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Forced waves in a uniform waveguide with distributed and localized dynamic structures attached

Leonid Slepyan

School of Mechanical Engineering, Tel Aviv University, P.O. Box 39040, Ramat Aviv 69978 Tel Aviv, Israel

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ABSTRACT

Waves are considered excited by a moving-oscillating load in a general uniform waveguide with uniformly distributed and localized dynamic structures. In its order, the load frequency and speed define the waveguide frequency. Asymptotic solutions are obtained based on three parameters corresponding to the exciting wave frequencies: the phase and group velocities and the *dispersion number* (the second derivative of the frequency). We also present descriptions of the quasi-front and the resonant waves. The waveguide response under the load determined allowed us to find the phase shift between the oscillating force and the (energy flux related) waveguide speed.

1. Introduction

1.1. Structure vs Microstructure

A structured waveguide is usually called one with a *microstructure*. Such terms, the microstructure, and nanostructure reflect the structure scale, micrometer, and nanometer, respectively. However, let the equations and additional conditions be formulated in terms of letters and symbols, be valid for arbitrary numerical values. In this case, the scale does not matter, and from the mechanics' point of view, the prefix *micro* in *microstructure* looks optional.

Let us consider how a process on a scale can be modeled in a different one. Let the material qualities, and hence the wave speeds be the same in the both scales. In this case, the scale relations for the length and time are the same

$$\mathbf{X} = \lambda \mathbf{x} \quad \Longrightarrow \quad T = \lambda t\,, \quad \lambda = \text{const}\,, \quad 0 < \lambda < \infty\,, \tag{1}$$

where $\mathbf{x}, t$ and $\mathbf{X}, T$ are the initial and new coordinate-time couples, respectively.

A point-mass–spring lattice is a natural model of such a microstructure. Its behavior can be modeled in a similar lattice of an arbitrary scale as in (1). The continuous, nonstructural medium model is only a long-wave approximation (long compared with the lattice cell size).

Slepyan (1967) considered the evolution of a wave in a cylindrical elastic shell with the attached dynamic *macro*-structure (modeled by uniformly distributed mass–spring oscillators). The same wave (with the corresponding time scale) can propagate in a continuous waveguide with the microstructure. Also, in this case, the wave evolution is scale-insensitive.

Where there is no such a self-similarity? It happens if a value, which should be changing under the scale change, is fixed. For example, it is the gravity force. The acceleration under the gravity (at the same height!) is a constant, while, as follows from (1), it is proportional to $1/\lambda$. Surface energy (fixed) is another example.

E-mail address: slepyanl@tauex.tau.ac.il.

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1.2. Some references and about the current paper

Milton and Willis (2007) introduced a theory of an elastic medium with a dynamic structure. Waves in various waveguides were the subject in many works. In particular, in Bigoni et al. (2013), Bigoni and Movchan (2002) and Mishuris et al. (2019, 2020), where the corresponding references are listed. Waves in a complex waveguide are topical in various applications. For example, Engelbrecht et al. (2020) and Peets et al. (2021) studied nerve signals propagation in a natural micro-structured waveguide. Eun Bok et al. (2018) discussed the role of a nontrivial structural dynamic layer as a water–air interface.

We consider 1D waves forced by a moving-oscillation load and propagating in a uniform waveguide with a dynamic structure. As a (micro) structure, different oscillating systems are assumed, in particular, a dynamic elastic foundation introduced. We obtain asymptotic solutions (and compare them with an 'exact' numerical calculation for a not too big time, Section 3.2. The analysis is based on the phase and group velocities at the points, where the oscillating-propagating load's line (on the $k-\omega$ plane) crosses the dispersion relation graph, Section 2.2. For a wave quasi-front area, Section 3.2, and in a resonant excitation, Section 3.3, where the load velocity coincides with the wave group velocity, the second derivative, $\mathrm{d}^2\omega(k)/\mathrm{d}k^2$, the *dispersion number*, is also involved. Such an asymptotic analysis is not applicable for the area under the load, where such asymptotic representation does not valid. The second derivative allowed us to demonstrate the phase shift between the oscillations of the load and the structure speed, which strongly affects the energy flux. (Such a shift is a well-known phenomenon in electricity.) We also investigate a response of a localized structure attached.

