}}_{\ell,k} \tilde{\mathbf{F}}_{\ell,k} \mathbf{\Lambda} \tilde{\mathbf{F}}_{i,j}^H \tilde{\mathbf{A}}_{i,j}^H \delta_{k,j} + \sigma^2 \mathbf{I} \delta_{\ell,i} \delta_{k,j}. \end{aligned} \tag{9}$$

Define the vectors and matrices

$$\begin{aligned} \tilde{\mathbf{r}}_k &\triangleq [\tilde{\mathbf{r}}_{1,k}^T, \tilde{\mathbf{r}}_{2,k}^T, \cdots \tilde{\mathbf{r}}_{L,k}^T]^T, \\ \tilde{\mathbf{r}} &\triangleq [\tilde{\mathbf{r}}_1^T, \tilde{\mathbf{r}}_2^T, \cdots \tilde{\mathbf{r}}_K^T]^T, \\ \tilde{\mathbf{R}}_k &\triangleq E\{\tilde{\mathbf{r}}_k \tilde{\mathbf{r}}_k^H\}, \\ \tilde{\mathbf{R}} &\triangleq E\{\tilde{\mathbf{r}} \tilde{\mathbf{r}}^H\}. \end{aligned} \tag{10}$$

The Gaussian probability density function of the data is given by

$$f_{\tilde{\mathbf{r}}}(\tilde{\mathbf{r}}|\mathbf{p}) = (\pi |\tilde{\mathbf{R}}|)^{-1} \exp\{-\tilde{\mathbf{r}}^H \tilde{\mathbf{R}}^{-1} \tilde{\mathbf{r}}\} \tag{11}$$

and the negative log-likelihood is given, up to an additive constant, by

$$\begin{aligned} L_f(\mathbf{p}) &= \log\{|\tilde{\mathbf{R}}|\} + \tilde{\mathbf{r}}^H \tilde{\mathbf{R}}^{-1} \tilde{\mathbf{r}} \\ &= \log\{\prod_{k=1}^{K} |\tilde{\mathbf{R}}_k|\} + \sum_{k=1}^{K} \tilde{\mathbf{r}}_k^H \tilde{\mathbf{R}}_k^{-1} \tilde{\mathbf{r}}_k. \end{aligned} \tag{12}$$

Using the definition,

$$\tilde{\mathbf{B}}_k \triangleq [\tilde{\mathbf{F}}_{1,k}^H \tilde{\mathbf{A}}_{1,k}^H, \cdots \tilde{\mathbf{F}}_{L,k}^H \tilde{\mathbf{A}}_{L,k}^H]^H \tag{13}$$

we get from (9) and (10) the relation

$$\tilde{\mathbf{R}}_k = \tilde{\mathbf{B}}_k \mathbf{\Lambda} \tilde{\mathbf{B}}_k^H + \sigma^2 \mathbf{I}. \tag{14}$$

We use the Woodbury matrix identity [12] and the matrix determinant lemma [13] to obtain

$$\tilde{\mathbf{R}}_k^{-1} = \sigma^{-2}(\mathbf{I} - \tilde{\mathbf{B}}_k (\sigma^2 \mathbf{\Lambda}^{-1} + \tilde{\mathbf{B}}_k^H \tilde{\mathbf{B}}_k)^{-1} \tilde{\mathbf{B}}_k^H), \tag{15}$$

$$|\tilde{\mathbf{R}}_k| = |\mathbf{\Lambda}^{-1} + \sigma^{-2} \tilde{\mathbf{B}}_k^H \tilde{\mathbf{B}}_k| |\mathbf{\Lambda}| |\sigma^2 \mathbf{I}|. \tag{16}$$

Further simplification is obtained by noting that

$$\tilde{\mathbf{B}}_k^H \tilde{\mathbf{B}}_k = \sum_{\ell=1}^{L} \tilde{\mathbf{F}}_{\ell,k}^H \tilde{\mathbf{A}}_{\ell,k}^H \tilde{\mathbf{A}}_{\ell,k} \tilde{\mathbf{F}}_{\ell,k} = L\mathbf{I}. \tag{17}$$

