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On the scale effect in the thin layer delamination problem

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Abstract. Influence of layer thickness on the stress distribution in the vicinity of a crack tip is examined, taking into account the fact that the conventional stress intensity factor concept becomes invalid if the thickness of the layer is not much more than the size of the fracture process zone. An eigen-problem is considered which is characterized by two asymptotes. The first is a near one; it is formed in a small vicinity of the crack tip in the layer thickness scale. The second asymptote is a far one in the same scale. The regions of validity of these asymptotes are determined and shown to depend upon layer thickness, material parameters and crack tip speed. The complete stress distribution in front of the crack is obtained, as well. Some conclusions are made concerning the stress distribution and energy release rate for the general problem. Mode III crack propagation is considered in detail.

1. Introduction

The problem of cracks along and within adhesive layers has been receiving much attention in the literature. Two papers (Rice [11]; Hutchinson [5]) have addressed the difficulties involved in dealing with interface crack problems; that is, when a crack is along the interface between two dissimilar media. In particular, the problem of a crack along the interface between a thin layer which is bonded to a substrate is of engineering importance as, for example, in coating or film delamination. This problem has been considered for the case of cracking of prestressed films (see Hutchinson and Suo [6]; Xu et al. [18]).

For layer decohesion, the problem of scaling requires consideration. This was recognized by Xu et al. [18]. In classical fracture mechanics theory, for a single finite length crack in an infinite, homogeneous, linear elastic body, two length scales, generally, enter into the problem; they include the crack length l and a length L associated with the load distribution. The stresses are found to be square root singular, corresponding to the eigen-problem for a semi-infinite crack in an infinite body. Indeed, only the stress intensity factor is essential for describing the stresses and displacements in the neighborhood of the crack tip.

An additional length scale arises in the case of a thin layer of thickness h which delaminates from a thick substrate (see Fig. 1). The main aim of this investigation is to more carefully examine this specific problem. In this case, if h is sufficiently small, the fracture process zone size R becomes important. For many fracture phenomena, this latter length scale is small compared to other length parameters. Note that the fracture process zone is the region in which micro-behavior is actually controlling crack propagation. Different fracture criteria have incorporated this parameter into their formulations. It appears for example, as the length of the Dugdale–Barenblatt cohesive zone (see Barenblatt [1]; Dugdale [2]; Rice [10]), the

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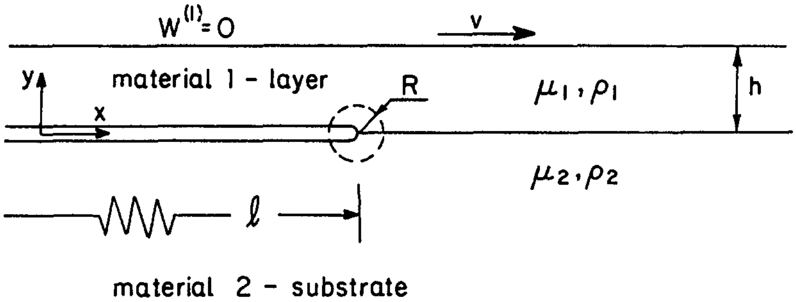

Fig. 1. Geometry of layer delamination from a substrate.

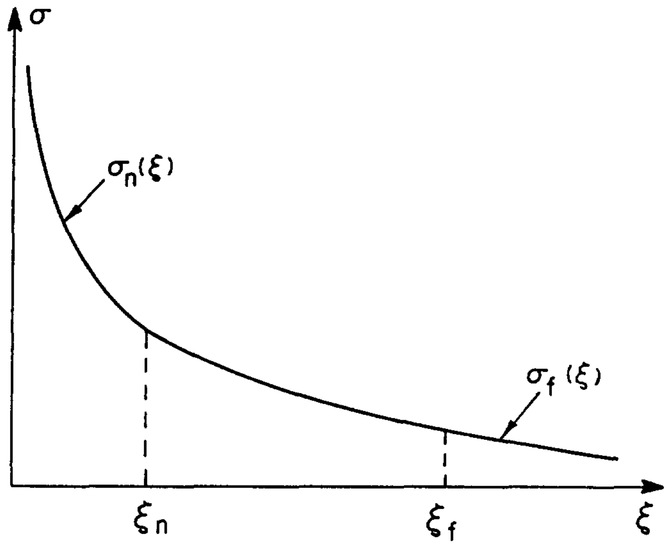

Fig. 2. Typical stress distribution ahead of a crack in the thin layer delamination problem.

length of the critical average stress zone of Neuber–Novozilov (see Neuber [8]; Novozilov [9]), and the length at which a critical strain is reached in the approach by McClintock [7].

In the usual fracture theory, the stress distribution in the process zone is controlled by some length parameters which are expected to be much greater than R. The more complicated problem of a thin layer, in which the parameter R cannot be neglected, is to be investigated here. The ratios h/l, h/L and h/R influence the stress behavior. A 'thin' layer is implied by $h/l \ll 1$ and $h/L \ll 1$. Hence, the only essential scaling parameter is $\lambda \equiv h/R$. Indeed, the layer thickness h influences the magnitude of the stress intensity factor. If $\lambda \gg 1$, the formulation of a crack propagation criterion based upon the stress intensity factor remains the same as if $h \to \infty$. If λ is not sufficiently large, however, the value of h influences not only the amplitude but also the form of the stress distribution within, as well as outside, the process zone. No longer are the stresses necessarily square root singular nor may the stress intensity factor be employed as a fracture criterion.

In this study, the crack is assumed to propagate steadily. Results will also be presented for a stationary crack; that is, a crack tip velocity $v = 0$. Two limiting cases of near and far fields are considered. The stress distribution in front of the crack tip and the crack opening displacements behind the crack tip may be determined in both cases. Let $\xi = x - vt$ be the crack tip coordinate in the x-direction where t is time. Close to the crack tip, for $\xi/h \to 0$, an asymptote for the stresses, for example, directly ahead of the crack in the near field may be found as $\sigma_n(\xi)$. On the other hand, a far field asymptote, $\sigma_f(\xi)$ for $\xi/h \to \infty$ may be determined. Both asymptotes correspond to the usual assumptions that $\xi/l \to 0$ and $\xi/L \to 0$. Thus, in both regimes square root singular behavior may exist with different stress intensity factors. For K-controlled fracture, one must determine which asymptote is responsible for crack propagation. When these do not control fracture, the complete stress distribution is required. These conditions are determined by the values of the ratios ξ_n/R and ξ_f/R where ξ_n and ξ_f are (see Fig. 2), respectively, validity limits of the asymptotes, namely

$$\sigma(\xi) \approx \sigma_n(\xi), \quad 0 < \xi < \xi_n,$$
$$\sigma(\xi) \approx \sigma_f(\xi), \quad \xi_f < \xi.$$

If $R \ll \xi_n$, the conventional stress intensity factor concept is valid and one may use the σ_n asymptote. For $R \gg \xi_f$, again the stress intensity factor is valid but from the far asymptote σ_f. Finally, if neither of these inequalities is satisfied, the exact stress distribution $\sigma(\xi)$ is required for predicting crack propagation.

In this investigation, both static and steadily propagating cracks between dissimilar linear elastic materials are considered. In Section 2, a limiting procedure for deriving near and far asymptotes is developed. In Section 3, energy methods are employed to determine the relationship between the inner and outer distributions of the stresses and crack opening displacements in the general problem. In particular, the mode III problem is considered in detail. In Section 4, by means of the Wiener–Hopf method, a closed form eigensolution is obtained. The asymptotic behavior of this solution is investigated in Section 5. Analysis of the full stress distribution for various material parameter combinations is presented in Section 6. The validity regions of the asymptotic solutions are determined.

2. Some asymptotes

Consider propagation of a straight semi-infinite crack in an elastic body. Geometric and mechanical parameters of the body are assumed to be constant in the direction of crack propagation x; they may be non-uniformly distributed normal to this direction. One may consider the eigensolution of a semi-infinite crack as the limit of the solution corresponding to the action of an external force. Namely, let the crack be subjected to an external force moving together with the crack so that a steady state problem is defined. As a result of the work done by the moving force, there is energy flux from the force to the crack tip. In a limiting process, the amplitude of the applied load approaches zero in any finite vicinity of the crack tip such that the energy release rate remains fixed. For example, for a concentrated force, the distance of the force from the crack tip d becomes infinite while its amplitude which is proportional to $\sqrt{d}$ also increases such that the energy release rate is fixed. In this way, the eigensolution corresponding to an energy source at infinity and the energy flux from infinity to the moving crack tip are obtained. The eigensolution and its asymptotes are derived in this paper. In this section, the exact mathematical concepts are presented.