2. Waveguide with a distributed dynamic structure

2.1. Formulation

We consider 1D forced waves in a *master* body equipped by a structure or being in contact with a medium. We assume that the combined system is perfect and stable (condition (*)).

Represent the force, $P(t,x)$, acting on the body from the structure in terms of the structure related fundamental solution, $R(t,x)$, corresponding to the body displacement, $u(t,x)$,

$$P(t,x) = -R(t,x) ** u(t,x)\,. \tag{2}$$

In particular, if there is no interconnections in the structure, there exists a point response of the structure. In this case, the convolution on x in above equation disappears, and

$$P(t,x) = -R(t) * u(t,x) = \int_0^t R(t-\tau)u(\tau,x)\,\mathrm{d}\tau\,. \tag{3}$$

In a general case, the dynamic equation for the body under both, the structure response and a moving-oscillating external force, is

$$(\mathcal{L}(t,x) + R(t,x)) **\ u(t,x) = Q(t,x) = Q_\eta(\eta)\mathrm{e}^{\mathrm{i}\omega_0 t}\,,\quad \eta = x - vt, \tag{4}$$

where $\mathcal{L}(t,x)$ is a linear operator, and $Q(t,x)$ is the external force moving with speed v and oscillating with frequency ω_0.

As far as the operator $\mathcal{L}$ contains an inertia term, with account of the attached structure it appears that the body-related inertia term receives an addition, as the above convolution

$$\varrho\frac{\partial^2 u(t,x)}{\partial t^2} \implies \left(\varrho\frac{\partial^2}{\partial t^2} + R(t)\,*\right)u(t,x)\,. \tag{5}$$

The master body dynamics can be considered separately, paying no attention to the structure but having modified inertia. It looks like a modified Newton's second law, as Graeme W. Milton and John R. Willis noted, Milton and Willis (2007).

We consider the wave excitation problem under zero initial conditions, namely: $u = 0$ $(t < 0)$, and the Laplace transform on time is just what we have to use together with the Fourier transform on x-coordinate. The Eq. (4) becomes

$$\mathcal{L}_+^{LF}(s,k)u^{LF}(s,k) = \frac{Q_\eta^{F_\eta}(k)}{s - \mathrm{i}(\omega_0 + kv)},$$
$$\mathcal{L}_+^{LF}(s,k) = \mathcal{L}^{LF}(s,k) + R^{LF}(s,k). \tag{6}$$

where

$$Q_\eta^{F_\eta}(k) = \int_{-\infty}^{\infty} Q_\eta(\eta)\mathrm{e}^{\mathrm{i}k\eta}\,\mathrm{d}\eta\,,\quad \eta = x - vt\,. \tag{7}$$

Taking into account that the original function corresponding to $1/(s - \mathrm{i}(\omega_0 + kv))$ is $\exp(\mathrm{i}(\omega_0 + kv)t)$ we see that the frequency $\omega_0 + kv$ corresponds to that detected by an unmoving observer. The difference from that detected by a moving observer, ω_0, is the well-known Doppler effect.

With respect to the dispersion relation we first note that, as follows from the condition (*), a zero point of the total operator, $\mathcal{L}_+^{LF}(s,k)$, with k real may correspond only to an imaginary s, $s = \mathrm{i}\omega(k)$. In other words, the sinusoidal complex wave, $\exp(\mathrm{i}(\omega(k)t - kx))$,

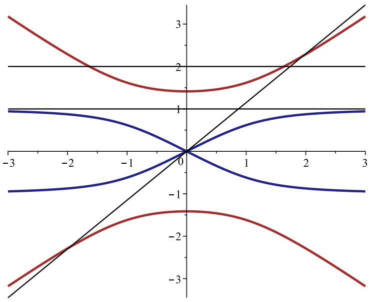

Fig. 1. Dispersion dependencies $\omega(k)$ for an elastic string with distributed oscillators. The two branches correspond to the non-dimensional equation $(k^2-\omega^2)(1-\omega^2)=\omega^2$. The upper branch can be called "The string under the oscillator's influence", and vice versa for the lower one. The straight lines correspond to some possible load's actions. The horizontal lines reflect to the non-moving oscillating load; the lower one tangent to the lower branch relates to the resonant excitation. The inclined line plotted for the moving ($v>0$) non-oscillating load.