This result indicates that $|\mathbf{R}_k|$ is independent of the emitter location. Define

$$\begin{aligned} \mathbf{\Gamma} &\triangleq (\sigma^2 \mathbf{\Lambda}^{-1} + \tilde{\mathbf{B}}_k^H \tilde{\mathbf{B}}_k)^{-1} \\ &= \text{diag}\{\frac{\lambda_1}{(\sigma^2 + L\lambda_1)}, \cdots \frac{\lambda_{N_1}}{(\sigma^2 + L\lambda_{N_1})}\}, \end{aligned} \tag{18}$$

where $\lambda_i = \mathbf{\Lambda}_{i,i}$. Thus, (15) becomes

$$\tilde{\mathbf{R}}_k^{-1} = \sigma^{-2}(\mathbf{I} - \tilde{\mathbf{B}}_k \mathbf{\Gamma} \tilde{\mathbf{B}}_k^H). \tag{19}$$

The cost function in (12) can be simplified and rewritten as

$$C_r(\mathbf{p}) = \sum_{k=1}^{K} \tilde{\mathbf{r}}_k^H \tilde{\mathbf{R}}_k^{-1} \tilde{\mathbf{r}}_k = \sigma^{-2} \sum_{k=1}^{K} \tilde{\mathbf{r}}_k^H (\mathbf{I} - \tilde{\mathbf{B}}_k \mathbf{\Gamma} \tilde{\mathbf{B}}_k^H) \tilde{\mathbf{r}}_k. \tag{20}$$

Instead of finding the minimum of (20) we can find the maximum of

$$\begin{aligned} \tilde{C}_r(\mathbf{p}) &= \sum_{k=1}^{K} \tilde{\mathbf{r}}_k^H \tilde{\mathbf{B}}_k \mathbf{\Gamma} \tilde{\mathbf{B}}_k^H \tilde{\mathbf{r}}_k \\ &= \sum_{k=1}^{K} \left\| \mathbf{\Gamma}^{1/2} \sum_{\ell=1}^{L} \tilde{\mathbf{F}}_{\ell,k}^H \tilde{\mathbf{A}}_{\ell,k}^H \tilde{\mathbf{r}}_{\ell,k} \right\|^2. \end{aligned} \tag{21}$$

The estimated emitter location is then given by

$$\hat{\mathbf{p}} = \underset{\mathbf{p}}{\text{argmax}} \{\tilde{C}_r(\mathbf{p})\}. \tag{22}$$

A possible algorithm is displayed in Algorithm 1.

In order to obtain some insight, consider the case of only $L = 2$ receivers. In this case, equation (21) and (22) become,

$$\hat{\mathbf{p}} = \underset{\mathbf{p}}{\text{argmax}} \sum_{k=1}^{K} \Re\{\tilde{\mathbf{r}}_{1,k}^H \tilde{\mathbf{A}}_{1,k} \tilde{\mathbf{F}}_{1,k} \mathbf{\Gamma} \tilde{\mathbf{F}}_{2,k}^H \tilde{\mathbf{A}}_{2,k}^H \tilde{\mathbf{r}}_{2,k}\}. \tag{23}$$

According to the above equation, the estimated position, $\hat{\mathbf{p}}$, maximizes the sum, over the intercepted signals, of the weighted cross-correlations of the received signals. The weights, represented by $\mathbf{\Gamma}$, correspond to the ratio between the signal power, $\mathbf{\Lambda}$, and the noise power, σ^2.

Note in passing that the proposed algorithm requires a two-dimensional search for the emitter location if its altitude is known or three dimensional search in the general case. However, the two-step methods require first an estimate of differential-delay and differential-Doppler for all intercept intervals and all the receivers and only then a search for the emitter location.

```
Define the area of interest and determine a suitable grid
of locations p_1, p_2 ··· p_g.
for j = 1 to g do
    Set C̃_r(p_j) = 0
    for k = 1 to K do
        Set G = 0
        for ℓ = 1 to L do
            Evaluate the delay, τ_{ℓ,k}, and Doppler, f_{ℓ,k},
            for a transmitter at p_j.
            Evaluate Ã_{ℓ,k}, F̃_{ℓ,k}
            Evaluate G = G + F̃^H_{ℓ,k} Ã^H_{ℓ,k} r̃_{ℓ,k}
        end
        Let C̃(p_j) = C̃(p_j) + ||Γ^{1/2} G||^2
    end
end
Find the grid point for which C̃ is the biggest. This grid
point is the estimated position.
```