First, consider the crack surfaces subjected to a traction vector such that the non-zero component of this vector $\sigma(\xi) = -\delta(\xi - \eta)$ where δ is the Dirac delta-function, $\xi < 0$ and $\eta < 0$. This component is assumed to act on the lower surface of the crack $y = -0$ with a corresponding component of opposite sign acting on the upper surface. Let the length h be a characteristic size of material or geometric parameters of the body in the y-direction. In particular, here h is taken to be layer thickness. The corresponding stress component ahead of the crack tip and crack face displacements behind the crack tip resulting from the delta function are denoted by $\sigma(\xi,\eta)$ where $\xi > 0$, $\eta < 0$ and $y = 0$ and $u^{(r)}(\xi,\eta)$ where $\xi < 0$, $\eta < 0$ and $r = 1,2$ representing the upper and lower materials, respectively, so that, $y = \pm 0$, correspondingly. From dimensional analysis, these functions may be expressed as

$$\sigma(\xi,\eta) = \frac{1}{\sqrt{-\xi\eta}}\, M\left(\frac{\xi}{\eta},\frac{\xi}{h}\right) \tag{1}$$

and

$$u^{(r)}(\xi,\eta) = \sqrt{\frac{\xi}{\eta}}\, N^{(r)}\left(\frac{\xi}{\eta},\frac{\xi}{h}\right). \tag{2}$$

It is assumed that the functions M and $N^{(r)}$ are defined so that the stresses and displacements in the neighborhood of the crack tip produce a bounded strain energy in any finite region. Further, energy flux from infinity under variation of crack tip position is precluded. Thus, the solution is unique.

If the crack is subjected to an arbitrary traction $\sigma(\eta)$, these variables may be expressed by means of superposition as

$$\sigma(\xi) = -\frac{1}{\sqrt{\xi}}\int_{-\infty}^{0}\sigma(\eta) M\left(\frac{\xi}{\eta},\frac{\xi}{h}\right)\frac{\mathrm{d}\eta}{\sqrt{-\eta}} \quad (\xi > 0) \tag{3}$$

and

$$u^{(r)}(\xi) = -\sqrt{-\xi}\int_{-\infty}^{0}\sigma(\eta) N^{(r)}\left(\frac{\xi}{\eta},\frac{\xi}{h}\right)\frac{\mathrm{d}\eta}{\sqrt{-\eta}} \quad (\xi < 0). \tag{4}$$

Note that in the specific case of a homogeneous elastic infinite plane in statics, the functions M and $N^{(r)}$ depend only on the first argument (see for example, Slepyan [17]) namely

$$M = \frac{1}{\pi}\left(1 - \frac{\xi}{\eta}\right)^{-1} \tag{5}$$

and

$$N^{(r)} = \pm\frac{\kappa+1}{4\pi\mu}\sqrt{\frac{\eta}{\xi}}\,\ln\left|\frac{\sqrt{-\xi}+\sqrt{-\eta}}{\sqrt{-\xi}-\sqrt{-\eta}}\right|, \tag{6}$$

where μ is the shear modulus, $\kappa = 3-4\nu$ for plane strain in modes I and II, $\kappa = 3$ for mode III and ν is Poisson's ratio.

In the general case, for $\xi \to 0$, one has

$$\sigma(\xi) \sim -\frac{1}{\sqrt{\xi}} M(0,0)\int_{-\infty}^{0}\sigma(\eta)\frac{\mathrm{d}\eta}{\sqrt{-\eta}} = \frac{K_m}{\sqrt{2\pi\xi}}, \tag{7}$$

where K_m is the stress intensity factor and $m = 1, 2$ and 3 depending upon the fracture mode. Thus,

$$\int_{-\infty}^{0} \sigma(\eta) \frac{\mathrm{d}\eta}{\sqrt{-\eta}} = -\frac{K_m}{M(0,0)\sqrt{2\pi}}. \tag{8}$$

Assuming that this integral exists, a set of tractions

$$\sigma(\eta) = -\frac{1}{\sqrt{L}} p\left(\frac{\eta}{L}\right), \tag{9}$$

is examined where L is a parameter. Substituting $\eta = \alpha L$ into (3) and (4), one has

$$\sigma(\xi) = \frac{1}{\sqrt{\xi}} \int_{-\infty}^{0} p(\alpha)\, M\left(\frac{\xi}{\alpha L}, \frac{\xi}{h}\right) \frac{\mathrm{d}\alpha}{\sqrt{-\alpha}} \tag{10}$$

and

$$u^{(r)}(\xi) = \sqrt{-\xi} \int_{-\infty}^{0} p(\alpha) N^{(r)}\left(\frac{\xi}{\alpha L}, \frac{\xi}{h}\right) \frac{\mathrm{d}\alpha}{\sqrt{-\alpha}}. \tag{11}$$

The traction $\sigma(\eta)$ in (9) extends along the negative half-axis, and its amplitude tends to zero as $L \to \infty$. The force is given by

$$\int_{-\infty}^{0} \sigma(\eta)\, \mathrm{d}\eta = -\sqrt{L} \int_{-\infty}^{0} p(\alpha)\, \mathrm{d}\alpha, \tag{12}$$

which becomes infinite if the integral on the right hand side is non-zero. Letting $L \to \infty$ in (10) and (11), one may obtain the eigensolution

$$\sigma(\xi) = \frac{M(0, \xi/h)}{M(0,0)} \frac{K_m}{\sqrt{2\pi\xi}} \tag{13}$$

and

$$u^{(r)} = \frac{N^{(r)}(0, \xi/h)}{M(0,0)} K_m \sqrt{\frac{-\xi}{2\pi}}. \tag{14}$$

It should be noted that the solution in (13) and (14) only differs by K_m as the applied traction $\sigma(\eta)$ changes. The functional form of the solution remains the same.

This eigensolution has two asymptotes. The first is the near one in the h-scale. It is valid for $\xi/h \to 0$, namely

$$\sigma \sim \sigma_n = \frac{K_m}{\sqrt{2\pi\xi}} \tag{15}$$

and

$$u^{(r)} \sim u_n^{(r)} = \frac{N^{(r)}(0,0)}{M(0,0)} K_m \sqrt{\frac{-\xi}{2\pi}}. \tag{16}$$

The second asymptote, σ_f, $u_f^{(r)}$, is the far one. It corresponds to the far field in the h-scale, where $\xi/h \to \infty$. In particular, if $M(0,\infty)$ and $N^{(r)}(0,\infty)$ are non-zero and finite, one has

$$\sigma \sim \sigma_f = \frac{M(0,\infty)}{M(0,0)} \frac{K_m}{\sqrt{2\pi\xi}} \tag{17}$$

and

$$u^{(r)} \sim u_f^{(r)} = \frac{N^{(r)}(0,\infty)}{M(0,0)} K_m \sqrt{\frac{-\xi}{2\pi}} . \tag{18}$$

However, the far asymptotes are not necessarily of the square root type. For example, in the case of a layer bonded to a half-plane which is considered in the sequel, $-\infty < y < h$, the last representation is not valid for $u_f^{(1)}$. Another example of this kind is shown in the next section.

It may be noted that these results are general. For an interface crack with mixed mode I and II deformation, the same considerations may be generalized with more than one stress and displacement component resulting.

Finally, the near and far asymptotes in the h-scale may both be considered as near asymptotes in the load-scale L for the problem with loaded crack surfaces. These asymptotes exist if the ratio L/h is sufficiently large so that the relation $h \ll \xi \ll L$ holds for some range of ξ.

3. Energy considerations

In this section, a generalization of known formula for the energy release rate is introduced. This formula for the energy released at the crack tip is obtained from the stress and displacement distributions in any finite vicinity of the crack tip on the crack propagation line. Both the near, as well as far asymptotes of these distributions may be employed in this calculation.

Let the stress intensity factor be given by K_m. Assume that the crack tip is traveling with speed v and that there are no harmonic waves which can propagate in the x-direction with phase velocity v. Since these waves are excluded, the energy release rate is equal to the work of the force per unit length of crack propagation, including the work of the forces at infinity. Two steady state problems are considered (see Fig. 3). The eigensolution of the problem in Fig. 3(a) is characterized by an energy release rate G and stress intensity factor given by $K_m = K_m^0$. Internal stresses ahead of the crack act on the lower part of the body $y = -0$ given by $\boldsymbol{\sigma}(x - vt)$ (see Fig. 3(a)). It may be noted that $\boldsymbol{\sigma}$ represents the stress components σ_{yx}, σ_{yy} or σ_{yz}. The solution to the second problem in Fig. 3(b) is described next. The cracks in these two problems differ by their crack tip coordinates $x = vt$ and $x = vt + a$ for the first (Fig. 3(a)) and the second (Fig. 3(b)) cases, respectively. In addition, the traction vector

$$\tilde{\boldsymbol{\sigma}} = \mp\Lambda\boldsymbol{\sigma}(x - vt), \tag{19}$$

is applied in the second problem as shown in Fig. 3(b) for $vt < x < vt + a$, $y = \pm 0$ where Λ is a parameter such that $0 \leqslant \Lambda \leqslant 1$. The traction $\boldsymbol{\sigma}(x - vt)$ is the same as that which acts on the lower part of the body, $y = -0$, in front of the crack in the first problem.