where $\omega(k)$ is the dispersion relation, and k is the wave number, satisfies the homogeneous Eq. (6). The structure respond term transformed is

$$R^{LF}(s,k)=\int_0^\infty\int_{-\infty}^\infty R(t,x)\mathrm{e}^{-st+\mathrm{i}kx}\,\mathrm{d}t\,\mathrm{d}x\,. \tag{8}$$

It corresponds to the sinusoidal waves in $0<t<\infty, -\infty<x<\infty$. Generally, with the first restriction absent, both integrals are from $-\infty$. We also take into account that, due to above mentioned symmetry, the perfect waveguide operator, $CL_+^{LF}(s,k)$, depends on s^2 and k^2.

Some dispersion relations are presented in Figs. 1–3.

2.2. Dispersion relations for the illustrative examples

2.2.1. The string with uniformly distributed oscillators

Consider a string equipped by uniformly distributed oscillators with no interconnections between the latter. The uniform Eq. (6) is

$$(\varrho s^2+Tk^2+R^L(s))u^{LF}(s,k)=0\,,\quad R^L(s)=\frac{\varkappa s^2}{s^2+\omega_{osc}^2}\,, \tag{9}$$

where $\varrho, T, \varkappa$ and ω_{osc} are the string mass per unit length, the tension force, the oscillator's spring stiffness and its frequency, res. From here, with $s\to\mathrm{i}\omega$, the (inverse) dispersion dependence follows as

$$k=\pm\sqrt{\frac{\omega^2}{c^2}-\frac{\varkappa\omega^2}{T(\omega^2-\omega_{osc}^2)}}\,, \tag{10}$$

where the wave speed $c=\sqrt{T/\varrho}$, and this relation is valid only for real k,ω.

The dispersion relation $\omega(k)$ expressed in non-dimensional form (all the parameter are taken equal to one) is plotted in Fig. 1.

2.2.2. The above problem, but for a bending beam

Next, consider the similar system for a bending beam. The Eq. (6) with Tk^2 replaced by Dk^4, where D is the bending stiffness of the beam, remains valid. The (inverse) dispersion relation becomes

$$k=\pm\left(\frac{m\omega^2}{D}-\frac{\varkappa\omega^2}{D(\omega^2-\omega_{osc}^2)})\right)^{1/4}\,. \tag{11}$$

It is shown (also as $\omega(k)$ and in the non-dimensional form) in Fig. 2

2.2.3. Dynamic elastic foundation

As another structure, consider a dynamic elastic foundation, a uniformly distributed supports without interconnections (as usually adopted in statics but with the account of inertia forces). Let $E_0, \varrho_0, c_0=\sqrt{E_0/\varrho_0}$ be the support material parameters (in usual notations), and l_0, b_0 and y be its length and width and the (normal to x) coordinate along it. The equation and boundary conditions in terms of the displacement $w(t,y)$ for an arbitrary x and the boundary conditions are

$$\varrho_0 b\frac{\mathrm{d}^2w(t,y)}{\mathrm{d}t^2}-E_0\frac{\mathrm{d}^2w(t,y)}{\mathrm{d}y^2}=0\,,\quad w(t,0)=0\,,\quad w(t,l)=u(t,x)\,, \tag{12}$$

where $u(t,x)$ is the transverse displacement the waveguide.

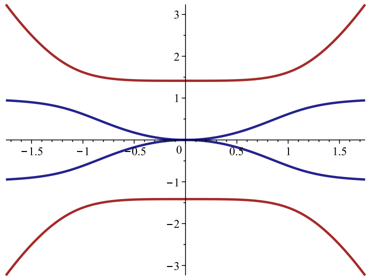

Fig. 2. Dispersion dependencies $\omega(k)$ for a bending beam with distributed oscillators. The equation is $(k^4-\omega^2)(1-\omega^2)=\omega^2$. The upper branch can be called as for the beam under the oscillators influence, and vice versa for the lower one.

