PROCEEDINGS A

Research

CrossMark
click for updates

Author for correspondence:
Leonid I. Slepyan
e-mail: leonid@eng.tau.ac.il

On the energy partition in oscillations and waves

Leonid I. Slepyan

School of Mechanical Engineering, Tel Aviv University, PO Box 39040, Ramat Aviv, 69978 Tel Aviv, Israel

A class of generally nonlinear dynamical systems is considered, for which the Lagrangian is represented as a sum of homogeneous functions of the displacements and their derivatives. It is shown that an energy partition as a single relation follows directly from the Euler–Lagrange equation in its general form. The partition is defined solely by the homogeneity orders. If the potential energy is represented by a single homogeneous function, as well as the kinetic energy, the partition between these energies is defined uniquely. For a steady-state solitary wave, where the potential energy consists of two functions of different orders, the Derrick–Pohozaev identity is involved as an additional relation to obtain the partition. Finite discrete systems, finite continuous bodies, homogeneous and periodic-structure waveguides are considered. The general results are illustrated by examples of various types of oscillations and waves: linear and nonlinear, homogeneous and forced, steady-state and transient, periodic and non-periodic, parametric and resonant, regular and solitary.

1. Introduction

For free linear oscillations and sinusoidal waves, it has long been recognized that kinetic and potential energies averaged over the period are equal to each other. In some partial cases, this is observed directly [1–4]. This fact is also confirmed by Whitham [5] for a non-specified linear sinusoidal wave. The statement is found as a 'side effect' of a version of the variation principle, which was introduced as the basis for the theory of slow-varying sinusoidal waves. In Whitham's considerations, the expression for a sinusoidal wave

$$\phi \sim \Re A \, e^{i(kx-\omega t)}, \qquad (1.1)$$

where A, k and ω are the slow-varying parameters, is substituted in the Lagrangian, neglecting derivatives of these parameters. Then, the Lagrangian is averaged over the period. Thereafter, the equipartition directly follows as a result of the variation of the wave amplitude.

THE ROYAL SOCIETY
PUBLISHING

Along with this, the energy partition in different areas of nonlinear dynamics and with different meanings is of interest [2,6,7]. In addition, it seems that even the linear case is not fully explored. The cases of systems with time-dependent parameters, forced motions and solitary waves are not discussed.

We consider this topic in terms of classical mechanics and obtain the partition relations for a class of generally nonlinear oscillations and waves, where the Lagrangian is represented by a sum of homogeneous functions of the displacements and their derivatives, possibly not only of first order. The Lagrangian can also depend explicitly on time and spatial coordinates. Starting from the Euler–Lagrange equation of a general view and taking into account Euler's theorem on homogeneous functions we obtain both the energy partition relation and the conditions defining the regions of averaging (this could be the oscillation period, if it exists, and not only it). The partition relation is fully defined by the homogeneity orders, regardless of other parameters of the system and the dynamic process. In the case where the potential energy is represented by a single homogeneous function, as well as the kinetic energy, the partition between these energies is defined uniquely.

For a steady-state solitary wave, where the potential energy consists of two functions of different orders, there are three different energy terms, and the single equality is not sufficient to obtain the desired energy partition. At the same time, the wave is exponentially localized, which allows us to use the Derrick–Pohozaev identity [8–10] as an additional relation.

The identity is based on the representation of the steady-state solitary wave as a static state. Note that, in the steady-state mode, the kinetic energy can be considered as such, as a function of $\dot{u} = \partial u / \partial t$, and also as a function of $u' = \partial u / \partial x$. In the latter case, it can be treated as an additional (negative) part of the potential energy. Different energy relations, one discussed in the present paper and another given by the Derrick–Pohozaev identity, follow from the static representation as a result of different variations. While the former corresponds to the displacement variation $\delta u = \varepsilon u$, the latter is a uniform x-extension/compression. The use of both relations allows us to separate the kinetic and potential energies completely.

First, we consider general relations for finite discrete systems. The relations obtained for the latter are applicable, with minor additions, to a finite continuous-material body and to an infinite continuous or discrete waveguide as discussed below. Then, some examples of free and forced oscillations are presented, which evidence that the linearity is neither a necessary nor a sufficient condition for the equipartition. Next, some examples of linear and nonlinear oscillations are presented, where the energy partition and the regions of averaging are clearly seen. Homogeneous and forced, steady-state and transient, periodic and non-periodic, regular, parametric and resonant oscillations are examined.

Further, the general relations are presented to be applicable for a finite continuous body and for homogeneous, periodic-structure and discrete waveguides. Some examples of linear and nonlinear, periodic, steady-state, transient and solitary waves are presented, where the energy partition is also demonstrated.

2. The main relations

Consider the Euler–Lagrange equations

$$\frac{\partial L}{\partial u_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{u}_i} = 0, \quad L = L(\boldsymbol{u}, \dot{\boldsymbol{u}}, t), \ i = 1, 2, \ldots, \tag{2.1}$$

where $\boldsymbol{u} = (u_1, u_2, \ldots)$. Multiplying the equations by u_i and integrating over an arbitrary segment, $t_1 \leq t \leq t_2$, we obtain (with summation on repeated indices)

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial u_i} u_i + \frac{\partial L}{\partial \dot{u}_i} \dot{u}_i \right) \mathrm{d}t + B_1 - B_2 = 0, \tag{2.2}$$

where

$$B_{1,2} = \frac{\partial L}{\partial \dot{u}_i} u_i \quad (t = t_{1,2}). \tag{2.3}$$

Thus,

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial u_i} u_i + \frac{\partial L}{\partial \dot{u}_i} \dot{u}_i \right) dt = 0 \tag{2.4}$$

if

$$B_1 = B_2. \tag{2.5}$$

Regarding this formulation, we note that the relation (2.4) follows from Hamilton's principle of least action with the variation δu replaced by u. This, however, entails a change in the additional conditions. Namely, in the variational formulation, the integration limits, $t_{1,2}$, are arbitrary under the condition $\delta u = 0$ at $t = t_{1,2}$, whereas, in the modified formulation, they are not arbitrary but must meet the condition (2.5).

We now suppose that the Lagrangian is a sum of homogeneous functions of u, $\dot{u}$

$$L(\lambda u, \lambda \dot{u}, t) = \lambda^{\nu_n} L_n(u, \dot{u}, t), \tag{2.6}$$

where ν_n is the homogeneity order. In this expression, ν_n can be not only an integer, and it is assumed that only a positive value can have a fractional exponent. For example, in the latter case, $|u_i|^\nu$ can be present in the Lagrangian, but not u_i^ν. Accordingly, in the homogeneity definition, we take $\lambda > 0$. It is also assumed that $|u_i|^\nu \geq 0$. Note that the homogeneity condition excludes possible non-uniqueness in the definition of the energies.

With reference to Euler's theorem on homogeneous functions

$$\frac{\partial f(x)}{\partial x_i} x_i = \nu f(x), \quad \text{if } f(\lambda x) = \lambda^\nu f(x) \tag{2.7}$$

and the condition (2.5), we can rewrite the relation (2.4) in the form

$$\int_{t_1}^{t_2} \nu_n L_n \, dt = 0. \tag{2.8}$$

The latter relation can serve for the determination of the energy partition. For example, let L be a difference between the kinetic and potential energies, $L = \mathcal{K} - \mathcal{P}$, where $\mathcal{K}$ and $\mathcal{P}$ are homogeneous functions of the orders μ and ν, respectively. In this case, the relation between the averaged energies

$$\langle \mathcal{K} \rangle = \frac{1}{\tau} \int_{t_1}^{t_1+\tau} \mathcal{K} \, dt, \quad \langle \mathcal{P} \rangle = \frac{1}{\tau} \int_{t_1}^{t_1+\tau} \mathcal{P} \, dt \quad (\tau = t_2 - t_1) \tag{2.9}$$

is

$$\frac{\langle \mathcal{K} \rangle}{\langle \mathcal{P} \rangle} = \frac{\nu}{\mu} \implies \langle \mathcal{K} \rangle = \frac{\nu \mathcal{E}}{\mu + \nu}, \quad \langle \mathcal{P} \rangle = \frac{\mu \mathcal{E}}{\mu + \nu}, \tag{2.10}$$

where $\mathcal{E} = \mathcal{K} + \mathcal{P}$ is the total energy. Note that the expressions (2.9) and (2.10), in terms of the total energy as a constant, are valid only for conservative systems.

In particular, in the linear case, where $\nu = \mu = 2$, the averaged energies are equal. Cases $\nu > \mu$ and $\nu < \mu$ correspond to hardening and softening nonlinearity, respectively. The potential energy vanishes as $\nu/\mu \to \infty$. In the limit, this case corresponds to the periodic perfect collisions of rigid particles.

Note that *in statics*, $\mathcal{K} = 0$, the boundary terms, $B_{1,2}$, vanish independently of $t_{1,2}$. It follows from (2.8) that

$$\nu_n \mathcal{P}_n = 0. \tag{2.11}$$

Thus, the energy is at zero if it is a homogeneous function; otherwise, if it is a sum of such functions this relation defines the energy partition (in this connection, see an example in §5c(i)).

It is remarkable that the energy ratio in (2.10) is equal to the homogeneity order ratio, regardless of the other parameters of the system and the dynamic process. In turn, the

homogeneity orders generally depend on the constitutive properties but not the material constants. In the case where the potential energy is represented by the sum of two or more homogeneous functions of different orders, the relation (2.8) is only one with respect to three or more terms. In this case, it is not sufficient for the determination of ratios between all terms, but it still allows one to draw some conclusions concerning the energy partition.