When $\Lambda = 1$, the first problem is recovered; when $\Lambda = 0$ the second problem becomes the same as the first but with crack tip coordinate $x = vt + a$. In the second problem there

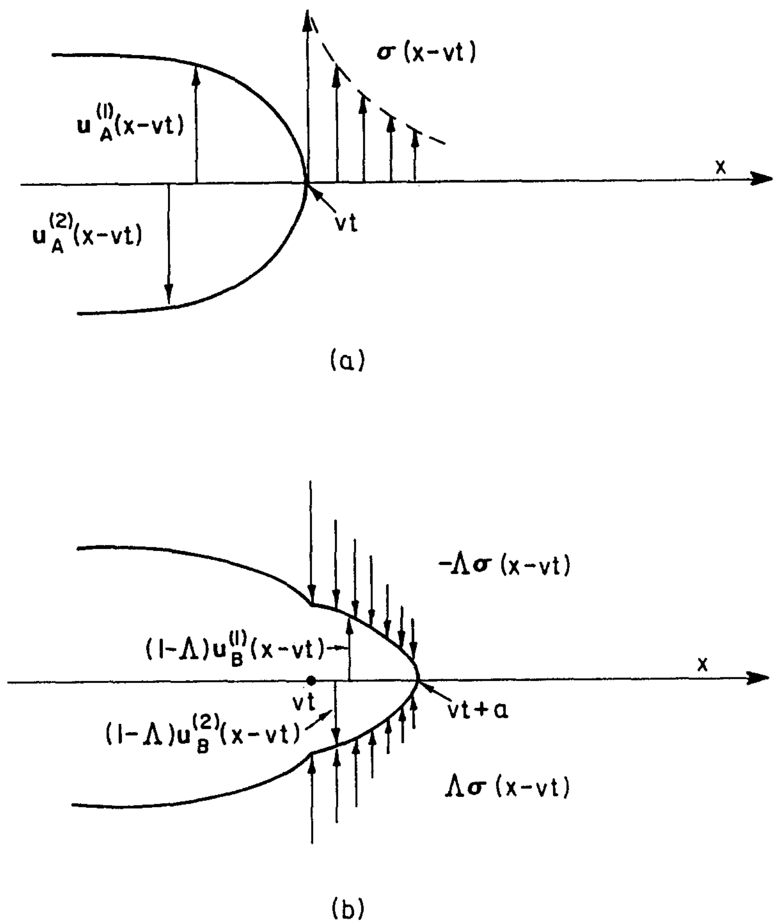

Fig. 3. Two steady state semi-infinite crack problems. (a) The crack tip coordinate is $x = vt$. There is energy flux from infinity. (b) The crack tip coordinate is $x = vt + a$ with the same energy flux from infinity and applied traction $\mp\Lambda\sigma(x - vt)$.

are two singular points: one at $x = vt$ with a stress intensity factor K_m^- and the other at $x = vt + a$ with a stress intensity factor K_m^+. The stress intensity factors as linear functionals of the traction are represented by

$$K_m = K_m^+ = (1 - \Lambda)K_m^0 \tag{20}$$

and

$$K_m = K_m^- = \Lambda K_m^0\ . \tag{21}$$

The crack face opening displacement in the interval $vt < x < vt + a$ can be described by

$$\tilde{\boldsymbol{u}} = (1 - \Lambda)\boldsymbol{U}_B(x - vt) = (1 - \Lambda)\boldsymbol{U}_A(x - vt - a), \tag{22}$$

where $\boldsymbol{U}_A = \boldsymbol{u}_A^{(1)} - \boldsymbol{u}_A^{(2)}$ is the crack face opening displacement in the first case and $(1 - \Lambda)\boldsymbol{U}_B = (1 - \Lambda)(\boldsymbol{u}_B^{(1)} - \boldsymbol{u}_B^{(2)})$ is the jump in the second case; the superscripts (1) and (2) represent the upper and lower crack faces, respectively.

The above mentioned condition of no waves with phase velocity v implies that there are no waves excited by the traction, so that, the influence of the traction is local. It follows from energy conservation that all work produced by the traction flows into the singular points. As a result, the sum of the energy fluxes radiated by the traction and the singular points $x = vt$ and $x = vt + a$, $y = 0$ is independent of Λ.

The energy release rate is a quadratic function of the stress intensity factor. Corresponding to $K_m = K_m^0$, the energy release rate is given by G. It may be observed that the energy released through the points $x = vt$ and $x = vt + a$ are, respectively, $\Lambda^2 G$ and $(1 - \Lambda)^2 G$. In addition, the energy release rate per unit time resulting from the traction is given by

$$\begin{aligned}
&\int_{vt}^{vt+a} \Lambda\boldsymbol{\sigma}(x - vt) \cdot (1 - \Lambda)\frac{\partial \boldsymbol{U}_B(x - vt)}{\partial t}\,\mathrm{d}x \\
&\quad = -\Lambda(1 - \Lambda)v \int_{vt}^{vt+a} \boldsymbol{\sigma}(x - vt) \cdot \frac{\partial \boldsymbol{U}_B(x - vt)}{\partial x}\,\mathrm{d}x \\
&\quad = \Lambda(1 - \Lambda)v \int_0^a \boldsymbol{\sigma}(\xi) \cdot \frac{\partial \boldsymbol{U}_A(\xi - a)}{\partial a}\,\mathrm{d}\xi, \qquad (23)
\end{aligned}$$

where $\xi = x - vt$. Since the total energy release rate is independent of Λ, it is possible to express the energy dissipation rate Gv as

$$Gv = \Lambda^2 Gv + (1 - \Lambda)^2 Gv + \Lambda(1 - \Lambda)v\frac{\mathrm{d}}{\mathrm{d}a}\int_0^a \boldsymbol{\sigma}(\xi) \cdot \boldsymbol{U}(\xi - a)\,\mathrm{d}\xi, \qquad (24)$$

where $\boldsymbol{U} \equiv \boldsymbol{U}_A$. Note that the limits of integration are assumed to be $(-0, a + 0)$. Since the convolution integral is actually from $-\infty$ to ∞, with $\boldsymbol{\sigma}(\xi) = 0$ for $\xi < 0$ and $\boldsymbol{U}(\xi - a) = 0$ for $\xi - a > 0$, so that

$$\frac{\mathrm{d}}{\mathrm{d}a}\int_0^a f(\xi, a)\,\mathrm{d}\xi = \int_0^a \frac{\partial}{\partial a} f(\xi, a)\,\mathrm{d}\xi$$

where $f(\xi) = \boldsymbol{\sigma}(\xi) \cdot \boldsymbol{U}(\xi - a)$.

Equation (24) is satisfied only when

$$G = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}a}\int_0^a \boldsymbol{\sigma}(\xi) \cdot \boldsymbol{U}(\xi - a)\,\mathrm{d}\xi. \qquad (25)$$

Since G does not depend upon a, it is possible to rewrite this equality as

$$G = \frac{1}{2a}\int_0^a \boldsymbol{\sigma}(\xi) \cdot \boldsymbol{U}(\xi - a)\,\mathrm{d}\xi, \qquad (26)$$

where the integral is proportional to the upper limit. This result is a version of the well known formula

$$\begin{aligned}
G &= \lim_{a\to 0} \frac{1}{2a}\int_0^a \boldsymbol{\sigma}(\xi) \cdot \boldsymbol{U}(\xi - a)\,\mathrm{d}\xi \\
&= \tfrac{1}{2} \lim_{a\to 0} \frac{\mathrm{d}}{\mathrm{d}a}\int_0^a \boldsymbol{\sigma}(\xi) \cdot \boldsymbol{U}(\xi - a)\,\mathrm{d}\xi, \qquad (27)
\end{aligned}$$

which by the same procedure was extended to dynamics by Slepyan [16]. It is important that the last result be valid for general conditions, including non-uniform crack propagation and other cases in which harmonic waves exist. Since the traction as a generalized function tends to zero with a, it cannot be the source of a wave. In the steady state case, the expression without the limit sign in (26) is more interesting to discuss. First, it shows that the result is independent of the integration interval and hence, valid for all values of a; second, under conditions to be described, this formula allows one to conclude that the energy release rate can be determined using the far asymptotes of the stress and crack opening displacement distributions, as well as the usual near asymptotes. It is well-known that the J-integral yields the energy release representation which is valid for both near and far fields. The discussed relation (26) essentially differs from the J-integral which is based on the infinitesimal variation whereas the above obtained formula (26) corresponds to the finite variation if $a > 0$.

It is useful to represent the first limit in (27) in another form as was done by Slepyan [16]. This may be done by applying to the convolution integral in (27) the one-sided Fourier transform

$$f^F(s) = \int_0^\infty f(a) \exp(isa)\, \mathrm{d}a \tag{28}$$

and the known theorem

$$\lim_{a \to 0} f(a) = \lim_{s \to +\infty} s f^F(is), \tag{29}$$

which is valid if the left-hand limit exists. As a result, one may write

$$G = \tfrac{1}{2} \lim_{s \to +\infty} s^2 \boldsymbol{\sigma}^F(is) \cdot \boldsymbol{U}^F(-is)\,. \tag{30}$$

In the same manner, it is possible to omit 'lim' for the steady state case, so that

$$G = \tfrac{1}{2} s^2 \boldsymbol{\sigma}^F(is) \cdot \boldsymbol{U}^F(-is)\,. \tag{31}$$

Note that (26), (27), (30) and (31) are valid not only for the functions $\boldsymbol{\sigma}$ and $\boldsymbol{U}$ of the square root type but also for any other functions which may be employed to model elastic bodies, such as discrete systems, beams, etc.