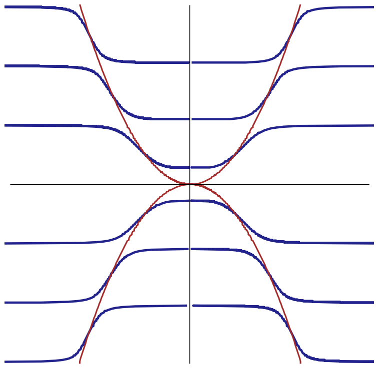

Fig. 3. Dispersion dependence $\Omega(O)$ for a bending beam on the dynamic elastic foundation, (15). The brown curves, $\Omega=\pm K^2$, correspond to the free beam ($b_0=0$). The horizontal asymptotes of the blue lines, $K\to\pm\infty$, reflect the resonant oscillations.

In the following, we consider the latter as an elastic bending beam, the uniform equation of each is

$$\varrho S\frac{\mathrm{d}^2u(t,x)}{\mathrm{d}t^2}+D\frac{\mathrm{d}^4u(t,x)}{\mathrm{d}x^4}-P(t)=0\,, \tag{13}$$

where S and D are the cross-section area and bending stiffness, respectively, and $P(t)$ is the (x-distributed force by which the support acts on the beam.

For the complex wave, $u(t,x)=\exp(\mathrm{i}(\omega t+kx))$, the above relations result in

$$w(t,y)=\frac{\sin(\omega y/c_0)}{\sin(\omega l/c_0)}\mathrm{e}^{\mathrm{i}\omega t}\,,$$
$$P=-\frac{\mathrm{d}w(t,y)}{\mathrm{d}y}E_0b_0=\frac{E_0b_0\omega}{c_0}\cot\left(\frac{\omega l}{c_0}\right)\mathrm{e}^{\mathrm{i}\omega t}\,. \tag{14}$$

Finally, we come to the (inverse) dispersion relation

$$K=\pm(\Omega(\Omega-\lambda\cot(\Omega))^{1/4}\,, \tag{15}$$

where only real values should be accepted, and the (non-dimensional) values are

$$\Omega=\frac{\omega l}{c_0}\,,\quad K=k\left(\frac{Dl^2}{\varrho Sc_0^2}\right)^{1/4}\,,\quad \lambda=\frac{\varrho_0bl}{\varrho S}\,. \tag{16}$$

The dispersion relation $\Omega(K)$ for $\lambda=1$ is plotted in Fig. 3.

Note that $K=0$ at the points where $\Omega=\lambda\cot(\Omega)$, and

$$K\to\pm\infty\quad(\Omega\to n\pi-0,n=0,\pm1,\pm2,\dots)\,. \tag{17}$$

Physically, the first case corresponds to the oscillation of the support with the (straight-line) beam. The second one appears at the resonant oscillations of the support. In these connections, note that the bending beam model is valid only for relatively long waves, that is, not too large wavenumbers. So, the latter may not increase unboundedly.

3. The forced wave

3.1. The main part of the wave

From the Eq. (6):

$$u^{LF}(s,k) = \frac{Q_\eta^{F_\eta}(k)}{\mathcal{L}_+^{LF}(s,k)(s-\mathrm{i}(\omega_0+kv))} \tag{18}$$

Let $s=\mathrm{i}\omega(k)$ be a zero point of $\mathcal{L}_+^{LF}(s,k)$. Asymptotically only real wave numbers are significant under which the second multiplier is also equal to zero

$$\omega(k) = \omega_0 + kv\,. \tag{19}$$

Denote one of them $k=k_*$.

Physically, this statement means those waves are excited, which frequencies coincide with the load frequency (otherwise, the load gives no energy to the wave to propagate). Formally, these points define asymptotic representations of the propagating waves excited by the load.

With this note in mind, we assume that the frequency ω_0 is real, and, with no loss of generality, the speed $v\geq 0$.

Taking into account the symmetry, represent

$$\mathcal{L}_+^{LF}(s,k) = \Phi(s,k)(s^2+\omega^2(k))\,. \tag{20}$$

Note that only first-order poles can exist for a non-growing function. So, $\Phi(s,k)\neq 0$ at $s=\pm\mathrm{i}\omega$.