The condition in (2.5) is satisfied in periodic oscillations

$$u(t + \tau) = u(t) \tag{2.12}$$

and also in the case where $u = 0$ at $t = t_{1,2}$. In addition, if the kinetic energy, being a homogeneous function of order μ with respect to u_i, $\dot{u}_i$, is such a function of order 2 with respect to $\dot{u}_i$

$$\mathcal{K}(\lambda u, \lambda \dot{u}) = \lambda^\mu \mathcal{K}(u, \dot{u}), \quad \mathcal{K}(u, \lambda \dot{u}) = \lambda^2 \mathcal{K}(u, \dot{u}), \tag{2.13}$$

then

$$B_{1,2} = B_{1,2}^0 u_i \dot{u}_i \tag{2.14}$$

and $B_1(B_2) = 0$ if $\dot{u} = 0$ at $t = t_1(t_2)$. Therefore, the condition (2.5) is also satisfied in the case of non-periodic oscillations, where u and/or $\dot{u}$ vanish at points. This can be seen in the below examples of oscillations and periodic and solitary waves. In the case of regular or irregular oscillations, it does not matter which, where the condition (2.5) is satisfied for each of the neighbouring points, $t_1 < t_2 < \cdots < t_m < \cdots$, the energy partition averaged over any interval between two such points is the same. So, the energy partition appears fixed for a large range of time.

Here, we have considered systems of a finite number of material points. The main relations presented here are applicable, with minor additions, to a finite continuous-material body and to an infinite continuous or discrete waveguide, as discussed in §§4 and 5.

While the energy partition is defined solely by the homogeneity orders, below we calculate the energy distribution function, $\mathcal{L}(t)$, as a function of time

$$\mathcal{L}(t) = \int_0^t v_n L_n(t)\, \mathrm{d}t, \tag{2.15}$$

assuming that $B_1 = 0$ at $t = t_1 = 0$. This function must vanish at $t = t_2$ (2.8) if the condition (2.5) is satisfied at that point. This representation allows us to demonstrate how the partition varies during the (t_1, t_2)-interval and how $\mathcal{L}(t)$ follows the statement (2.8) (also see the expression in (2.14)). As a rule, dimensionless quantities are used, because this simplification does not affect the partition.

## 3. Discrete systems

(a) The linearity and equipartition

Three examples are presented below which evidence that the linearity is neither a necessary nor a sufficient condition for the equipartition. First, consider a piecewise linear equation

$$m\ddot{u}(t) + [\mathcal{P}_+ H(u) + \mathcal{P}_- H(-u)]u(t) = 0, \tag{3.1}$$

where H is the Heaviside step function and $\mathcal{P}_\pm$ are different positive constants. The corresponding Lagrangian, kinetic and potential energies and homogeneity orders are, respectively,

$$L = \mathcal{K} - \mathcal{P}, \quad \mathcal{K}(t) = \frac{m\dot{u}^2(t)}{2}, \quad \mathcal{P}(t) = [\mathcal{P}_+ H(u) + \mathcal{P}_- H(-u)]\frac{u^2(t)}{2}, \quad \mu = \nu = 2. \tag{3.2}$$

Equation (3.1) defines periodic oscillations. The displacement $u(t)$, speed $\dot{u}(t)$ and the energy distribution function $\mathcal{L}(t)$ (2.15) are presented in figure 1 for $\mathcal{P}_+ = 1, \mathcal{P}_- = 5$ and initial conditions $u(0) = 0, \dot{u}(0) = 1$.

5

rspa.royalsocietypublishing.org Proc. R. Soc. A **471**: 20140838
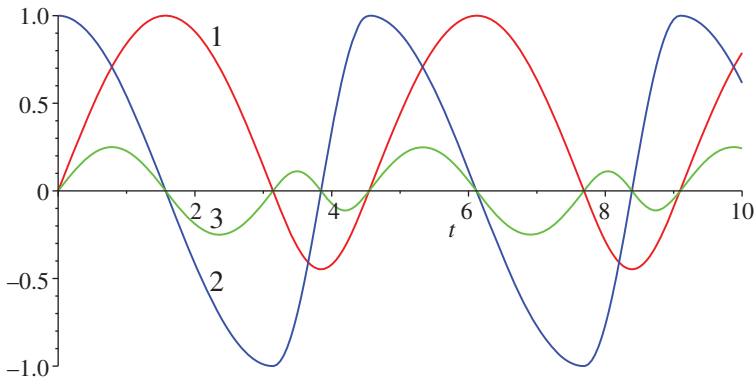
Figure 1. Piecewise linear oscillator. Displacements $u(t)$ (1), speed $\dot{u}(t)$ (2) and the energy distribution function $\mathcal{L}(t)/2$ (3) for $\mathcal{P}_+ = 1, \mathcal{P}_- = 5$. It is seen that $\mathcal{L} = 0$ at points where $u = 0$ and where $\dot{u} = 0$. (Online version in colour.)

It can be seen that the energies averaged over the period, $\langle \mathcal{K} \rangle$ and $\langle \mathcal{P} \rangle$ (in this case, period $\tau = \left(1 + 1/\sqrt{5}\right)\pi \approx 4.5465556$), are equal

$$\mathcal{L}(\tau) = \int_0^\tau 2L(t)\,dt = \int_t^{t+\tau} 2L(t)\,dt = 2\tau[\langle \mathcal{K} \rangle - \langle \mathcal{P} \rangle] = 0. \tag{3.3}$$

Next, we consider a nonlinear equation related to centrosymmetric oscillations of a bubble in unbounded, perfect, incompressible liquid, where the added mass is proportional to the bubble radius cubed. Let the (dimensionless) energies be

$$\mathcal{K} = \tfrac{1}{2}r^3(t)\dot{r}^2(t), \quad \mathcal{P} = \tfrac{1}{5}r^5(t), \quad \mu = \nu = 5 \ \text{(the radius } r(t) \geq 0). \tag{3.4}$$

The corresponding dynamic equation is

$$r(t)\ddot{r}(t) + \tfrac{3}{2}\dot{r}^2(t) + r(t)^2 = 0. \tag{3.5}$$

The radius $r(t)$ and the energy distribution function $\mathcal{L}(t)$ under conditions $r(0) = 2, \dot{r}(0) = 0$ are plotted in figure 2.

It is seen that the equipartition also takes place in this strongly nonlinear situation.

Next, consider forced resonant oscillations as an inhomogeneous linear problem

$$\left.\begin{aligned}
&\ddot{u} + u = P\sin t, \quad \mathcal{K} = \frac{\dot{u}^2}{2} \ (\mu = 2) \\[2mm]
\text{and} \quad &\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2, \quad \mathcal{P}_1 = \frac{u^2}{2} \ (\nu_1 = 2), \quad \mathcal{P}_2 = -Pu\sin t \ (\nu_2 = 1).
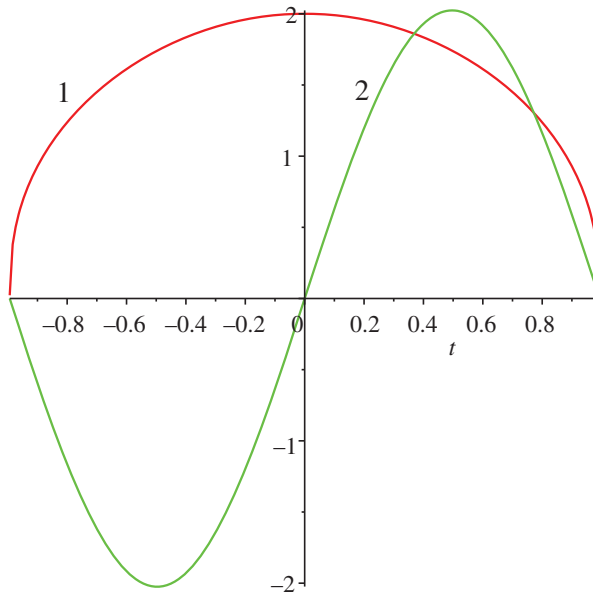\end{aligned}\right\} \tag{3.6}$$

It follows that under zero initial conditions

$$u = \frac{P}{2}(\sin t - t\cos t). \tag{3.7}$$

The displacements and the energy distribution function $\mathcal{L}(t)$ are plotted in figure 3 for $P = 1$.