With this as a basis, a relation between the near and far asymptotes is determined. The energy release rate in (26) may be rewritten as

$$G = \frac{1}{2} \int_0^1 \boldsymbol{\sigma}(a\zeta) \cdot \boldsymbol{U}[a(\zeta - 1)]\, \mathrm{d}\zeta \tag{32}$$

where $\zeta = \xi/a$. As $a \to 0$, the value of the energy release rate G is attained. Then, it is possible to replace the stress and jump in the crack face displacements by their near field asymptotes $\boldsymbol{\sigma}_n$ and $\boldsymbol{U}_n$ provided their convolution is independent of a. Thus, the equality in (32) may be replaced by

$$G = \frac{1}{2} \int_0^1 \boldsymbol{\sigma}_n(a\zeta) \cdot \boldsymbol{U}_n[a(\zeta - 1)]\, \mathrm{d}\zeta. \tag{33}$$

Next, consider the far asymptotes σ_f and U_f which are assumed to be extrapolated to $\xi = 0$ provided that the convolution of the type in (32) is independent of a and thus

$$G_f = \frac{1}{2}\int_0^1 \sigma_f(a\zeta)\cdot U_f[a(\zeta-1)]\,\mathrm{d}\zeta, \tag{34}$$

is constant. Assume the integrands in (32) and (34) are integrable uniformly with respect to a, so that these relations may be represented as

$$G = \tfrac{1}{2}\lim_{\varepsilon\to 0}\lim_{a\to\infty}\int_\varepsilon^{1-\varepsilon} \sigma(a\zeta)\cdot U[a(\zeta-1)]\,\mathrm{d}\zeta \tag{35}$$

and

$$G_f = \tfrac{1}{2}\lim_{\varepsilon\to 0}\lim_{a\to\infty}\int_\varepsilon^{1-\varepsilon} \sigma_f(a\zeta)\cdot U_f[a(\zeta-1)]\,\mathrm{d}\zeta. \tag{36}$$

In this case, one can see that $G_f \to G$ when $a \to \infty$, but these values are independent of a. Hence, $G_f = G$ and the energy release rate can be expressed by the far asymptote, namely

$$G = \frac{1}{2a}\int_0^a \sigma_f(\xi)\cdot U_f(\xi - a)\,\mathrm{d}\xi. \tag{37}$$

Thus, under the above mentioned condition of integrability, a relation exists between the near and far asymptotes given by

$$\int_0^a \sigma_n(\xi)\cdot U_n(\xi - a)\,\mathrm{d}\xi = \int_0^a \sigma_f(\xi)\cdot U_f(\xi - a)\,\mathrm{d}\xi. \tag{38}$$

Assume the functions $\sigma_n(\xi)$, $U_n(\xi)$, $\sigma_f(\xi)$ and $U_f(\xi)$ are of the square root type, so that they are characterized by stress intensity factors; the far field factors may differ from the near field factors as was seen in Section 2. In this case, the validity of the equality in (38) is assured provided the integrands are integrable and both sides of the relation are independent of a. This equality yields the relation between these factors as was previously noted in [6] for interfacial or sub-interface cracks.

However, the relation in (38) is valid in a more general case, as well. Consider, for example, the static problem of a centrally located semi-infinite crack in a non-uniform elastic plane under mode III deformation. The distribution of the shear modulus is assumed to be of the power type in the y-direction, namely

$$\mu = \mu_0\left(\frac{|y|}{h}\right)^\alpha, \quad -1 < \alpha < 1, \tag{39}$$

where μ_0 is a constant with dimensions of stress and h is a length parameter. Note that for modes I and III, the stress distribution for a constant load applied to the faces of a finite length crack with $\alpha > 0$ was determined by Sih and Chen [15]. Here, the governing equation for the displacement w is given by

$$|y|^\alpha\frac{\partial^2 \mathrm{w}}{\partial x^2} + \frac{\partial}{\partial y}\left(|y|^\alpha\frac{\partial \mathrm{w}}{\partial y}\right) = 0. \tag{40}$$

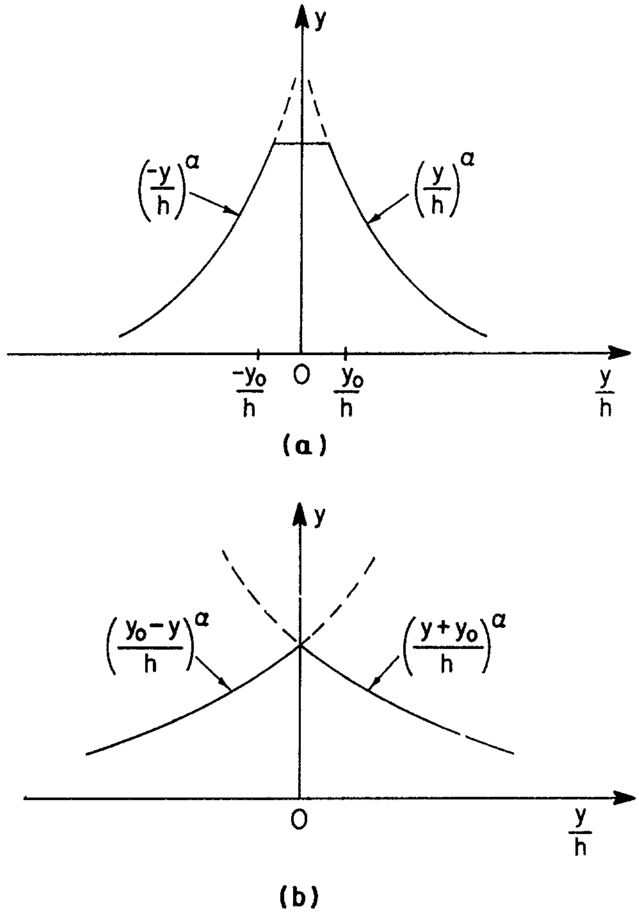

Fig. 4. Possible shear moduli distributions with $\alpha < 0$.

Applying the Fourier integral transform in x, one has the solution

$$\mathrm{w}^F(s,y) = \mathrm{sgn}(y)\, A(s)|y|^\nu K_\nu(|sy|), \tag{41}$$

where s is assumed to be real, $A(s)$ is an unknown function, $\nu = (1-\alpha)/2$ and K_ν is the modified Bessel function of the third kind. The Fourier transform of the stress is seen to be

$$\sigma_{yz}^F = \mu_0 \left(\frac{|y|}{h}\right)^\alpha \frac{\mathrm{dw}(s,y)}{\mathrm{d}y} = -\mu_0 A(s) \left(\frac{|y|}{h}\right)^{\nu+\alpha} h^\nu |s| K_{\nu+\alpha}(|sy|). \tag{42}$$

For $y \to \pm 0$

$$\sigma_{yz}^F(s,0) = -\mu_0 h^{-\alpha} A(s)\Gamma(\nu+\alpha)\left(\frac{|s|}{2}\right)^\nu \tag{43}$$

and

$$\mathrm{w}^F(s,\pm 0) = \pm\tfrac{1}{2} A(s)\Gamma(\nu)\left(\frac{|s|}{2}\right)^{-\nu}, \tag{44}$$

where Γ is the Gamma-function. Thus, along the x-axis, the transforms of the stress and displacement are related by

$$\sigma_{yz}^F = \mp\mu_0 \left(\frac{2}{h}\right)^\alpha \frac{\Gamma(\nu+\alpha)}{\Gamma(\nu)} |s|^{1-\alpha}\, \mathrm{w}^F . \tag{45}$$

Next, by employing a technique presented by Slepyan [16] for development of eigensolutions to such problems, it is possible to show that the transforms of the stress in front of the crack and the crack opening displacement can be expressed as

$$\sigma_{yz}^F = B\left(\frac{h}{2}\right)^{-(1/2)\alpha} (0-is)^{-(\nu+\alpha)} \tag{46}$$

and

$$\mathrm{w}^F = \pm\frac{B}{\mu_0}\left(\frac{h}{2}\right)^{(1/2)\alpha} \frac{\Gamma(\nu)}{\Gamma(\nu+\alpha)}(0+is)^{-(1+\nu)}, \tag{47}$$

where B is a constant. Using the expression for the energy release rate in (31), one can obtain

$$G = \frac{B^2}{\mu_0}\frac{\Gamma(\nu)}{\Gamma(\nu+\alpha)}, \tag{48}$$

where it may be recalled that $\nu = (1-\alpha)/2$.

This result may be considered as an example of use of the expressions for the energy release rate in (26) or (31) for stresses and crack opening displacements, which are not of the square root type. On the other hand, the above solution may be employed as the far asymptote for more realistic problems. Let the dependence (39) be valid only at a distance along the x-axis, say, for $|y| > y_0 > 0$, and the layer $|y| < y_0$ be of a finite, non-zero modulus (see Fig. 4(a)). Another possibility for the shear modulus distribution is shown in Fig. 4(b). In these cases if $\alpha \neq 0$, the derived solution is the far asymptote. For the energy release rate calculation, the near asymptote of the square root type is not required.