Algorithm 1: A Possible Implementation of the DPD Algorithm

3. Cramér Rao Bound for Random Signals

The Cramér-Rao Bound is a lower theoretical bound on the covariance of any unbiased estimator. The bound is given by the inverse of the Fisher Information Matrix (FIM). According to [16], for zero-mean, complex, Gaussian data vectors with covariance matrix $\mathbf{R}$ the elements of the FIM are given by

$$[\mathbf{J}]_{i,j} = \mathrm{tr}\left\{\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\psi_i}\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\psi_j}\right\}, \quad (24)$$

where ψ_i is the i-th element of the unknown parameter vector. In our case, the parameter vector is the transmitter coordinates vector. Thus, $\psi_1 = x$, $\psi_2 = y$. Assume that the signal spectrum is flat and its bandwidth is W [rad/sec]. We can show that the 2 FIM is given by

$$\begin{aligned}
\mathbf{J}_{1,1} &= \Phi\,\mathrm{tr}\Big\{W^2[\mathbf{V}_\tau^{(x)}]^T\mathbf{V}_\tau^{(x)} + (2\pi T)^2[\mathbf{V}_f^{(x)}]^T\mathbf{V}_f^{(x)}\Big\},\\
\mathbf{J}_{2,2} &= \Phi\,\mathrm{tr}\Big\{W^2[\mathbf{V}_\tau^{(y)}]^T\mathbf{V}_\tau^{(y)} + (2\pi T)^2[\mathbf{V}_f^{(y)}]^T\mathbf{V}_f^{(y)}\Big\},\\
\mathbf{J}_{1,2} &= \Phi\,\mathrm{tr}\Big\{W^2[\mathbf{V}_\tau^{(x)}]^T\mathbf{V}_\tau^{(y)} + (2\pi T)^2[\mathbf{V}_f^{(x)}]^T\mathbf{V}_f^{(y)}\Big\}
\end{aligned} \quad (25)$$

where $\mathbf{V}_\tau^{(x)}$ and $\mathbf{V}_\tau^{(y)}$ represent the derivative of the delays w.r.t. x and the derivative of the delays w.r.t. y, respectively. Similarly, $\mathbf{V}_f^{(x)}$ and $\mathbf{V}_f^{(y)}$ represent the derivative of the Doppler shift w.r.t. x and the derivative of the Doppler shifts w.r.t. y, respectively. See (**??**) and (**??**) for more details. Further,

$$\Phi \triangleq \frac{WT\,SNR}{12\pi}. \quad (26)$$

This result is pleasing since $(12\pi)/(W^3T\,SNR)$ is recognized as the Cramér-Rao Lower Bound on time delay estimation and $3/(\pi WT^3SNR)$ is recognized as the Cramér-Rao Lower Bound on differential Doppler estimation.

4. Time and Frequency Synchronization Errors

So far we have assumed perfect time and frequency synchronization of all receivers. That means that the clocks at all platforms have exactly the same time and the same frequency. In practice this is impossible even if GPS and atomic clocks are used. It is therefore of interest to examine the sensitivity of the localization precision to time/frequency errors. Although a brute force small error analysis is possible we choose here to obtain the result using asymptotic reasoning.

As explained in the introduction, the two-step approach and the single-step approach are *asymptotically* equivalent. Thus, the error covariance is asymptotically the same. The two-step approach in our case consist of first estimating the differential Doppler (DD) and the time difference (TD) and then using the results for localization. The estimation error of DD and TD is affected directly by the synchronization errors of the receivers. Therefore the TD estimation error variance is obtained by adding to the error variance caused by the noise the variance of synchronization errors

$$\sigma_{TD}^2 \triangleq \frac{12\pi}{W^3T\,SNR} + \sigma_\tau^2, \quad (27)$$

and the DD estimation error is similarly given by

$$\sigma_{DD}^2 \triangleq \frac{3}{\pi WT^3SNR} + \sigma_f^2, \quad (28)$$

where σ_τ^2 and σ_f^2 are the variance of the time synchronization error between receivers and σ_f^2 is the variance of the frequency synchronization errors.

Finally, the localization error covariance is given by the inverse of the matrix whose elements are defined by

$$\begin{aligned}
\tilde{\mathbf{J}}_{1,1} &= \mathrm{tr}\Big\{\sigma_{TD}^{-2}[\mathbf{V}_\tau^{(x)}]^T\mathbf{V}_\tau^{(x)} + \sigma_{DD}^{-2}[\mathbf{V}_f^{(x)}]^T\mathbf{V}_f^{(x)}\Big\},\\
\tilde{\mathbf{J}}_{2,2} &= \mathrm{tr}\Big\{\sigma_{TD}^{-2}[\mathbf{V}_\tau^{(y)}]^T\mathbf{V}_\tau^{(y)} + \sigma_{DD}^{-2}[\mathbf{V}_f^{(y)}]^T\mathbf{V}_f^{(y)}\Big\},\\
\tilde{\mathbf{J}}_{1,2} &= \mathrm{tr}\Big\{\sigma_{TD}^{-2}[\mathbf{V}_\tau^{(x)}]^T\mathbf{V}_\tau^{(y)} + \sigma_{DD}^{-2}[\mathbf{V}_f^{(x)}]^T\mathbf{V}_f^{(y)}\Big\}.
\end{aligned} \quad (29)$$

This concludes this section.