Thus, there is no equipartition between the kinetic and potential energies in this inhomogeneous linear problem

$$\langle \mathcal{K} \rangle = \langle \mathcal{P}_1 \rangle + \tfrac{1}{2}\langle \mathcal{P}_2 \rangle < \langle \mathcal{P} \rangle = \langle \mathcal{P}_1 \rangle + \langle \mathcal{P}_2 \rangle. \tag{3.8}$$

6

rspa.royalsocietypublishing.org *Proc. R. Soc. A* **471**: 20140838

Figure 2. The bubble radius $r(t)$ (1) and the energy distribution function $\mathcal{L}(t)/2$ (2). (Online version in colour.)
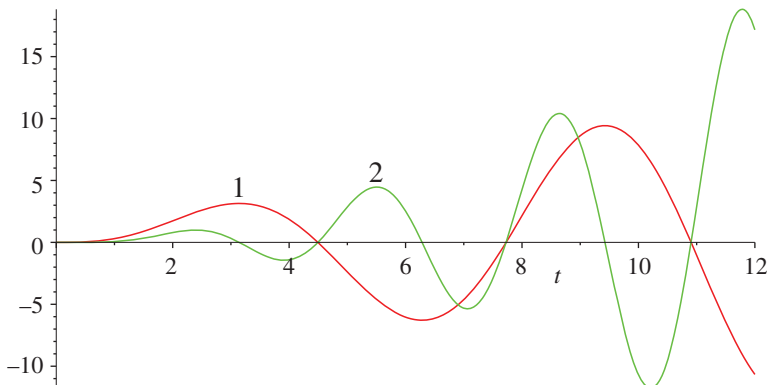
Figure 3. Inhomogeneous problem: the ordinary resonance. The displacement $u(t)$ (1) and the energy distribution function $\mathcal{L}(t)$ (2). It is seen that zeros of the latter correspond to zeros of $u(t)\dot{u}(t)$. (Online version in colour.)
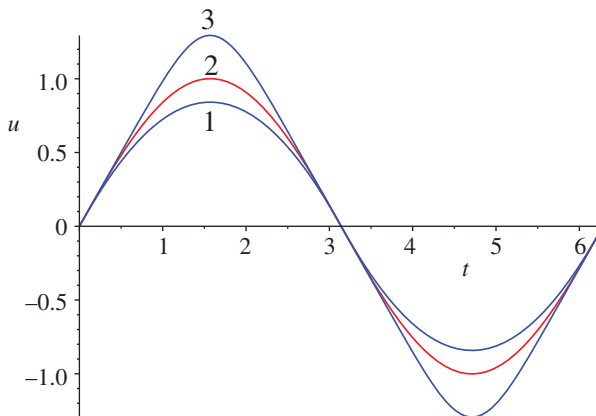
(b) Simplest example of unequal partition

Consider a ball of mass M thrown upwards with speed $\dot{u}(0) = v_0$. The energies are

$$\mathcal{K} = \frac{M\dot{u}^2}{2}, \quad \mathcal{P} = Mgu \ (\mu = 2, \ v = 1), \tag{3.9}$$

where g is the acceleration of gravity. Here, $u = 0$ at $t = t_1 = 0$ and $\dot{u} = 0$ at $t = t_2 = v_0/g$. It follows from (2.8) that the energies averaged over the (t_1, t_2)-segment must satisfy the relation

$$\langle \mathcal{P} \rangle = 2\langle \mathcal{K} \rangle, \tag{3.10}$$

7

rspa.royalsocietypublishing.org Proc. R. Soc. A 471: 20140838

Figure 4. The oscillators: $\nu = \frac{6}{5}$, $\mathcal{P}_0 = 0.6152$ (1); $\nu = 2$, $\mathcal{P}_0 = \frac{1}{2}$ (2); and $\nu = 6$, $\mathcal{P}_0 = 0.1067$ (3). Displacements under initial conditions $u = 0$, $\dot{u} = 1$ for each of the oscillators. (Online version in colour.)

which can also be obtained by direct calculations. Indeed,

$$
\left.
\begin{aligned}
\dot{u} &= v_0 - gt, \quad u = v_0 t - \frac{gt^2}{2}, \\
\langle \mathcal{K} \rangle &= \frac{1}{2t_2} \int_0^{t_2} M(v_0 - gt)^2 \, \mathrm{d}t = \frac{Mv_0^2}{6} \\
\langle \mathcal{P} \rangle &= \frac{1}{t_2} \int_0^{t_2} Mg \left(v_0 t - \frac{gt^2}{2} \right) \mathrm{d}t = \frac{Mv_0^2}{3} = 2\langle \mathcal{K} \rangle.
\end{aligned}
\right\}
\tag{3.11}
$$

and

(c) Linear and nonlinear oscillators

Consider oscillators with kinetic and potential energies

$$
\mathcal{K} = m\frac{\dot{u}^2}{2}, \quad \mathcal{P} = \mathcal{P}_\pm \left(\frac{|u|}{u_0} \right)^\nu,
\tag{3.12}
$$

where the coefficients $\mathcal{P}_+$ and $\mathcal{P}_-$ correspond to positive and negative values of u, respectively, and can be different, $(m, \mathcal{P}_\pm, u_0, \nu) > 0$. For these systems, the ratio of the kinetic-to-potential averaged energies is equal to $\nu/2$. Plots of the displacements and energies for $\nu = 6/5$, 2 and 6 are shown in figures 4–7. To have the same period, 2π, for all these cases, we take $\mathcal{P}_+ = \mathcal{P}_- = 0.6152, 1/2, 0.1067$ for $\nu = 6/5$, 2 and 6, respectively, and $m = u_0 = 1$, $\mathcal{E} = 1/2$ for each of them. It is seen that zeros of $\mathcal{L}(t)$ correspond to the condition (2.5) with (2.14). Note that an analytical solution exists corresponding to the above equation (3.12). However, here, we are not interested in it, because the energy partition is defined solely by the homogeneity orders.

(d) Two-degrees-of-freedom nonlinear system

Consider a mass-spring chain (figure 8) with energies

$$
\mathcal{K} = \frac{m_1 \dot{u}_1^2}{2} + \frac{m_2 \dot{u}_2^2}{2}, \quad \mathcal{P} = \mathcal{P}_{1\pm} \left(\frac{|u_1|}{u_0} \right)^\nu + \mathcal{P}_{2\pm} \left(\frac{|u_1 - u_2|}{u_0} \right)^\nu.
\tag{3.13}
$$

Plots of $u_1(t), u_2(t)$ and $\mathcal{L}(t)$ for $m_1 = m_2 = u_0 = 1$, $\mathcal{P}_1 = \mathcal{P}_2 = \frac{1}{4}$ and $\nu = 4$ based on the corresponding dynamic equations

$$
\frac{\mathrm{d}^2 u_1}{\mathrm{d}t^2} = -(u_1 - u_2)^3, \quad \frac{\mathrm{d}^2 u_2}{\mathrm{d}t^2} = (u_1 - u_2)^3 - u_2^3
\tag{3.14}
$$

are shown in figure 9. Agreement with relation (2.8) and with relation (2.5) with (2.14) is demonstrated.

8

rspa.royalsocietypublishing.org *Proc. R. Soc. A* **471**: 20140838
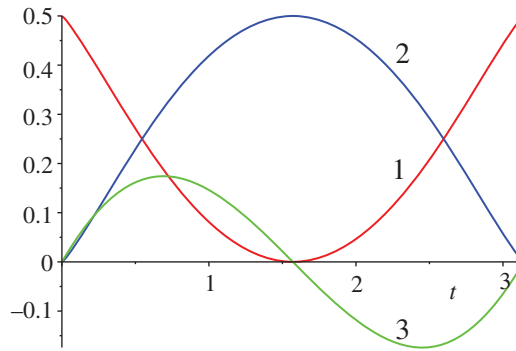
Figure 5. Nonlinear oscillator, $\nu = \frac{6}{5}$, $\mathcal{P}_0 = 0.6152$. Kinetic energy (1), potential energy (2) and the energy distribution function $\mathcal{L}(t)/2$ (3). (Online version in colour.)
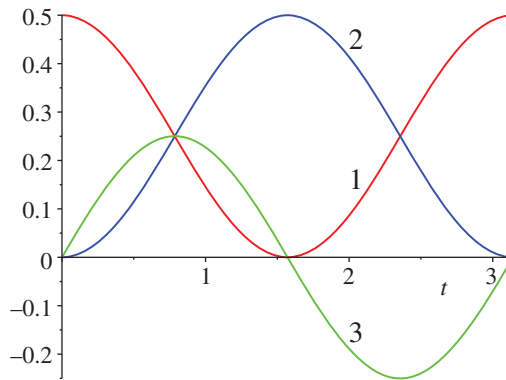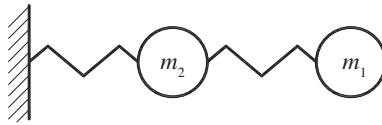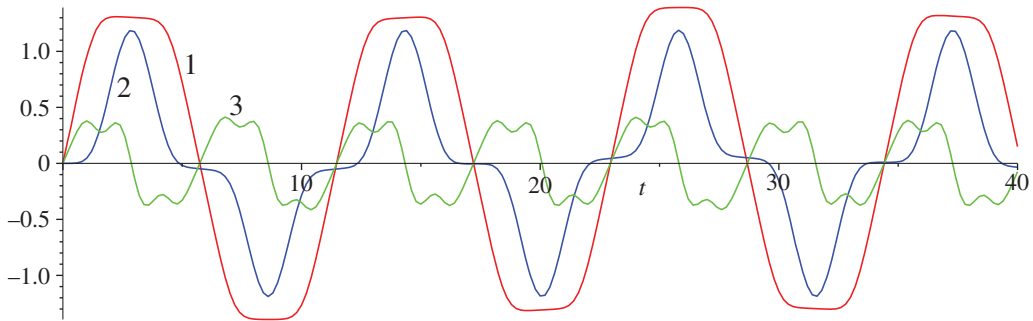
Figure 6. Linear oscillator, $\nu = 2$, $\mathcal{P}_0 = \frac{1}{2}$. Kinetic energy (1), potential energy (2) and the energy distribution function $\mathcal{L}(t)/2$ (3). (Online version in colour.)
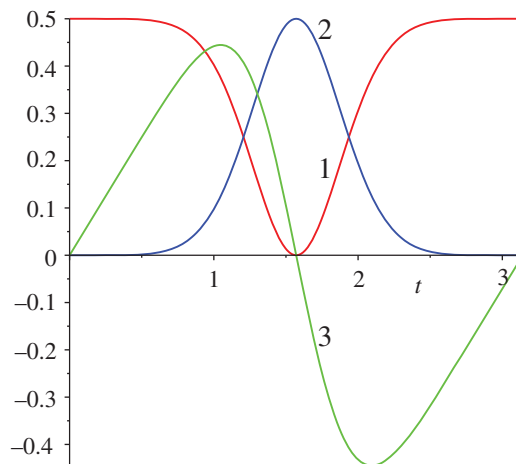
Figure 7. Nonlinear oscillator, $\nu = 6$, $\mathcal{P}_0 = 0.1067$. Kinetic energy (1), potential energy (2) and the energy distribution function $\mathcal{L}(t)/2$ (3). (Online version in colour.)