The two poles, $s=\mathrm{i}\omega(k)$ and $s=\mathrm{i}(\omega_0+kv)$, united with $k\to k_*$ give us

$$\begin{aligned} u^F(t,k) &\sim \frac{Q_\eta^{F_\eta}(k_*)}{\Phi^{LF}(\mathrm{i}\omega(k_*),k_*)2\omega(k_*)}U^F(t,k)\,,\\ U^F(t,k) &= \frac{\mathrm{e}^{\mathrm{i}\omega(k)t}-\mathrm{e}^{\mathrm{i}(\omega_0+kv)t}}{\omega_0+kv-\omega(k)} \sim \frac{\mathrm{e}^{\mathrm{i}v_g(k_*)\xi t}-\mathrm{e}^{\mathrm{i}v\xi t}}{(v-v_g(k_*))\xi}\,\mathrm{e}^{\mathrm{i}\omega(k_*)t}\quad (\xi = k-k_*), \end{aligned} \tag{21}$$

and $v_g(k_*)=\mathrm{d}\omega/\mathrm{d}k$ is the group velocity ($k=k_*$). Next,

$$\begin{aligned} U(t,x) &\sim A\mathrm{i}\mathrm{e}^{\mathrm{i}(\omega(k_*)t-k_*x)}\quad (t\to\infty)\,,\\ A &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}(v_g(k_*)t-x)\xi}-\mathrm{e}^{\mathrm{i}(vt-x)\xi}}{(v-v_g(k_*))\mathrm{i}\xi}\,\mathrm{d}\xi\\ &= \frac{1}{2(v-v_g(k_*))}(\mathrm{sign}(v_g(k_*)t-x)-\mathrm{sign}(vt-x))\,. \end{aligned} \tag{22}$$

This asymptotic representation is valid for $v\neq v_g$ (the equality corresponds to a resonant wave) and for $|v_gt-x|\to\infty$ and $|vt-x|\to\infty$, that is outside the wave front (quasi-front) and the load area. Below we consider the resonant wave and the wave in these two special areas.

Thus, under the above restrictions, the wave, corresponding to a single point of the dispersion dependence, places between rays $x=vt$ and $x=v_gt$.

3.2. The wave with a quasi-front

To describe the wave in a wave front (quasi-front) area, that is in a vicinity of the ray $x=v_g(k_*)t$, we should take into account the second derivative of the frequency, or the first nonzero higher derivative

$$\omega(k)-\omega(k_*) \sim v_g(k_*)\xi + \varkappa_n\xi^n\,,\quad \varkappa_n = \frac{1}{n!}\frac{\mathrm{d}^n\omega(k)}{\mathrm{d}k^n}\,,\quad k=k_*\,, \tag{23}$$

where $n\geq 2$ is minimal one for which $\varkappa_n\neq 0$.

The amplitude A (22) becomes (we assume here that $v<v_g$)

$$\begin{aligned} A\to A_+ &= \frac{1}{2\pi(v-v_g(k_*))}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}((v_g(k_*)t-x)\xi+\varkappa_n\xi^n t)}-\mathrm{e}^{\mathrm{i}(vt-x)\xi}}{\mathrm{i}\xi}\,\mathrm{d}\xi\\ &= \frac{1}{2\pi(v-v_g(k_*))}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}(\zeta_n\xi+\varkappa_n\xi^n)}-\mathrm{e}^{-\mathrm{i}\eta t^{-1/n}\xi}}{\mathrm{i}\xi}\,\mathrm{d}\xi\,. \end{aligned} \tag{24}$$

Here

$$\zeta_n = \frac{v_gt-x}{t^{1/n}}\,,\quad \eta = x-vt\,. \tag{25}$$

For $n = 2$ we find

$$A_+ = \frac{1}{2(v - v_g(k_*))}(A_{+1} + A_{+2}),$$
$$A_{+1} = \int_{-\infty}^{\infty} \frac{e^{i\varkappa_2\xi^2}\sin(\zeta_2\xi)}{\pi\xi}\,d\xi, \quad A_{+2} = \int_{-\infty}^{\infty} \frac{\sin(\xi\eta/\sqrt{t})}{\pi\xi}\,d\xi. \tag{26}$$