5. Numerical Examples

In this section we provide a numerical example of the algorithm performance. We simulated two platforms whose speed is 300 meter/second and a stationary transmitter as depicted in Figure 1. The distance between the platforms is 1000 meter along the x axis and 50 meters along the y axis. The transmitter is located at coordinates $2000, 2000$ meters. The first receiver intercepts the signal when it is located at (0,0),(1000,0) and (2000,0) meters. The second receiver intercepts the signal when it is located at (1000,50), (2000,50) and (3000,50) meters. The transmitter carrier frequency is 1 GHz and the signal bandwidth is 262.144 KHz with flat

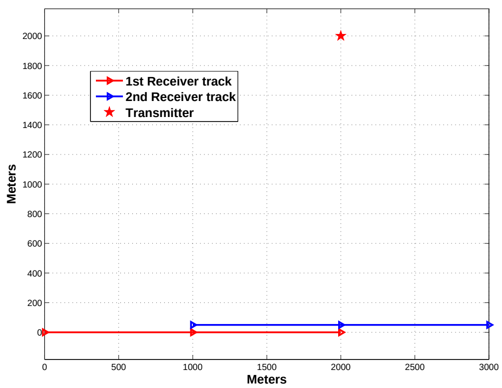

Fig. 1. Receivers track and emitter location

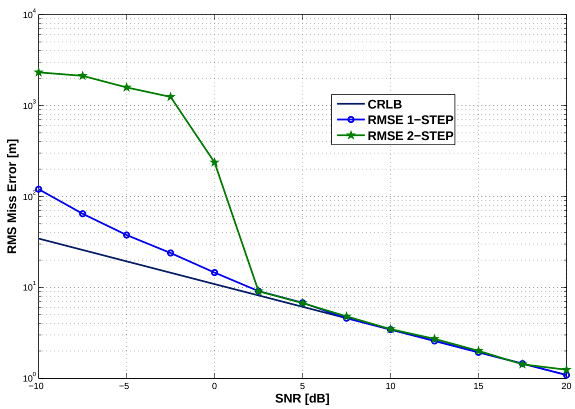

Fig. 2. RMSE of the DPD method (single step), the 2-step method and the Cramér-Rao Lower Bound versus SNR.

spectrum. The signal observation time at each interception point is 3.9 milliseconds.

Figure 2 shows the Root Mean Square miss distance as a function of Signal to Noise Ratio (SNR). The data is based on 200 Monte-Carlo trials for each data point. The performance of the advocated method (single-step a.k.a. DPD) the performance of the conventional, two-step method and the Cramér-Rao Lower Bound . Observe that for high SNR the DPD Root Mean Square miss distance is the same as that of the two step method. However, for low SNR the DPD provides much better performance.

Figure 3 shows the Root Mean Square miss distance as a function of the receivers frequency error. The SNR is held at 20dB, the timing error is held at 50 nano-second (standard deviation) and the frequency error is changed between 1 Hz and 10 KHz (standard deviation). We observe that the analysis predicts a degradation as the error increases with saturation when the frequency error stops to affect the error since it is dominated by the differential delay. The simulation agrees with the theory for small errors and diverge from the

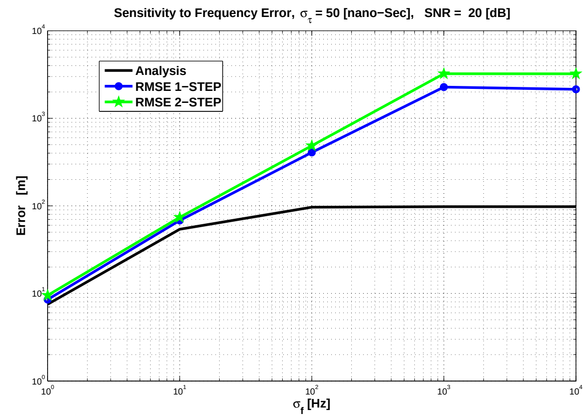

Fig. 3. RMSE of the DPD method (single step), the 2-step method and the theoretically predicted accuracy versus the receivers frequency error standard deviation.

prediction when the errors are large. Obviously, similar plots can be obtained for timing errors, receiver location errors and receivers velocity errors.