Figure 8. Nonlinear mass-spring system of two degrees of freedom.

Figure 9. Nonlinear mass-spring system of two degrees of freedom. Displacements $u_1(t)$ (1), $u_2(t)$ (2) and $\mathcal{L}(t)/2$ (3); $\nu = 4$. Zeros of $\mathcal{L}(t)$ at the beginning and the end of the visible period, $t = 0$ and $t \approx 34.5$, appear in accordance with condition (2.5) with (2.14). In addition, a number of intermediate zeros can be observed. (Online version in colour.)
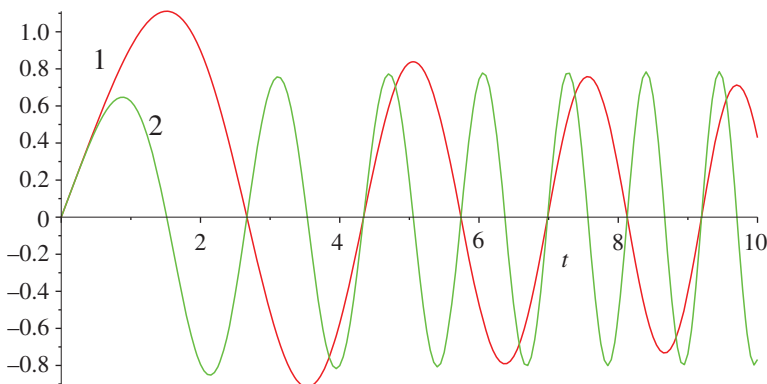
Figure 10. Oscillator with linearly increasing stiffness. The displacement $u(t)$ (1) and the energy distribution function $\mathcal{L}(t)$ (2). It is seen that zeros of the latter correspond to zeros of $u(t)\dot{u}(t)$. (Online version in colour.)

(e) Oscillators under variable stiffness

Consider oscillators whose potential energy depends explicitly on t. First, let the energies be

$$\mathcal{K} = \frac{\dot{u}^2}{2}, \quad \mathcal{P} = \frac{tu^2}{2}, \quad \nu = \mu = 2. \tag{3.15}$$

It follows from (2.4) that, in spite of the aperiodicity of the oscillations, there is the energy equipartition. It concerns the kinetic and potential energies averaged over any interval, t_1, t_2, where $B_1 = B_2$ (2.5). The corresponding dynamic equation is

$$\ddot{u} + tu = 0. \tag{3.16}$$

Its solution, $u(t)$, under conditions $u(0) = 0$, $\dot{u}(0) = 1$ and the energy distribution function $\mathcal{L}(t)$ (2.15), are plotted in figure 10. It is seen that zeros of $\mathcal{L}(t)$ coincide with zeros of $u(t)\dot{u}(t)$, as they should.
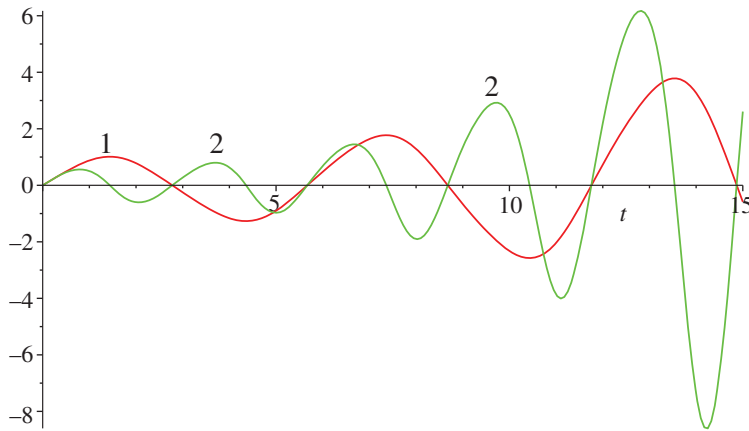
Figure 11. Parametric resonance. The displacement $u(t)$ (1) and the energy distribution function $\mathcal{L}(t)$ (2). It is seen that zeros of the latter correspond to zeros of $u(t)\dot{u}(t)$. (Online version in colour.)

Next, we consider parametric resonance based on Mathieu's differential equation. The energies are

$$\mathcal{K} = \frac{\dot{u}^2}{2}, \quad \mathcal{P} = \frac{(1 + 0.5\sin 2t)u^2}{2}, \quad \nu = \mu = 2. \tag{3.17}$$

The corresponding equation is

$$\ddot{u} + (1 + 0.5\sin 2t)u = 0. \tag{3.18}$$

Its transient solution, $u(t)$, under conditions $u(0) = 0$, $\dot{u}(0) = 1$ and the energy distribution function $\mathcal{L}(t)$ (2.15), are plotted in figure 11. It is seen that zeros of $\mathcal{L}(t)$ coincide with zeros of $u(t)\dot{u}(t)$, as they should.

4. A finite continuous body

In this case, the Lagrangian depends, in addition, on derivatives with respect to the spatial variables. The corresponding Euler–Lagrange equation multiplied by $u(x, t)$ is now integrated by parts over both the time-segment (t_1, t_2) and the body material volume, Ω. As a result, we obtain the relation (2.8), where L is the Lagrangian incorporating the energy of the whole body including its boundaries, whereas the condition (2.5) becomes

$$\int_{\Omega} (B_2 - B_1)\, \mathrm{d}\Omega = 0. \tag{4.1}$$

Note that the integration limits, in the energy partition relation (2.8), correspond to the boundary terms annihilation condition (2.5) (or (4.1)). In contrast, no condition is imposed on the boundary terms associated with the integration by parts over Ω. Such terms reflect the energy located on body boundaries.

As an example consider the collision of a linearly elastic rod, $0 < x < 1$, with a rigid obstacle. For this linear problem the (dimensionless) energies are
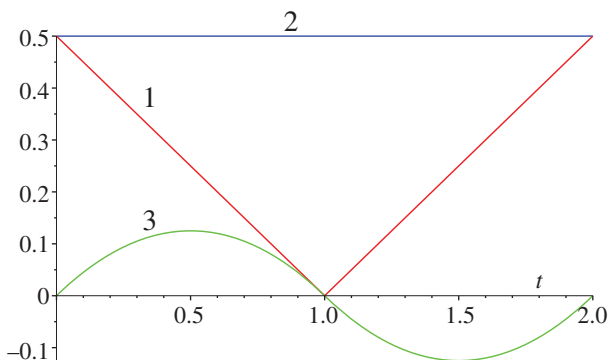
$$\mathcal{K} = \frac{1}{2}\int_0^1 \dot{u}^2\, \mathrm{d}x, \quad \mathcal{P} = \frac{1}{2}\int_0^1 (u')^2\, \mathrm{d}x, \tag{4.2}$$

and the homogeneity orders $\mu = \nu = 2$. The corresponding one-dimensional wave equation is

$$\ddot{u}(x, t) - u''(x, t) = 0, \quad (0 < x < 1) \tag{4.3}$$

with additional conditions

$$u(x, 0) = 0, \quad \dot{u}(x, 0) = 1, \quad u(0, t) = 0 \ (0 < t < 2), \quad u'(1, t) = 0. \tag{4.4}$$

11

rspa.royalsocietypublishing.org Proc. R. Soc. A 471: 20140838

Figure 12. Collision of a rod with a rigid obstacle. The kinetic energy (1), the total energy (2) and the energy distribution function $\mathcal{L}(t)$ (3). Zeros of the latter are at $t = 0$ ($u(x, 0) = 0$), $t = 1$ ($\dot{u}(x, 1) = 0$) and $t = 2$ ($u(x, 2) = 0$). (Online version in colour.)

The collision period: $0 < t < 2$. Dependencies for the energies, $\mathcal{K}$ and $\mathcal{P} = \mathcal{E} - \mathcal{K}$ (the total energy $\mathcal{E} = 1/2$), and the energy distribution function $\mathcal{L}(t)$ are plotted in figure 12.