Next

$$\Re A_{+1} = \text{FresnelC}(z) + \text{FresnelS}(z),$$
$$\Im A_{+1} = \text{sign}(\varkappa_2)(\text{FresnelC}(z) - \text{FresnelS}(z)),$$
$$A_{+2} = 1, \quad z = \frac{\zeta_2}{\sqrt{2\pi|\varkappa_2|}}. \tag{27}$$

With reference to (21), (22) and (24)–(27), we obtain

$$u(t,x) \sim \frac{Q_\eta^{F_\eta}(k_*)}{\Phi^{LF}(i\omega(k_*),k_*)2\omega(k_*)}U(t,x),$$
$$\Re U(t,x) \sim \frac{1}{2(v_g(k_*) - v)}((1 + \Re A_{+1})\sin(\omega(k_*)t - k_*x)$$
$$+ \Im A_{+1}\cos(\omega(k_*)t - k_*x)),$$
$$\Im U(t,x) \sim -\frac{1}{2(v_g(k_*) - v)}((1 + \Re A_{+1})\cos(\omega(k_*)t - k_*x)$$
$$- \Im A_{+1}\sin(\omega(k_*)t - k_*x)). \tag{28}$$

To compare the above explicit asymptotic representation (28) with that obtained by numerically calculations, consider waves in a bending beam excited by an oscillation load. The equation is

$$\frac{d^2u(t,x)}{dt^2} + \frac{d^4u(t,x)}{dx^4} = Q_0\delta(x)\cos(\omega_0 t), \tag{29}$$

where the length and time units are used as the inertia radius, r, of the cross-section area of the beam, and r/c, respectively(c is the sound speed in the beam material). After the Fourier transform on x the equation becomes

$$\frac{d^2u^F(t,k)}{dt^2} + k^4u^F(t,k) = Q_0\cos(t) \quad (\omega_0 = 1). \tag{30}$$

It follows

$$u^F(t,k) = Q_0\frac{\sin(k^2t)}{k^2} * \cos(t) = Q_0\frac{\cos(t) - \cos(k^2t)}{k^4 - 1}. \tag{31}$$

Thus, we have two expressions

$$u_{num} = \frac{Q_0}{\pi}\int_0^\infty \frac{\cos(t) - \cos(k^2t)}{k^4 - 1}\cos(kx)\,dk, \tag{32}$$

and the analytical expression (28) with

$$k_* = \omega_0 = \omega(k_*) = 1, v_g = \varkappa_2 = 1, Q = Q_0, \Phi^{LF}(i\omega(k_*),k_*) = 1, v = 0. \tag{33}$$

Under these conditions

$$u(t,x) = \frac{Q_0}{8}((1 + \Re A_{+1})\sin(t - x) + \Im A_{+1}\cos(t - x)), \quad \Im u(t,x) = 0. \tag{34}$$

The analytical and numerical results for $t = 100$ ($\approx$ 17 periods of oscillations) are shown in Fig. 4 and Fig. 5, respectively.

3.3. Resonant wave

The resonance arises when the load velocity coincides with the wave group velocity, $v = v_g$. In this case, also the higher derivative of the frequency, as in Eq. (23), should be involved in the asymptotic description of the wave. The function $U^F(t,k)$ (21) becomes

$$U^F(t,k) \sim \frac{\exp(iv\xi t)(1 - \exp(i\varkappa_n(k_*)\xi^n t))}{-i\varkappa_n(k_*)\xi^n}e^{i\omega(k_*)t}, \tag{35}$$

and

$$U^F(t,x) \sim A_{res}e^{i(\omega(k_*)t - k_*x)},$$
$$A_{res} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(i\xi(vt - x))(1 - \exp(i\varkappa_n(k_*)\xi^n t))}{-i\varkappa_n(k_*)\xi^n}\,d\xi$$

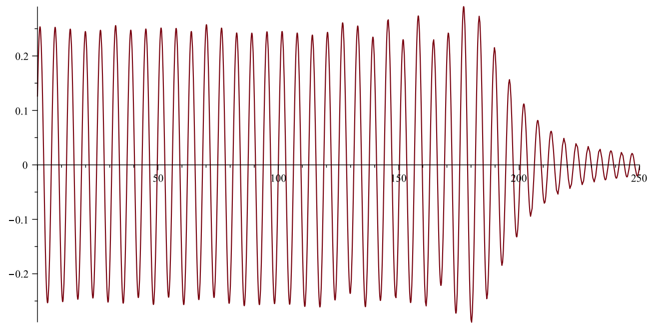

Fig. 4. The wave in a bending beam. Asymptotic representation following from the general expressions (27), (28) with nondimensional parameters $\omega_0 = k_* = \eta_2 = 1, v_g = 2, t = 100$.