Now consider the mode III eigenproblem that is being investigated below in which $v = 0$, i.e., a standing crack. Let μ_1 and μ_2 be the shear moduli of the layer and half-space, respectively (see Fig. 1). The near asymptotes are characterized by the stress intensity factor $K_3 \equiv k$, so that

$$\sigma_{yz} \sim \frac{k}{\sqrt{2\pi\xi}}, \quad u_z^{(1)} \sim \frac{2k}{\mu_1}\sqrt{\frac{-\xi}{\pi}}, \quad u_z^{(2)} \sim -\frac{2k}{\mu_2}\sqrt{\frac{-\xi}{\pi}}, \tag{49}$$

as $\xi \to 0$. From (26), the energy release rate is found to be

$$G = \frac{1}{4}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)k^2. \tag{50}$$

At the same time, the influence of the layer on the far asymptotes σ_f and $u_f^{(2)}$ is negligible and the displacement $u_f^{(1)}$ is bounded as a result of the fixed displacement condition on the layer boundary; these do not contribute to the energy release rate. As a result, the energy release rate may be expressed as

$$G = \frac{K^2}{4\mu_2}, \tag{51}$$

where K is the far field stress intensity factor. Equating (50) and (51) leads to

$$K = \sqrt{1 + \frac{\mu_2}{\mu_1}}\; k. \tag{52}$$

Similarly, employing known relationships (see, for example Freund [3]) one may write for steady state propagation

$$G = \frac{1}{4}\left(\frac{1}{\mu_1 a_1} + \frac{1}{\mu_2 a_2}\right) k^2 = \frac{K^2}{4\mu_2 a_2}, \tag{53}$$

where $a_r^2 = 1 - v^2/c_r^2$, the shear wave speed $c_r^2 = \mu_r/\rho_r$, ρ_r is the density and $r = 1, 2$ represents the upper and lower materials, respectively. Consequently,

$$K = \sqrt{1 + \frac{\mu_2 a_2}{\mu_1 a_1}}\; k, \tag{54}$$

for $v < \min\,(c_1, c_2)$.

4. Determination of the eigensolution

In this section, a general approach is employed for a semi-infinite crack propagating steadily along the interface of an elastic layer of thickness h and a dissimilar elastic half-space (Fig. 1). Only mode III deformation is considered. In the fixed coordinate system (x, y, z) with $y = 0$ being the interface, the governing equations for the antiplane problem include the equilibrium equations

$$\frac{\partial \sigma_{xz}^{(r)}}{\partial x} + \frac{\partial \sigma_{yz}^{(r)}}{\partial y} = \rho_r \frac{\partial^2 \mathrm{w}^{(r)}}{\partial t^2}, \tag{55}$$

the constitutive equations

$$\sigma_{xz}^{(r)} = 2\mu_r \varepsilon_{xz}^{(r)}, \quad \sigma_{yz}^{(r)} = 2\mu_r \varepsilon_{yz}^{(r)}, \tag{56}$$

and the strain-displacement relations

$$\varepsilon_{xz}^{(r)} = \frac{1}{2}\frac{\partial \mathrm{w}^{(r)}}{\partial x}, \quad \varepsilon_{yz}^{(r)} = \frac{1}{2}\frac{\partial \mathrm{w}^{(r)}}{\partial y}. \tag{57}$$

Here, the only non-zero displacement is $\mathrm{w}^{(r)}(x, y, t) \equiv u_z^{(r)}(x, y, t)$. Recall that the index $r = 1, 2$, denotes the materials of the layer and half-space, respectively; μ_r are their elastic moduli and ρ_r are their densities. The upper boundary of the layer $y = h$ is assumed to be clamped, so that

$$\mathrm{w}^{(1)}(x, h, t) = 0. \tag{58}$$

Stresses and displacements on the uncracked interface $x - vt > 0$ satisfy the continuity conditions

$$\begin{aligned} \sigma_{yz}^{(1)}(x,0,t) &= \sigma_{yz}^{(2)}(x,0,t), \\ \mathrm{w}^{(1)}(x,0,t) &= \mathrm{w}^{(2)}(x,0,t). \end{aligned} \tag{59}$$

In order to obtain an eigensolution, the first step is to construct a solution for some non-zero loading. Then, a limiting procedure is employed in which the loading is applied along an infinitely long boundary with its amplitude approaching zero. To this end, it is convenient to take a load applied to the crack faces in the form

$$\sigma_{yz}^{(r)}(x,0,t) = -\sigma_0 \exp[(x-vt)/L], \quad x - vt < 0, \tag{60}$$

where L is a decay length.

The stress intensity factor for the viscoelastic analog of the formulated boundary value problem (55) through (60) was calculated by Ryvkin and Banks–Sills [13]). Since several steps in the solution here are the same as in that paper, they are only summarized. After introducing the moving coordinate

$$\xi = x - vt, \tag{61}$$

the Fourier transform

$$f^F(s,y) = \int_{-\infty}^{\infty} f(\xi,y)\exp(is\xi)\,\mathrm{d}\xi, \tag{62}$$

is employed. In (62), s is a complex variable. The inverse transform may be carried out along the contour $\mathcal{L}$ located on the real axis $\mathrm{Re}(s) = 0$. The problem is reduced to the Wiener–Hopf equation

$$H^+(s) = P(s)W^-(s) - H^-(s), \quad s \in \mathcal{L}, \tag{63}$$

where '+' and '−' functions represent, as usual, one-sided transforms analytic in the upper ($\mathrm{Im}(s) \geqslant 0$) and lower ($\mathrm{Im}(s) \leqslant 0$) half-planes, correspondingly. Namely,

$$H^+(s) = \int_0^{\infty} \sigma(\xi)\exp(is\xi)\,\mathrm{d}\xi, \tag{64}$$

is the transformed unknown interface stress in front of the crack and $\sigma(\xi) \equiv \sigma_{yz}^{(1)}(\xi,0) = \sigma_{yz}^{(2)}(\xi,0)$

$$H^-(s) = \int_{-\infty}^{0} \sigma(\xi)\exp(is\xi)\,\mathrm{d}\xi = -\frac{\sigma_0}{is+b}, \tag{65}$$

is the transformed applied stress of (60) and $b = 1/L$; and

$$W^-(s) = \int_{-\infty}^{0} W(\xi)\exp(is\xi)\,\mathrm{d}\xi, \tag{66}$$

is the transformed crack opening displacement with $W(\xi) \equiv w^{(1)}(\xi,0) - w^{(2)}(\xi,0)$. The coefficient $P(s)$ in (63) is given by

$$P(s) = -\mu_2\gamma_2 Q^{-1}(s) \tag{67}$$

where

$$Q(s) = 1 + \frac{\mu_2\gamma_2}{\mu_1\gamma_1}\tanh(\gamma_1 h), \tag{68}$$

$$\gamma_1 = a_1 s, \quad \gamma_2 = a_2\sqrt{s+i0}\sqrt{s-i0}. \tag{69}$$

Recall that $a_r^2 = 1 - v^2/c_r^2$ and $c_r^2 = \mu_r/\rho_r$. The expressions $s + i0$ and $s - i0$ in (69) are to be interpreted as that branch cut for the function $\sqrt{s}$ which is taken along the negative and positive imaginary axes, respectively. Consequently, the real part of the function $\gamma_2(s)$ will always be positive.

The form of the function $P(s)$ in (67) permits the Wiener–Hopf equation (63) to be factored more simply than that in [13]. The solution of the homogeneous problem

$$P^+(s) = P(s)P^-(s), \quad s \in \mathcal{L}, \tag{70}$$

is given by

$$P^\pm(s) = P_1^\pm(s)P_2^\pm(s), \tag{71}$$

where

$$P_1^+(s) = \sqrt{s+i0}, \quad P_1^-(s) = -\frac{Q(\infty)}{\mu_2 a_2\sqrt{s-i0}}, \tag{72}$$

$$P_2^\pm(s) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln P_2(\tau)}{\tau - s}\,\mathrm{d}\tau\right], \tag{73}$$

$$P_2(s) = Q(\infty)Q^{-1}(s) \tag{74}$$

and

$$Q(\infty) = \lim_{\mathrm{Re}(s)\to\infty} Q(s) = 1 + \frac{\mu_2 a_2}{\mu_1 a_1}. \tag{75}$$

Following Ryvkin and Banks–Sills [12], it is easy to verify that the function $P_2(s)$ which was chosen satisfies the necessary conditions of continuity on the real axis, has an index equal to zero and that $\ln P_2(s)$ tends to zero as $|s| \to \infty$. Therefore, as defined by the Cauchy type integral, the functions $P_2^\pm(s)$ will be analytic in their corresponding half-planes. Employing this result, one can easily factor the inhomogeneous equation (63) by rewriting it in the form

$$\frac{H^+(s)}{P^+(s)} + R^+(s) = \frac{W^-(s)}{P^-(s)} - R^-(s), \tag{76}$$

where

$$R^+(s) = \frac{i\sigma_0}{s - ib}\left[\frac{1}{P^+(s)} - \frac{1}{P^+(ib)}\right], \tag{77}$$

$$R^-(s) = \frac{i\sigma_0}{(s - ib)P^+(ib)}. \tag{78}$$

The asymptotic behavior of the unknown transforms of the stress $H^+(s)$ and displacement jump $W^-(s)$ for $|s| \to \infty$ is defined by the usual assumption that the local strain energy is bounded. Employing, finally, the generalized Liouville theorem yields the solution

$$H^+(s) = -R^+(s)P^+(s), \tag{79}$$

$$W^-(s) = R^-(s)P^-(s)\,. \tag{80}$$

Now that this particular solution to the inhomogeneous problem has been determined, to obtain the eigensolution, a general limiting procedure described in Section 2 is employed. In accordance with (9), it is assumed that

$$\sigma_0 = \frac{p}{\sqrt{L}} \tag{81}$$

where p may be viewed as an arbitrary constant having dimensions of the stress intensity factor and the decay length parameter L tends to infinity; so that, the amplitude of the stress approaches zero. On the other hand, it follows from (60) that the load is applied along an infinitely long boundary, i.e. along the crack faces. Thus, this limit corresponds to the eigensolution. Substitution of (81) into (77) through (80) with $L \to \infty$ produces, after some manipulations, the desired result (recall that $b = 1/L$). The expressions for the stress and displacement jump transforms are found to be

$$H_e^+(s) = \frac{p}{Q^{1/2}(\infty)}\exp(i\tfrac{1}{4}\pi)\frac{P_2^+(s)}{\sqrt{s+i0}} \tag{82}$$

and

$$W_e^-(s) = -\frac{p\,Q^{1/2}(\infty)}{\mu_2 a_2}\exp(i\tfrac{1}{4}\pi)\frac{P_2^-(s)}{(s-i0)^{3/2}}, \tag{83}$$

where the functions $P_2^{\pm}(s)$ are given in (73) and (74). Note that the subscript e denotes eigensolution.