Next, we consider oscillations of a nonlinearly elastic rod, $0 < x < l$. Let the energies be

$$\mathcal{K} = \int_0^l \frac{\dot{u}^2}{2} \, dx, \quad \mathcal{P} = \int_0^l \frac{(u')^4}{4} \, dx, \quad \mu = 2, \quad \nu = 4. \tag{4.5}$$

We solve the corresponding dynamic equation

$$\ddot{u} - 3(u')^2 u'' = 0 \tag{4.6}$$

by separation of variables, $u(x, t) = X(x)T(t)$, which results in equations

$$(X')^2 X'' + \frac{\lambda}{3} X = 0, \quad \ddot{T} + \lambda T^3 = 0, \tag{4.7}$$

where λ is an arbitrary constant. Note that these equations are solvable analytically. To simplify our considerations, we take values of λ and additional conditions and obtain the corresponding length of the rod and the oscillation amplitude. Namely, we take

$$\lambda = 1, \quad X(0) = X'(l) = T(0) = 0, \quad X'(0) = \dot{T}(0) = 1 \tag{4.8}$$

and obtain the rod length and the oscillation period as

$$l = 1.4674161382, \quad \tau = 3.118169. \tag{4.9}$$

The dependencies of the displacements and energies on the coordinate and time and the energy distribution function $\mathcal{L}(t)$ are presented in figures 13–15.
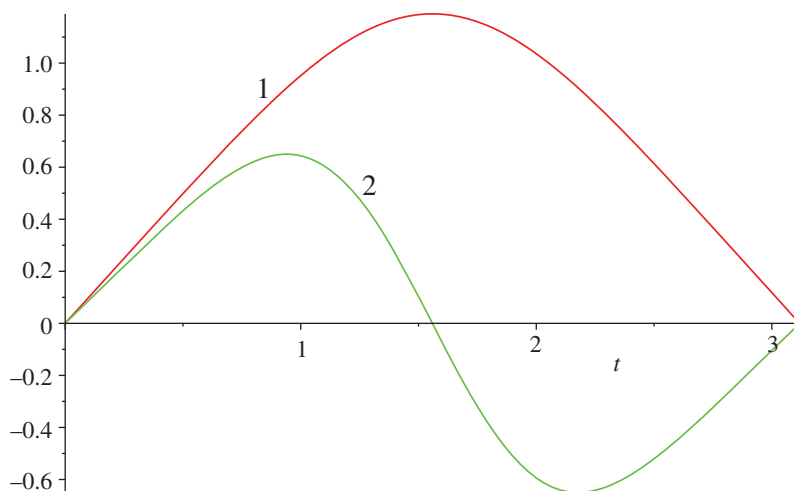
5. Waves

In the case where the total energy of a wave is infinite, we have to choose a finite segment of the waveguide (x_1, x_2), similar to (t_1, t_2) in time, to avoid boundary terms in integration by parts over this segment. Namely, in the conversions

$$\left. \begin{aligned} \int_{x_1}^{x_2} u_i \frac{d}{dx} \frac{\partial L}{\partial u_i'} \, dx &= -\int_{x_1}^{x_2} \frac{\partial L}{\partial u_i'} u_i' \, dx + \frac{\partial L}{\partial u_i'} u_i |_{x_1}^{x_2} \\ \int_{x_1}^{x_2} u_i \frac{d^2}{dx^2} \frac{\partial L}{\partial u_i''} \, dx &= \int_{x_1}^{x_2} \frac{\partial L}{\partial u_i''} u_i'' \, dx + \frac{d}{dx} \frac{\partial L}{\partial u_i''} u_i |_{x_1}^{x_2} - \frac{\partial L}{\partial u_i''} u_i' |_{x_1}^{x_2} \end{aligned} \right\} \tag{5.1}$$

and so on, the segment should be chosen such that the boundary terms vanish. In this case, the energy partition relation (2.8) remains valid with respect to the energies averaged over the space–time region ($x_1 < x < x_2, t_1 < t < t_2$).

12

rspa.royalsocietypublishing.org *Proc. R. Soc. A* **471**: 20140838
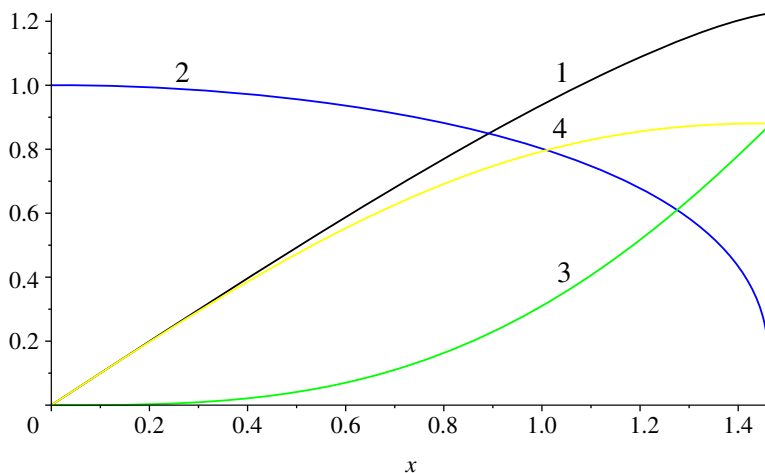
Figure 13. Oscillations of a nonlinearly elastic rod. Functions $X(x)$ (1), $X'(x)$ (2), $\int_0^x X^2(x)\,dx$ (3) and $\int_0^x (X')^4(x)\,dx$ (4). (Online version in colour.)

Figure 14. Oscillations of a nonlinearly elastic rod. Functions $T(t)$ (1) and $\mathcal{L}(t)$ (2). (Online version in colour.)

However, in some classes of waves, which are considered below, the averaging over one variable, t or x, appears to be sufficient.
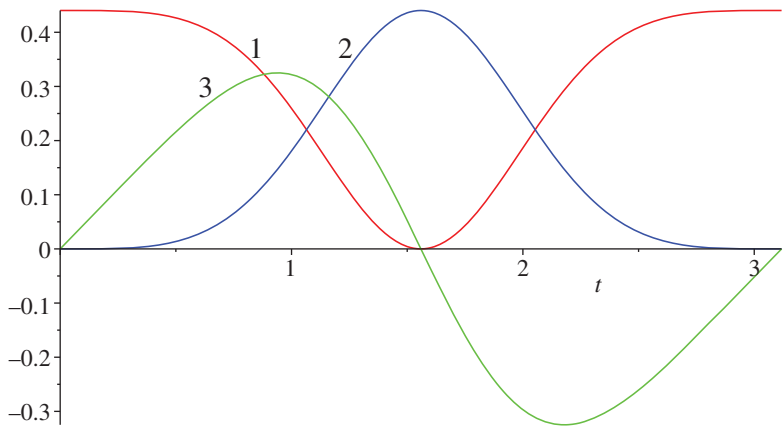
(a) Waves in a homogeneous waveguide

For waves depending only on one variable, $\eta = x - vt$, propagating in such a waveguide, averaging over the t- or x-period, that is, averaging over the η-period, is sufficient. So, in this case, the above relation (2.8) is valid with respect to any cross-section of the waveguide.
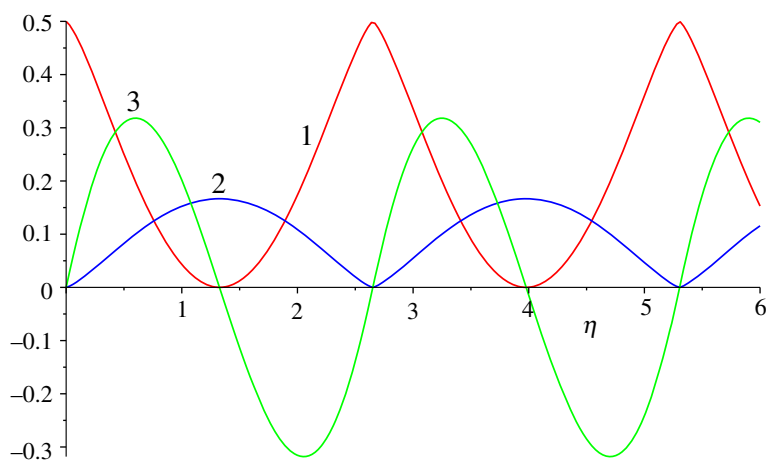
(i) Nonlinearly elastic beam

Consider a wave in a beam where the energies are

$$\mathcal{K} = \frac{\dot{u}^2(\eta)}{2}, \quad \mathcal{P} = \frac{(u''(\eta))^4}{4}, \quad \eta = x - vt. \tag{5.2}$$

13

rspa.royalsocietypublishing.org Proc. R. Soc. A **471**: 20140838

Figure 15. Oscillations of a nonlinearly elastic rod. The kinetic energy (1), potential energy (2) and $\mathcal{L}(t)$ (3). (Online version in colour.)

Figure 16. Periodic wave in a nonlinearly elastic beam. Kinetic energy (1), potential energy (2) and the energy distribution function $\mathcal{L}(t)$ (3) are calculated for $v = 4$, $v = 1$, $u(0) = 0$, $\dot{u}(0) = 1$. (Online version in colour.)

For a periodic wave, the dynamic equation

$$v^2 u''(\eta) + ((u''(\eta))^3)'' = 0 \tag{5.3}$$

can be reduced to

$$u''(\eta) + [v^2 u(\eta)]^{1/3} = 0. \tag{5.4}$$

The kinetic and potential energies and the energy distribution function $L(\eta)$ calculated based on this equation are presented in figure 16.
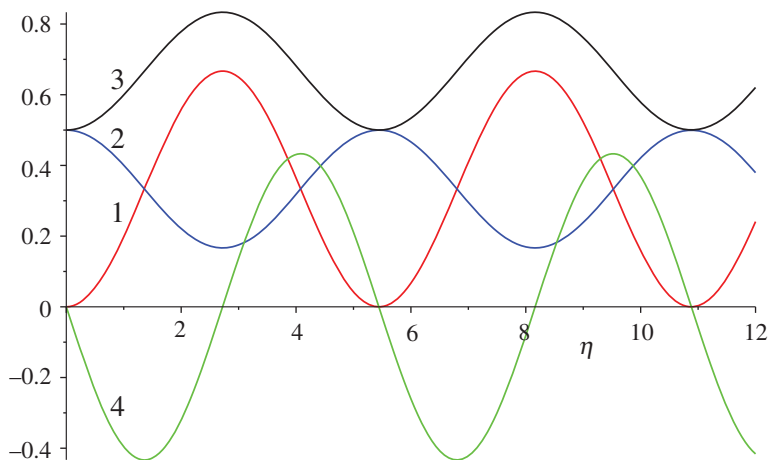
(ii) Linear and nonlinear Klein–Gordon equations