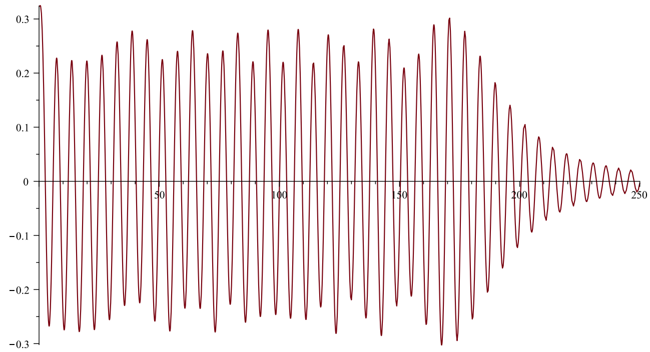

Fig. 5. The same function as in Fig. 4, but the 'exact' one obtained numerically.

$$= \frac{t^{1-1/n}}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}\xi\zeta_n)(1-\exp(\mathrm{i}\varkappa_n(k_*)\xi^n)}{-\mathrm{i}\varkappa_n(k_*)\xi^n} \,\mathrm{d}\xi\,, \quad \zeta_n = \frac{v_g t - x}{t^{1/n}}\,. \tag{36}$$

In particular, for $n = 2$

$$A_{res} = \frac{\sqrt{t}}{\varkappa_2} A^0\,, \quad \Re A^0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos(\xi\zeta)(\sin(\varkappa_2(k_*)\xi^2)}{\varkappa_2(k_*)\xi^2} \,\mathrm{d}\xi$$
$$= z_0(\mathrm{FresnelS}(z) - \mathrm{FresnelC}(z)) + \frac{1}{2\pi}(\cos^2(z_0) + \sin^2(z_0))\,,$$
$$\Im A^0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos(\xi z_0)(1-\cos(\xi^2)}{\xi^2} \,\mathrm{d}\xi\,, \quad z_0 = z\sqrt{\frac{\pi}{2}}\,, \quad z = \frac{x - v_g t}{\sqrt{2\pi\varkappa_2 t}}\,. \tag{37}$$

3.4. The phase shift and the energy flux from the force to the waveguide

For the load area, $x = vt$ and in its narrow neighborhood, the corresponding growing exponent disappears, and the asymptotic analysis does not hold anymore. The wave at the load can be obtained by direct calculations based on the original equation. In particular, in the case of an unmoving oscillating force acting on the beam, (29), (32), we have

$$u(t,0) = u_{num}(t,0) = \frac{Q_0}{\pi} \int_0^{\infty} \frac{\cos(t) - \cos(k^2 t)}{k^4 - 1} \,\mathrm{d}k\,, \tag{38}$$

The rate of the energy flux from the concentrated force to the beam,

$$N(t) = \frac{Q_0}{\pi} \int_0^{\infty} \frac{\sin(t) - k^2\sin(k^2 t)}{1 - k^4} \cos(t)\,\mathrm{d}k\,, \tag{39}$$

is plotted in Fig. 6.

One can see that the energy flux rate oscillates from - 0.05 to 0.3. The negative flux arises due to a phase shift between the sinusoids of the load and the beam, Fig. 7.

The calculations show that the energy flux rate (39) coincides with that in the propagating waves, as it should be.

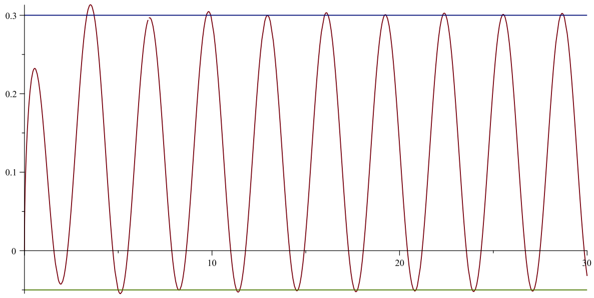

Fig. 6. The rate of the energy flux from the concentrated force to the beam.