The stress distribution along the interface in front of the crack tip and the displacement jump behind it are given by the inverse Fourier transforms, namely

$$\left\{\begin{array}{c}\sigma(\xi)\\ W(\xi)\end{array}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\begin{array}{c}H_e^+(s)\\ W_e^-(s)\end{array}\right\}\exp(-is\xi)\,\mathrm{d}s. \tag{84}$$

It is important to emphasize that the thickness of the layer h is the only length parameter in the obtained eigensolution.

In the next section, the near and far field solutions are determined.

5. Limiting cases

The near ($\xi/h \to 0$) and the far ($\xi/h \to \infty$) asymptotes for the stress and displacement jump are defined by the asymptotic behavior of their transforms for $|s| \to \infty$ and $s \to 0$, respectively. For the near asymptotes, (82) and (83) yield

$$H_e^+(s) \sim \frac{p}{Q^{1/2}(\infty)}(-is)^{-1/2} \tag{85}$$

and

$$W_e^-(s) \sim \frac{p\,Q^{1/2}(\infty)}{\mu_2 a_2}\,(is)^{-3/2}, \tag{86}$$

where $|s| \to \infty$ in respective half-planes. Consequently,

$$\sigma(\xi)_{\xi\to+0} = \frac{p}{\sqrt{\pi Q(\infty)\xi}} \tag{87}$$

and

$$W(\xi)_{\xi\to-0} = \frac{2p}{\mu_2 a_2}\sqrt{\frac{Q(\infty)\xi}{\pi}}. \tag{88}$$

The expression in (87) for the near asymptote of the stress is in agreement with the results in [13]. The non-dimensional stress intensity factor is given in their Eqn. (58) for the case of a crack propagating between an elastic strip bonded to an elastic substrate of a dissimilar media which is subjected to the same boundary conditions given in (58) and (60). Note that a factor $1/\pi$ is missing in their (58) before the integral. In the terminology of this investigation, this is, of course, the near field stress intensity factor. Taking the limit in their expression as $l \to \infty$ yields the same distribution as given in (87). It should be pointed out that in [13], l was the decay length parameter of the load instead of L here.

On the other hand, for the far asymptote, in the vicinity of the point $s = 0$, the transforms are found to be

$$H_e^+(s) \sim p\,(-is)^{-1/2}, \quad \text{Im}(s) \to +0 \tag{89}$$

and

$$W_e^-(s) \sim \frac{p}{\mu_2 a_2}\,(is)^{-3/2}, \quad \text{Im}(s) \to -0; \tag{90}$$

so that, the far asymptotes are given by

$$\sigma(\xi)_{\xi\to+\infty} = \frac{p}{\sqrt{\pi\xi}} \tag{91}$$

and

$$W(\xi)_{\xi\to-\infty} = \frac{2p}{\mu_2 a_2}\sqrt{\frac{\xi}{\pi}}. \tag{92}$$

From the results in (87) and (91), the relationship between the near and far field stress intensity factors is found to be

$$K = \sqrt{1 + \frac{\mu_2 a_2}{\mu_1 a_1}}\,k. \tag{93}$$

As expected, this expression coincides with that determined by energy considerations in (54).

It may be noted that the stress in the far field in (91) does not depend upon material properties. This point was mentioned in [6]. Intuitively, since the layer is considered to be vanishingly thin, this problem becomes that of a semi-infinite crack in a homogeneous, elastic body. Alternately, since the layer is negligible, in accordance with Saint-Venant's principle, the interface may be viewed as clamped. On the other hand, the expression for the near asymptote, depends upon the ratio of elastic moduli and shear wave speeds. Finally, employing the near and far asymptotes, the results may be corroborated by calculating the energy release rates. In this way, one may show that

$$G_n = G_f, \tag{94}$$

where G_n and G_f are the energy release rates defined by the near and far fields coinciding with (53).

Using the asymptotic results obtained in this section, it is worthwhile to rewrite the eigensolution for the stresses in the form of the general expression in (13). Setting s equal to s/ξ in (84), one can find

$$\sigma(\xi) = \frac{K}{\sqrt{2\pi\xi}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{P_2^+(s)}{Q^{1/2}(\infty)\sqrt{s+i0}}\exp[i(\tfrac{1}{4}\pi - s)]\,\mathrm{d}s, \tag{95}$$

where the function $P_2^+(s)$ is given in (73) with

$$P_2(\tau) = Q(\infty)\left[1 + \mathrm{sgn}(\tau)\,\frac{\mu_2 a_2}{\mu_1 a_1}\tanh\left(a_1\tau\frac{h}{\xi}\right)\right]^{-1}. \tag{96}$$

6. Stress distribution

As pointed out previously, in some cases, knowledge of asymptotic behavior is insufficient to predict catastrophic behavior. Instead, the complete stress distribution along the interface in front of the crack tip is required.

To obtain this distribution, it is necessary to evaluate the double integral with infinite limits in the general expression (84). Since for large τ, the function $P_2(\tau)$ is found to be

$$P_2(\tau)_{|\tau|\to\infty} = 1 + 2Q(\infty)^{-1}\exp\left(-2a_1\tau\frac{h}{\xi}\right) + O\left[\exp\left(-4a_1\tau\frac{h}{\xi}\right)\right], \tag{97}$$

where O represents order of magnitude; so that, the inner integral in (73) for $P_2^+(s)$ converges exponentially. However, the integrand in the outer integral behaves for large s as $O[s^{-1/2}\exp(-is\xi)]$ making it very inconvenient for numerical calculations. To overcome this obstacle, the integration path $\mathcal{L}$ in the outer integral is deformed from the real axis to the lower half-plane by employing the Cauchy theorem. By analytical continuation of the function $H_e^+(s)$ into the lower half-plane through the Wiener–Hopf equation (76) and symmetric properties of the integrand, it is possible to write

$$\sigma(\xi) = \frac{1}{\pi}\,\mathrm{Re}\left[\int_{\mathcal{L}_1} P(s)W_e^-(s)\exp(-is\xi)\,\mathrm{d}s\right], \quad s \in \mathcal{L}_1. \tag{98}$$

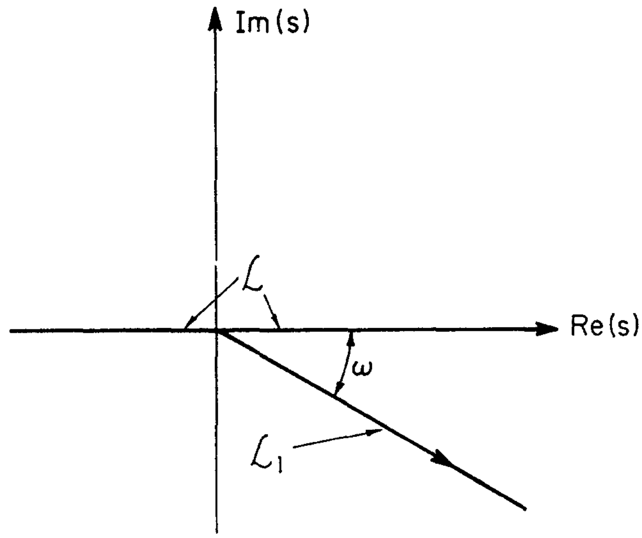

Fig. 5. Integration path $\mathcal{L}_1$ for the stress given in (98).

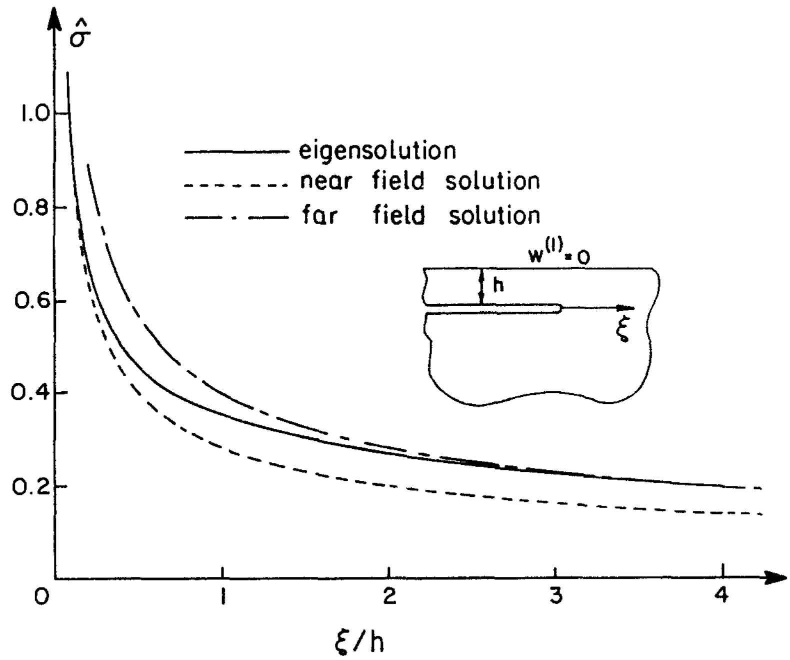

Fig. 6. The non-dimensional stress directly in front of the crack tip for a standing crack a distance h from the fixed boundary in a homogeneous body.