6. Conclusions

In this paper we proposed an algorithm for geolocation of an emitter observed by moving receivers. The transmitted signal is modeled as a random Gaussian signal and the localization is based on the delay and Doppler frequency shift that affect the observed signal. The proposed algorithm is the exact maximum likelihood. It is a single step estimator as opposed to two step methods that estimate the delay and the frequency shift in a first step and estimate the location in a second step. We demonstrated that the proposed method is superior to two step methods for low signal to noise ratio. We also presented closed form expressions for the Cramér-Rao Lower Bound Lower Bound. The expressions are simple and intuitive. We also provided sensitivity analysis for timing and frequency errors between receivers. An extension to receivers location/velocity errors is straightforward.

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References

[1] P. M. Schultheiss, and E. Weinstein, “Estimation of Differential Doppler Shifts,” *J. Acoust. Soc. Am.*, Vol. 66, No. 5, pp. 1412-1419, Nov. 1979.
[2] P. C. Chestnut, “Emitter location accuracy using TDOA and differential Doppler,” *IEEE Trans. on Aerospace and Electronic Systems*, Vol. AES-18, No. 2, pp. 214–218, Mar. 1982.
[3] D. J. Torrieri, “Statistical theory of passive location systems,” *IEEE Trans. on Aerospace and Electronic Systems*, Vol. AES-20, No. 2, pp. 183–198, Mar. 1984.
[4] B. Friedlander, “On the Cramér Rao Bound for Time Delay and Doppler Estimation,” *IEEE Trans. on Information Theory*, Vol. IT-30, No. 3, pp. 575-580, May 1984 .

[5] N. Levanon, "Interferometry against differential Doppler: performance comparison of two emitter location airborn systems," *IEE Proceedings*, Vol. 136, Pt. F, No. 2, pp. 70-74, Apr. 1989.

[6] K. Becker, "An efficient method of passive emitter location," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 28, No. 4, pp. 1091–1104, Oct. 1992.

[7] S. Stein, "Differential delay/Doppler ML estimation with unknown signals," *IEEE Trans. on Signal Processing*, Vol. 41, No. 8, pp. 2717-2719, Aug. 1993.

[8] K. C. Ho, and Y. T. Chan, "Geolocation of a known altitude object from TDOA and FDOA measurements," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 33, No. 3, pp. 770–783, Jul. 1997.

[9] D. P. Haworth, N. G. Smith, R. Bardelli, and T. Clement, "Interference localization for EUTELSAT satellites; The first european transmitter location system," *International Journal of Satellite Communications*, Vol. 15, No. 4, pp. 155–183, Sep. 1997

[10] J. Mason, "Algebraic two satellite TOA/FOA position soluion on an ellipsoidal earth," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 40, No. 3, pp. 1087–1092, Jul. 2004.

[11] A. Amar, and A.J. Weiss, "Localization of Radio Emitters Based on Doppler Frequency Shifts," *IEEE Transactions on Signal Processing,* vol. 56, no. 11, pp. 5500-5508, Nov. 2008.

[12] M. A. Woodbury, "Inverting modified matrices," *Statistical Research Group, Memo.* Rep. no. 42, Princeton University, Princeton, N. J., 1950.

[13] D. A. Harville, *Matrix Algebra From a Statistician's Perspective,* Springer, 2008.

[14] P. Stoica and T. Söderström, "On reparametrization of loss functions used in estimation and the invariance principle," *Signal Processing*, Volume 17, Issue 4, pp. 383-387, August 1989.

[15] A. L. Swindlehurst, and P. Stoica "Maximum Likelihood Methods in Radar Array Signal Processing," *Proceedings of the IEEE*, VOL. 86, NO. 2, pp. 421-441, Feb. 1998.

[16] S. M. Kay, *Fundamentals of Statistical Signal Processing - Estimation Theory, Volume I,* Prentice Hall, 1993.

[17] C. R. Rao, *Linear Statistical Inference and Its Applications*, 2nd Ed., New York, NY: John Wiley & Sons, 2002.

[18] R. M. Gray, "Toeplitz and Circulant Matrices: A review," Stanford University, Technical Report, 2000.

[19] F.W. Sun, et. al. "On the Convergence of the Inverse of Toeplitz Matrices and its Application," *IEEE Transactions on Information Theory,* vol. 49, no. 1, pp. 180-190, Jan. 2003.