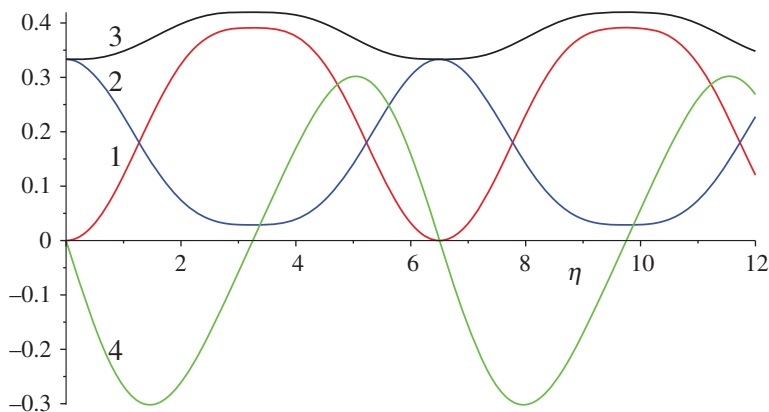Consider periodic waves, $u(\eta)$, $\eta = x - vt$, with the energies

$$K = \frac{\dot{u}^2}{2} = \frac{v^2(u')^2}{2}, \quad \mathcal{P} = \frac{|u'|^v}{v} + \frac{|u|^v}{v}. \tag{5.5}$$

The corresponding equation is

$$(v^2 - (v-1)|u'|^{v-2})u'' + |u|^{v-2}u = 0. \tag{5.6}$$

Figure 17. Klein–Gordon linear equation, $\nu = 2$. Kinetic energy (1), potential energy (2), total energy (3) and the energy distribution function $\mathcal{L}(t)$ (4). (Online version in colour.)
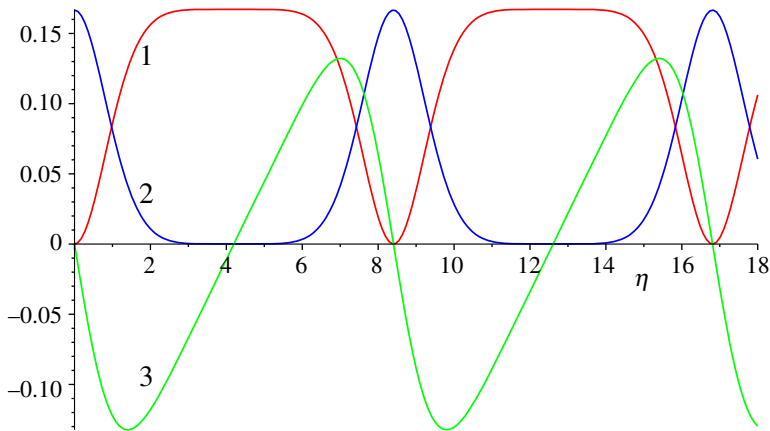

Figure 18. Klein–Gordon nonlinear equation, $\nu = 3$. Kinetic energy (1), potential energy (2), total energy (3) and the energy distribution function $\mathcal{L}(t)$ (4). (Online version in colour.)

The energies and the energy distribution function $\mathcal{L}(\eta)$ are plotted for $\nu = 2$ (the linear wave), 3 and 6 in figures 17–19, respectively. Note that, in the case of waves, the total energy varies during the period.

(b) A wave in periodic structures

For a periodic structure, we consider a Floquet–Bloch wave, where the displacements at discrete points $x = an + x_0$, $0 \leq x_0 < a$, depend on x_0 and $\eta = an - vt$, and possibly other variables, but not on integer n and time separately (here a is the spatial period). This means that each of the cross sections with the same x_0 describes the same periodic trajectory, which differs only by a shift in time. We assume that (generalized) displacements of the cross sections corresponding to a fixed x_0, say, $x_0 = 0$, defined those for the other cross sections.

In the considered case of an infinite waveguide, the energy of a periodic wave is also infinite. However, in the case of a finite radius of the interaction, where the nth cross section is directly connected with a finite number of the others (with the same x_0), the displacement $\boldsymbol{u}_n$ appears only

15

rspa.royalsocietypublishing.org   Proc. R. Soc. A **471**: 20140838

Figure 19. Klein–Gordon nonlinear equation, $v = 6$. Kinetic energy (1), potential energy (2) and the energy distribution function $\mathcal{L}(t)$ for the eighth cell (3). (Online version in colour.)

in a finite number of the infinite series representing the Lagrangian. Accordingly, the averaging (2.8) should correspond to the structure period

$$\int_{t_1}^{t_2} \int_a v_m L_m \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{5.7}$$

where a is the spatial period of the structure. Note that the spatial integral becomes a sum in the case of a discrete structure.

As an example consider a linear mass-spring chain. The (non-dimensional) energies are

$$\mathcal{K} = \tfrac{1}{2}\dot{u}_n^2, \quad \mathcal{P} = \tfrac{1}{4}(u_{n+1} - u_n)^2 + \tfrac{1}{4}(u_{n-1} - u_n)^2. \tag{5.8}$$

In this case, $\mu = \nu = 2$ and there is equipartition. This result also follows from the dynamic equation

$$\ddot{u}_n = u_{n+1} + u_{n-1} - 2u_n, \tag{5.9}$$

which is satisfied by the wave with the corresponding dispersion relation

$$u_n = \exp[\mathrm{i}(\omega t - kn)], \quad \omega = \pm 2 \sin \frac{|k|}{2}. \tag{5.10}$$

From this and (5.8), we find

$$\mathcal{K} = \frac{\omega^2}{2} = \mathcal{P} = 2 \sin^2 \frac{k}{2}. \tag{5.11}$$

Waves in nonlinear chains are considered in §5c.

(c) Solitary wave

In the case of a solitary wave, the partition of its total energy is considered, and the integration limits become infinite, $t_{1,2} = \mp\infty$.
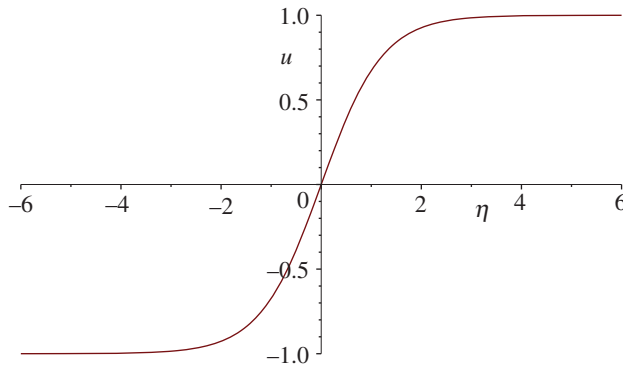
(i) Phi-four equation

Consider the so-called Phi-four equation [10,11]

$$\ddot{u}(x,t) - u''(x,t) - u(x,t) + u^3(x,t) = 0, \tag{5.12}$$

where dots and primes still mean the derivatives with respect to t and x, respectively. Equation (5.12) is satisfied by a kink

$$u(x,t) = U(\eta) = \pm\tanh\left(\frac{\eta}{\sqrt{2(1 - v^2)}} + \phi_0\right), \quad \eta = x - vt, \tag{5.13}$$

16

rspa.royalsocietypublishing.org *Proc. R. Soc. A* **471**: 20140838

Figure 20. The kink. (Online version in colour.)

where the arbitrary constants, v ($v^2 < 1$) and ϕ_0, are the kink's speed and the 'initial' phase, respectively. The corresponding energies are

$$\left.\begin{array}{c} \mathcal{K} = \dfrac{\dot{u}^2(\eta)}{2} = v^2 \dfrac{(u'(\eta))^2}{2}, \quad \mathcal{P} = \mathcal{P}_0 + \mathcal{P}_2 + \mathcal{P}_4 \\[3mm] \mathcal{P}_0 = \dfrac{1}{4}, \quad \mathcal{P}_2 = \dfrac{(u'(\eta))^2}{2} - \dfrac{u^2(\eta)}{2}, \quad \mathcal{P}_4 = \dfrac{u^4(\eta)}{4}, \end{array}\right\} \tag{5.14}$$

and

where the value of the arbitrary constant $\mathcal{P}_0$ ($v = 0$) is chosen to limit the potential energy.