Here $\mathcal{L}_1$ is some contour in the lower half-plane defined by the equality $s = \rho \exp(-i\omega)$, where $0 \leqslant \rho \leqslant \infty$ and the angle ω is chosen so that the integral in (98) converges rapidly (see Fig. 5). The expression in (98) may be written only if the function $P(s)$ has no singularities in the domain between contours $\mathcal{L}$ and $\mathcal{L}_1$. Existence of ω may be verified from the properties of $P(s)$ on the real axis; the optimal value of this parameter is determined during the numerical calculations. As a result of this procedure, the integrand in the outer integral also decays

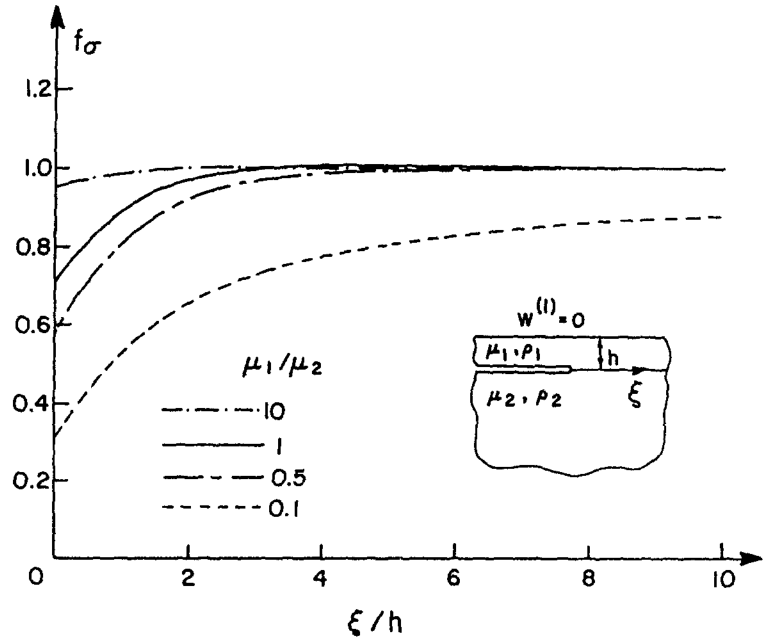

Fig. 7. For various values of the elastic moduli ratio μ_1/μ_2, the non-dimensional stress along the interface in front of the crack tip for a crack tip speed $v/c_2 = 0.5$. The shear wave speeds of the two materials are equal, namely $c_1/c_2 = 1$.

exponentially as $O[\exp(-|s|\xi \sin\omega)]$ for large $|s|$. Hence, the stress distribution in front of the crack given by (98) may easily be obtained numerically.

Next, a non-dimensional stress is defined as

$$\hat{\sigma} = \frac{\sqrt{h}}{K}\sigma(\xi), \tag{99}$$

which may be viewed as a function of four non-dimensional parameters, namely

$$\hat{\sigma} = \hat{\sigma}(\xi/h, \mu_1/\mu_2, c_1/c_2, v/c_2). \tag{100}$$

For several choices of these non-dimensional parameters, specific cases are solved completely to illustrate the results. It is reasonable to begin the numerical analysis with the important specific case of a standing crack ($v/c_2 = 0$) in a homogeneous material ($\mu_1/\mu_2 = c_1/c_2 = 1$). A plot of the stress $\hat{\sigma}$ which corresponds to the fundamental eigensolution is shown as the solid line in Fig. 6. The near and far square root asymptotes are indicated by different dashed lines. From (93), it follows that the outer stress intensity factor $K = \sqrt{2}k$. It may be noted that the near asymptote defining the k-controlled zone is valid to within 10 percent accuracy for distances from the crack tip which are less than half the layer thickness. Consequently, if the distance h between the crack and the fixed boundary is sufficiently small, usual fracture criteria for some materials based on a k-concept will be invalid.

To examine the influence of the non-dimensional parameters on the stress distribution, it is convenient to define another non-dimensional quantity as

$$f_\sigma = \frac{\sqrt{2\pi\xi}}{K}\,\sigma(\xi). \tag{101}$$

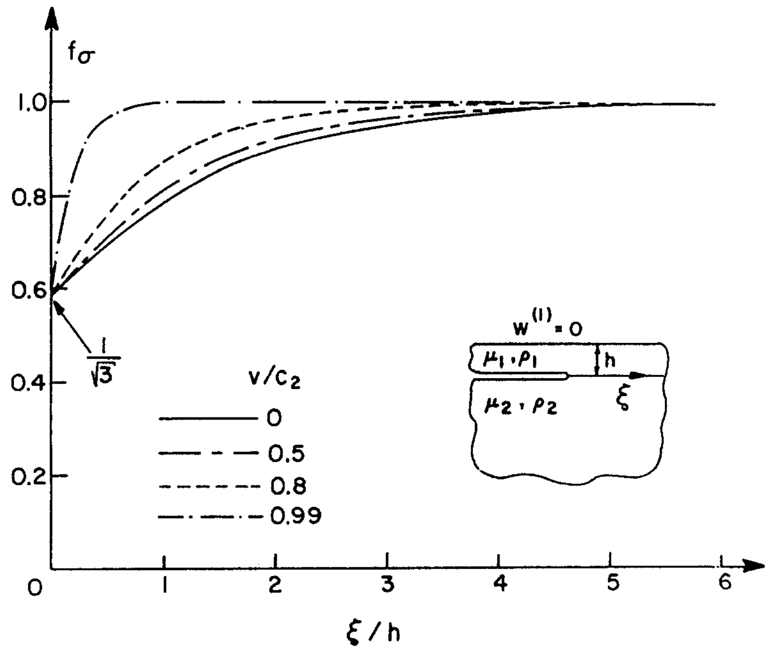

Fig. 8. For various values of the relative crack tip speed v/c_2, the non-dimensional stress along the interface in front of the crack tip for the elastic moduli ratio $\mu_1/\mu_2 = 0.5$. The shear wave speeds of the two materials are equal, namely $c_1/c_2 = 1$.

Maintaining the ratio of the shear wave speeds equal, namely $c_1/c_2 = 1$, the influence of the elastic moduli ratio on this non-dimensional stress distribution is illustrated in Fig. 7. The non-dimensional crack tip speed is chosen to be $v/c_2 = 0.5$ and $\mu_1/\mu_2 = 0.1, 0.5, 1.0, 10$. As mentioned previously, the stress in the far field does not depend upon the properties of the layer. Consequently,

$$\lim_{\xi/h\to\infty} f_\sigma = 1, \tag{102}$$

for all curves. The behavior of the stress for $\xi/h \to 0$ as defined by the near asymptote in accordance with (93) and (101) yields

$$\lim_{\xi/h\to 0} f_\sigma = \left(1 + \frac{\mu_2 a_2}{\mu_1 a_1}\right)^{-1/2}. \tag{103}$$

As the ratio μ_1/μ_2 increases with other parameters held constant, the value of f_σ for $\xi/h \to 0$ also increases. For the limiting case $\mu_1/\mu_2 \to \infty$, this value approaches unity since the near and far asymptotes coincide. As the ratio μ_1/μ_2 decreases, the layer becomes less stiff. Consequently, the stress approaches the far asymptote more slowly and the value f_σ for $\xi/h = 0$ defined by the near asymptote decreases. For the degenerate case of $\mu_1/\mu_2 = 0$, f_σ is zero; that is, the layer has disappeared and the amplitude of the singularity tends to zero.

Influence of the relative crack tip speed on the non-dimensional stress distribution is depicted in Fig. 8. The crack tip speeds are chosen to be $v/c_2 = 0, 0.5, 0.8$ and 0.99, the elastic moduli ratio $\mu_1/\mu_2 = 0.5$ and the ratio of the shear wave speeds $c_1/c_2 = 1$. Since the shear wave speeds are equal, the near asymptote in (103) is independent of the relative

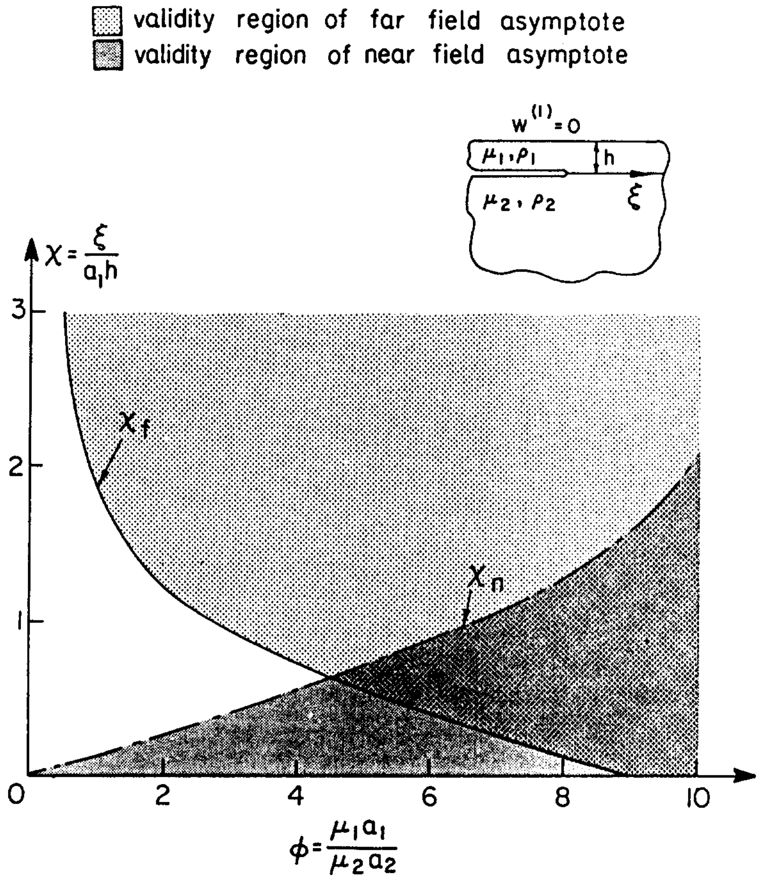

Fig. 9. Plot of nondimensional validity region χ ahead of the crack tip as a function ot the nondimensional modulus ratio ϕ for the near and far field asymptotic solutions.

crack tip speed v/c_2 and depends only upon the shear moduli ratio; so that, $f_\sigma = 1/\sqrt{3}$ for all curves for $\xi/h = 0$ as may be observed in Fig. 8. If the abscissa in Fig. 8 is chosen to be $\xi/(a_1 h)$ with $c_1 = c_2$, it may be observed from (95), (96) and (101) that the curves will coincide.

Knowledge of validity limits of the near and far asymptotes for different material parameter combinations is of great importance for implementation of different fracture criteria. Before carrying out corresponding numerical analysis, it is worthwhile to note from (75), (95), (96) and (101) that the function f_σ may be considered to be a function of two non-dimensional parameters; that is

$$f_\sigma = f_\sigma(\chi, \phi), \tag{104}$$

where $\chi = \xi/(a_1 h)$ is a normalized distance from the crack tip and $\phi = (\mu_1 a_1)/(\mu_2 a_2)$ is a normalized elastic moduli ratio. Employing the limiting values of the function f_σ given in (102) and (103), one can write the conditions defining validity regions of the near and far asymptotes in the following form

$$|f_\sigma - \left(1 + \frac{1}{\phi}\right)^{-1/2}| < \delta \quad \text{for } 0 < \chi < \chi_n, \tag{105}$$

$$|f_\sigma - 1| < \delta \quad \text{for } \chi_f < \chi < \infty. \tag{106}$$

Here δ is the desired accuracy and values $\chi_n = \xi_n/(a_1 h)$ and $\chi_f = \xi_f/(a_1 h)$ denote the limits of validity to be determined. Recall the meaning of the values ξ_n and ξ_f illustrated in Fig. 2.

The relations in (105) and (106) may be considered as definitions of the implicit functions $\chi_n(\phi)$ and $\chi_f(\phi)$. Graphs of these functions for the accuracy $\delta = 0.05$ are presented in Fig. 9. For small ϕ, when the layer is less stiff than the substrate, the regions of validity of near and far asymptotes degenerate, so that $\chi_n \to 0$ and $\chi_f \to \infty$. As the layer becomes stiffer, the problem tends to the limiting case of the clamped half-plane for which near and far asymptotes coincide. This explains the fact that both regions extend and consequently χ_n increases and χ_f decreases. The value of the parameter $\phi = \phi_0$ corresponding to the case when the near and far asymptote approach each other with 5 percent accuracy can be derived from

$$1 - \left(1 + \frac{1}{\phi_0}\right)^{-1/2} = 0.05, \tag{107}$$

and is found to be $\phi_0 = 9.25$. For all values of $\phi \geqslant 9.25$, the two asymptotes agree to within 5 percent of each other. For the value $\phi \approx 4.5$, the upper limit of the region of validity of the near asymptote is equal to the lower limit of that of the far asymptote; for larger ϕ, overlapping takes place. It should be emphasized that the curves in Fig. 9 are derived from universal relations and may be employed for determining validity limits of the near and far asymptotes for an arbitrary set of material parameters, namely, $\mu_1, \mu_2, \rho_1, \rho_2$ and h. Note that a study of the influence of the problem parameters on the region of near asymptote validity was presented in a recent paper by Huang and Gross [4]. They considered a finite crack in a homogeneous plane under a stress wave loading.

7. Discussion and conclusions

In this investigation, the stress distribution in front of the crack tip and the crack opening displacement were determined for a mode III crack along the interface of a layer bonded to a substrate. The eigensolution was obtained when both materials were linear elastic. In addition, near and far field asymptotes for these quantities were derived. The regions of asymptotic solution validity were determined.

The considerations concerning the eigensolution corresponds to a load applied to a large portion of the crack faces such that, the applied load is negligible in a substantial (in the h-scale) vicinity of the crack tip. For a load concentrated near the crack tip, the problem is not an eigenproblem. In this case, the near asymptotes of the stress and crack opening displacement exist since any distributed load appears ‘long’. However, the concept of the far asymptote may become invalid. As a result, the solution of such problems lose their universality by which the derived eigensolution is characterized. The energy release rate determined by Schovanec and Walton [14] for two loaded cracks located at $x < 0, y = \pm h$ in a homogeneous viscoelastic media may be considered as an example of such a situation. In that case, the energy release rate turns out to be a function of h/L. It tends to the result obtained in (51) when $h/L \to 0$. Note that from symmetry, this problem corresponds to that treated in this investigation for linear elastic material.

The solution derived in this investigation differs from the known eigensolution for an unbounded body only by a multiplying function of the non-dimensional distance from the crack tip ξ/h. For mode III, this relation may be expressed as

$$\sigma(\xi) = \frac{k_3}{\sqrt{2\pi\xi}} F_3\left(\frac{\xi}{h}\right), \tag{108}$$

where

$$F_3(0) = 1, \tag{109}$$

and

$$F_3(\infty) = \sqrt{1 + \frac{\mu_2 a_2}{\mu_1 a_1}}. \tag{110}$$

The parameters $a_1 = a_2 = 1$ when $v = 0$. For the important specific case of a crack propagating parallel to the boundary of a homogeneous half-plane, it was found that $F_3(\infty) = \sqrt{2}$.

One may see that the far field stress intensity factor exceeds the near one. It is a result of the fact that, in contrast to the near field, in the far field, there is energy flux only from the half-plane. Consequently, routine fracture analysis based on the near stress intensity factor may lead to an overestimation of the allowed loading.

For modes I and II, the considerations become somewhat more complicated. Namely, let the far asymptote correspond to mode I. Since h is finite, the stress state near the crack tip will correspond to a situation where both modes I and II are present. In accordance with the expression in (13)

$$\sigma = \frac{k_1}{\sqrt{2\pi\xi}} F_{11}\left(\frac{\xi}{h}\right) + \frac{k_2}{\sqrt{2\pi\xi}} F_{21}\left(\frac{\xi}{h}\right), \tag{111}$$

where

$$F_{11}(0) = F_{21}(0) = 1 \tag{112}$$

and

$$F_{11}(\infty) = \sqrt{2}\left[1 + \left(\frac{k_2}{k_1}\right)^2\right], \quad F_{21}(\infty) = 0. \tag{113}$$

The functions F_{11}, F_{21} and F_3 are universal in the sense that they are the same for any 'long' loading and are independent of layer thickness. Derivation of these functions seems to be the next step in this topic.

A fracture criterion is of importance when the usual concept of the near asymptote fails (i.e., $R > \xi_n$). As a matter of fact, this question cannot be resolved in the framework of the continuous elastic body model. Nevertheless, some remarks may be made concerning use of the result obtained here for formulation of a fracture criterion. The energy flux is the same whether it is defined by the near or the far field asymptotes. Hence, the double asymptotic solution has no influence on this criterion formulation. However, application of the energy criterion is doubtful in the case $R > \xi_n$. The K-concept is applicable for $R \ll \xi_n$ or $R \gg \xi_f$. In the first case, one may employ the near field stress intensity factor k; whereas in the second case, K may be employed.

One possibility in a more complicated situation when R assumes a moderate value in which $R > \xi_n$ is to employ the full field stress distribution in the Neuber–Novozilov (Neuber [8]; Novozilov [9]) criterion. This criterion is based on averaged stresses in front of the crack tip. Nevertheless, the results obtained in the form of the stress distribution in the region between the limits of validity of the near and far field asymptotes ξ_n/h and ξ_f/h and values of these limits, themselves, appear to be essential data for estimation of applicability of different fracture criteria and for formulation of new ones.

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