While the moving kink corresponds to the transition from one zero-energy state to another, the particle velocity,

$$\dot{u}(\eta) = \mp \frac{v}{\sqrt{2(1-v^2)}} \left[\cosh^2\left(\frac{\eta}{\sqrt{2(1-v^2)}} + \phi_0 \right) \right]^{-1}, \tag{5.15}$$

represents a true solitary wave. As a result, $u\dot{u} \to 0$ ($\eta \to \pm\infty$) and our statement is valid with respect to the total energies of the wave

$$\int_{-\infty}^{\infty} \mathcal{L}(\eta)\,\mathrm{d}\eta = 0, \quad \mathcal{L} = 2(\mathcal{K} - \mathcal{P}_2) - 4\mathcal{P}_4. \tag{5.16}$$

The kink, the solitary wave and the energies as functions of η are shown in figures 20 and 21. Far away from the transition region, there is a static state, where, in accordance with (2.11) and (5.16), $\mathcal{P} = 0$, $\mathcal{P}_2 = -u^2(\eta)/2 = -2\mathcal{P}_4$.
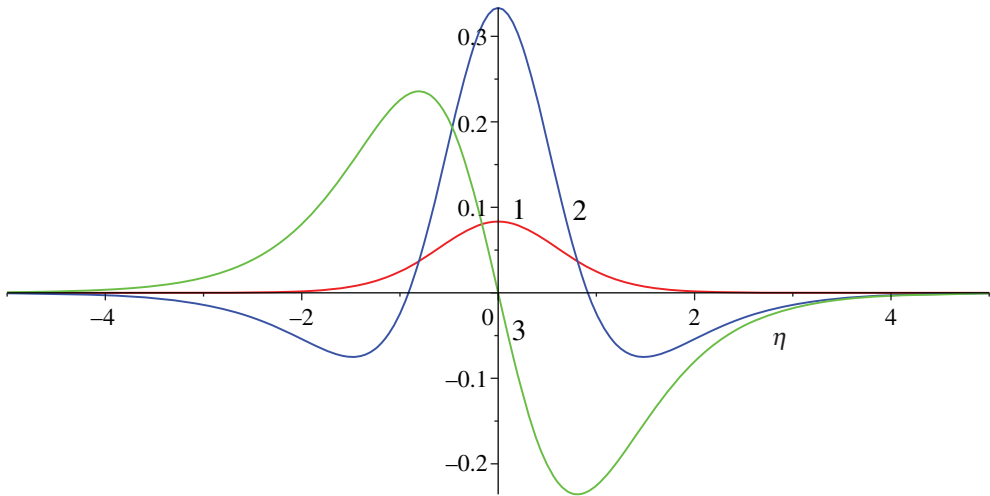
(ii) Derrick–Pohozaev identity

In the problem (5.16), there are three unknown energy terms, $\mathcal{K}$, $\mathcal{P}_2$ and $\mathcal{P}_4$. The single equality (2.8) is not sufficient to obtain the $\mathcal{K} - \mathcal{P}$ relation but only the $\mathcal{K} - (\mathcal{P}_2 + 2\mathcal{P}_4)$ one. Fortunately, for such a solitary steady-state wave, the Derrick–Pohozaev identity can be used as an additional energy relation. The identity is based on the representation of the steady-state solitary wave as a static state. Note that, in the steady-state mode, the kinetic energy can be considered as such, as a function of $\dot{u} = \partial u/\partial t$, and also as a function of $u' = \partial u/\partial x$,

$$\mathcal{K} = \frac{\dot{u}^2}{2} \equiv \mathcal{P}_- = \frac{v^2(u')^2}{2}. \tag{5.17}$$

In the latter case, it can be treated as an additional (negative) part of the potential energy. Different energy relations, one discussed in this paper and another given by the Derrick–Pohozaev identity, follow from the static representation as a result of different variations. While the former corresponds to the displacement variation $\delta u = \varepsilon u$, the latter is a uniform x-extension. The use of both relations allows us to separate the kinetic and potential energies completely.

17

rspa.royalsocietypublishing.org Proc. R. Soc. A 471: 20140838

Figure 21. The energies, $\mathcal{K}$ (1), $P_2 + 2P_4$ (2) and the energy distribution function $\mathcal{L}(\eta)$ (3) for the kink, $v = 1/2$. (Online version in colour.)

The Derrick–Pohozaev identity is found for the nonlinear Klein–Gordon equation with a general function of u (see [8,9] and [10, sect. 7.4]). Derived for a multi-dimensional case, it reads in one dimension as the equality of two parts of the energy, one as a function of u and another as a function of u'. In our terms, the identity is

$$\mathcal{E}_1 = \mathcal{E}_2, \quad \mathcal{E}_1 = \frac{1 - v^2}{2}(u')^2, \quad \mathcal{E}_2 = \frac{1}{4}(1 - u^2)^2. \tag{5.18}$$

The solution (5.13) satisfies this equality. Thus, we now have two independent relations, one in (5.16) and another in (5.18). Using both of them, we can completely separate the kinetic and potential energies. The partition between the total energies and the energies themselves is found as

$$\int_{-\infty}^{\infty} \mathcal{K}\,\mathrm{d}\eta = \frac{v^2}{2 - v^2} \int_{-\infty}^{\infty} \mathcal{P}\,\mathrm{d}\eta = \frac{v^2}{\sqrt{2(1 - v^2)}} \quad (-1 < v < 1). \tag{5.19}$$

The kinetic and total potential energy graphs are shown in figure 22. The equality between the total energies holds with respect to the energy densities, $\mathcal{K}$ and $\mathcal{P}$. So

$$\mathcal{K}(\eta) = \frac{v^2}{2 - v^2}\mathcal{P}(\eta) \quad (-1 < v < 1). \tag{5.20}$$
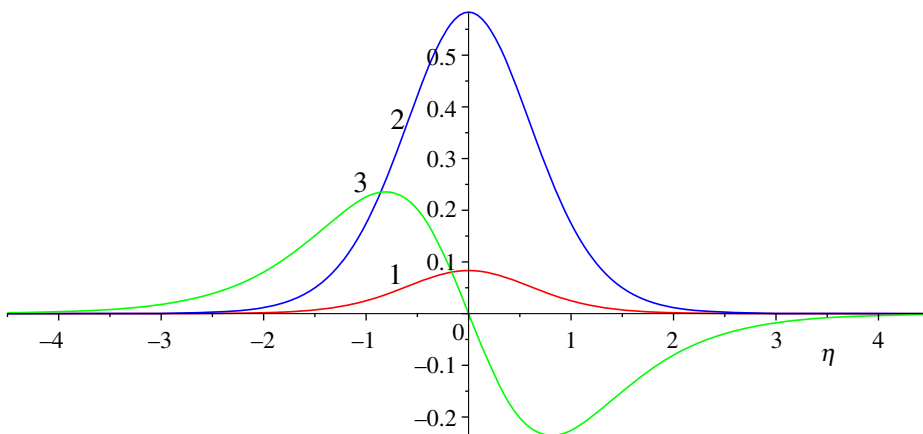
(iii) Fermi–Pasta–Ulam problem

Next, we consider a transient problem for a wave excited by a pulse applied to the same but 'semi-infinite' linear chain. No solitary wave can exist in the latter, and we study this problem to compare the result with that for a nonlinear chain first examined by Fermi $et\ al.$ [12]. The displacements of the first 20 masses under unit pulse acting on the first left-hand mass and also the corresponding speeds are presented in figures 23 and 24, respectively.

The wave itself, especially the particle velocity wave, and the energy distribution function, $\mathcal{L}(t)$, can be compared with those in the Fermi–Pasta–Ulam (FPU) problem [12] for the same but nonlinear chain (figures 25 and 26), where a stable solitary wave arises (compare figure 27 with figure 28), and with the 'pure nonlinear' one corresponding to the potential energy without linear terms. In a version of the FPU problem, the energies are taken as

$$\mathcal{K} = \frac{1}{4}\dot{u}_n^2, \quad \mathcal{P} = \frac{1}{4}(u_{n+1} - u_n)^2 + \frac{1}{4}(u_{n-1} - u_n)^2 + \frac{1}{2}(u_{n+1} - u_n)^4 + \frac{1}{2}(u_{n-1} - u_n)^4. \tag{5.21}$$
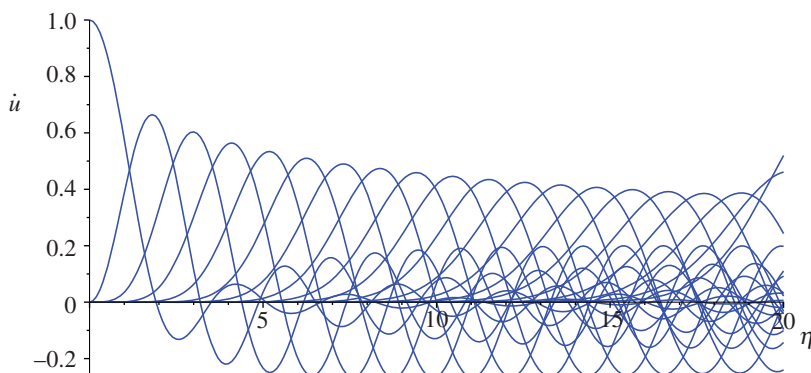
Here, $\mu = v_1 = 2$, $v_2 = 4$ and $\int_{-\infty}^{\infty}(\mathcal{K} - \mathcal{P}_1 - 2\mathcal{P}_2)\,\mathrm{d}t = 0$.

18

rspa.royalsocietypublishing.org *Proc. R. Soc. A* **471**: 20140838

Figure 22. Functions $\mathcal{K}$ (1), $\mathcal{P} = 7\mathcal{K}$ (2) and the energy distribution function $\mathcal{L}(\eta)$ (3) for the kink, $v = 1/2$. (Online version in colour.)
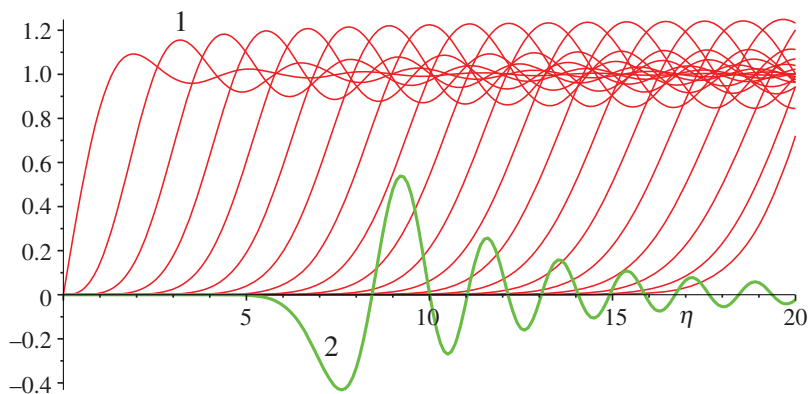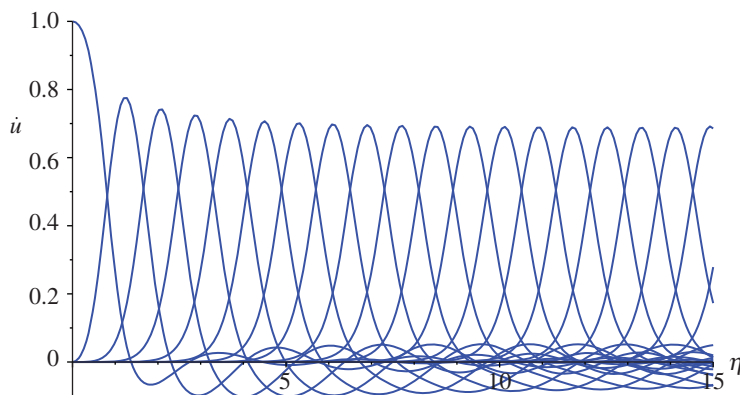

Figure 23. Transient wave in the linear chain under a pulse. The displacements of the masses (1) and $20\mathcal{L}(t)$ for the eighth cell (2). (Online version in colour.)


Figure 24. Transient wave in the linear chain under a pulse. The decreasing wave of particle velocities. (Online version in colour.)

19

rspa.royalsocietypublishing.org Proc. R. Soc. A 471: 20140838
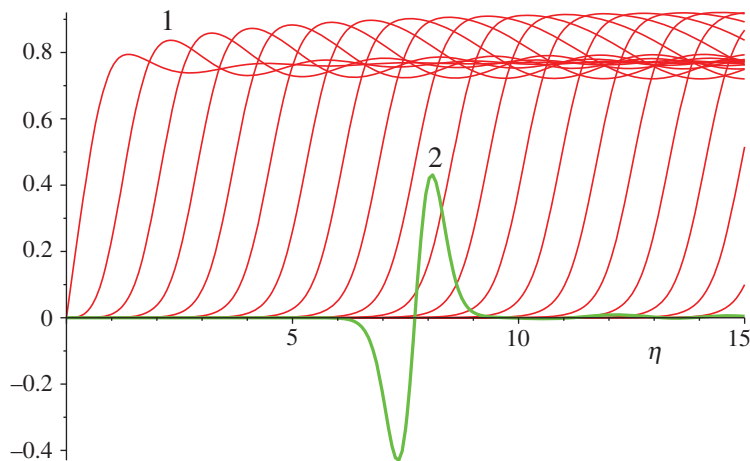
Figure 25. Transient wave in the nonlinear chain under a pulse (FPU problem). The displacements of the masses (1) and $8\mathcal{L}(t)$ for the 10th cell (2). (Online version in colour.)

Figure 26. Transient wave in the nonlinear chain under a pulse (FPU problem). The establishing solitary wave of particle velocities. (Online version in colour.)
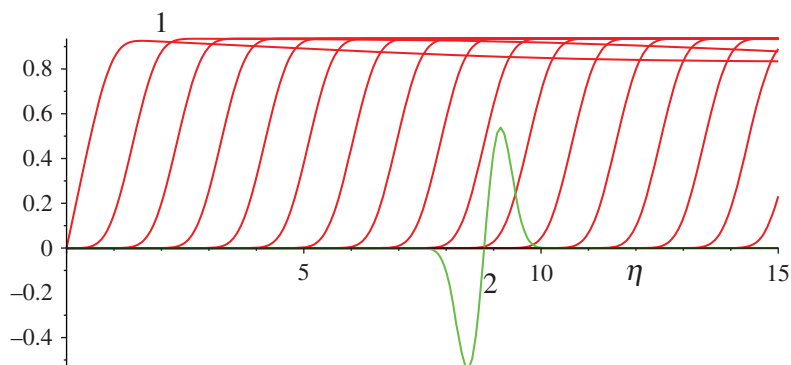
Figure 27. Solitary wave in the nonlinear chain with no quadratic terms in the expression of the potential energy (5.21). The displacements of the masses (1) and $4\mathcal{L}(t)$ for the 10th cell (2). (Online version in colour.)
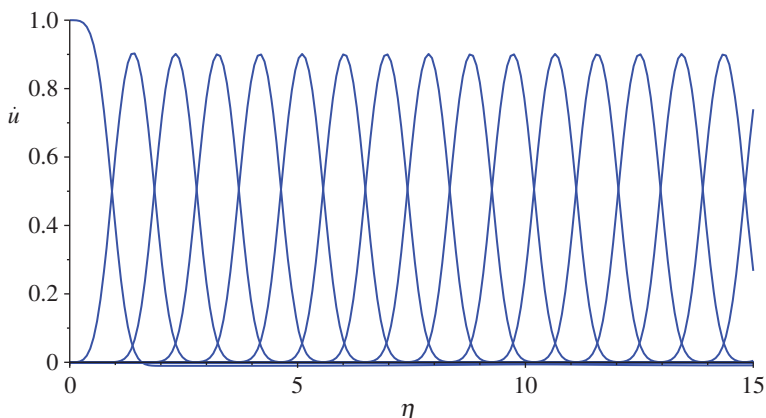
Figure 28. Solitary wave in the nonlinear chain with no quadratic terms in the expression of the potential energy (5.21). The particle velocities. (Online version in colour.)

A fast-established strongly localized solitary wave appears in the nonlinear chain with quadratic terms in the expression of the potential energy (5.21) removed. In this case, $\mathcal{K} = 2\mathcal{P}$. The calculation results are presented in figures 27 and 28.

6. Conclusion

The energy partition obeys the relation derived for the case where the Lagrangian is represented as a sum of homogeneous functions of the displacements and their derivatives (explicit dependencies on time and spacial coordinates are not excluded). If the kinetic and potential energies are entirely homogeneous the homogeneity orders define the partition uniquely.

The equipartition corresponds to the case where the kinetic and potential energies are of the same homogeneity order. In particular, this is true for homogeneous linear problems where both energies are of second order. At the same time, the linearity is neither a necessary nor a sufficient condition for the equipartition.

The energy partition corresponds to the energies averaged over a region satisfying the condition (2.5). Thus, the integration can correspond not only to the period, if it exists, but also to the end points, where $u = 0$ and, in the case (2.14), where $\dot{u} = 0$. This is especially important for non-periodic oscillations and waves, which can exist in transient problems and under time-varying parameters as discussed above. For a solitary wave the total energies are considered, and the averaging integral is taken over the whole axis (or a half-axis).

The energy partition relation is valid as far as the related general conditions are satisfied. So, it is applicable to any specific problem of the corresponding class, and there is no need for an examination of any specific problem in more detail. Nevertheless, we have presented various examples to show how the partition varies during the time interval of averaging, and also to demonstrate the validity of the relation and the conditions relating to the averaging. These examples show that the relation is valid for various problems: linear and nonlinear, steady state and transient, conservative and non-conservative, homogeneous and forced oscillations, periodic and solitary waves.

In the case of the steady-state solitary wave, where the potential energy consists of two functions of different orders, the static Derrick–Pohozaev identity is involved along with the dynamic energy partition relation derived in this paper. The use of both of them allowed us to find the desired kinetic-to-potential energy relation.

21

rspa.royalsocietypublishing.org *Proc. R. Soc. A* **471**: 20140838

References

1. Mask LR, Jay BE. 1967 The partition of energy in standing gravity waves of finite amplitude. *J. Geophys. Res.* **72**, 573–581. (doi:10.1029/JZ072i002p00573)
2. Lighthill J. 1978 *Waves in fluids*. Cambridge, UK: Cambridge University Press.
3. Antenucci JP, Imburger J. 2001 Energetics of long internal gravity waves in large lakes. *Limnol. Oceanogr.* **46**, 1760–1773. (doi:10.4319/lo.2001.46.7.1760)
4. Falnes J. 2007 A review of wave-energy extraction. *Mar. Struct.* **20**, 185–201. (doi:10.1016/j.marstruc.2007.09.001)
5. Whitham GB. 1974 *Linear and nonlinear waves*. New York, NY: John Wiley & Sons.
6. Cardozo TM, Nascimento MAC. 2009 Energy partitioning for generalized product functions: the interference contribution to the energy of generalized valence bond and spin coupled wave functions. *J. Chem. Phys.* **130**, 104102-1–8. (doi:10.1063/1.3085953)
7. Korycansky DG. 2013 Energy conservation and partition in CTH impact simulations. In *Proc. 44th Lunar and Planetary Science Conf., Woodlands, TX, 18–22 March 2013*, p. 1370. Houston, TX: Lunar and Planetary Science Institute.
8. Derrick G. 1964 Comments on nonlinear wave equations as models for elementary particles. *J. Math. Phys.* **5**, 1252–1254. (doi:10.1063/1.1704233)
9. Pohozaev S. 1965 Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Sovet Math. Docl.* **6**, 1408–1411.
10. Dodd RK, Eilbeck JC, Gibbon JD, Morris HC. 1984 *Solitons and nonlinear wave equations*. New York, NY: Academic Press.
11. Zwillinger D. 1997 *Handbook of differential equations*, 3rd edn. Boston, MA: Academic Press.
12. Fermi E, Pasta J, Ulam S. 1955 Studies of nonlinear problems. I. Los Alamos report LA-1940. Published later in *Collected papers of Enrico Fermi* (ed. E Segr). Chicago, IL: University of Chicago Press (1965).