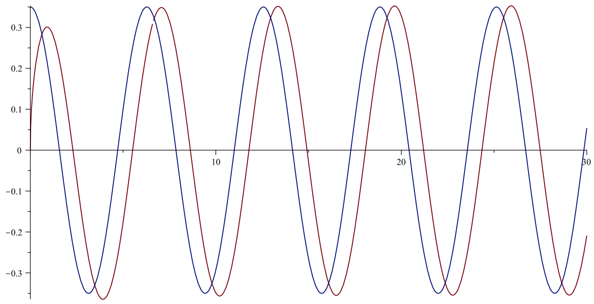

Fig. 7. The phase shift between the sinusoids of the load, $\sin(t)$ (navy), and the beam oscillation speed (brown).

4. Localized dynamic obstacle

Consider a one-dimensional waveguide with a dynamic structure attached at a point. Without loss of generality we can take the latter as the coordinate origin, $x = 0$. Let a sinusoidal wave, excited fare from the zero point, at $x \ll 0$, and propagating along the structure-free waveguide, manifests itself at $x = 0$ by oscillations as

$$u_0(t, 0) = A_0 \cos(\omega_0 t)\,, \tag{40}$$

and let $u_1(t)$ be the additional oscillations of the waveguide caused by the structure response, that is by the force $P(t)$ acting from the structure on the waveguide. We describe the dynamics at this point in terms of the Green's function for waveguide, $G(t)$, and the fundamental solutions for the structure, $R(t)$, such that

$$u_1(t) = G(t) * P(t)\,, \quad P(t) = -R(t) * u(t)\,, \quad u(t) = u_0(t) + u_1(t)\,. \tag{41}$$

We assume (for simplicity) that the wave u_0 (40) comes to the localized structure at $t = 0$, while the latter and the waveguide at $x \ge 0$ are at rest at $t < 0$. Using the Laplace transform we rewrite the above relations as follows

$$u_0^L(s) = \frac{A_0}{s - \mathrm{i}\omega_0}\,, \quad u_1^L(s) = G^L(s)P^L(s)\,, \quad P^L(s) = -R^L(s)u^L(s)\,. \tag{42}$$

The solution follows from the last two relations

$$\begin{aligned} P^L(s) &= -\frac{R^L(s)}{1 + R^L(s)G^L(s)}\,\frac{A_0 s}{s^2 + \omega_0^2}\,, \\ u_1^L(s) &= -\frac{R^L(s)G^L(s)}{1 + R^L(s)G^L(s)}\,\frac{A_0 s}{s^2 + \omega_0^2}\,. \end{aligned} \tag{43}$$

While $R^L(s)$ is defined separately by the structure, the Green function $G^L(s)$ follows from the waveguide equation. We express the latter as in the previous sections as

$$\begin{aligned} &\mathcal{L}^{LF}(s, k)u^{LF}(s, k) = Q^{LF}(s, k) - R^L(s)u^L(s, 0)\,, \\ &u^L(s, 0) = u_0^{LF} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{P^L(s)u^L(s, 0)}{\mathcal{L}^{LF}(s, k)}\,\mathrm{d}k\,. \end{aligned} \tag{44}$$

(Remain that $Q^{LF}(s, k)$ is the Laplace–Fourier transform of the load, $Q(t, x)$.) It follows that

$$G^L(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\mathcal{L}^{LF}(s, k)}\,\mathrm{d}k\,. \tag{45}$$

The wave $u_1(t,x)$ at $x<0$ and $x>0$ is now defined by the Eq. (45).

For example, in the simplest case of a string with an oscillator,

$$R^L(s) = \frac{\varkappa s^2}{s^2+\omega_{osc}^2}\,,$$
$$G^L(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{\varrho s^2+Tk^2} = \frac{1}{2\sqrt{\varrho T s^2}}\,, \tag{46}$$

where $\omega_{osc} = \sqrt{\varkappa/m}$, and ϱ, T, $\varkappa$ and m are the string's mass per unit length and the tension force, the spring stiffness and mass of the oscillator.

Finally, from the expressions (43) and (46) we find

$$P(t) \sim \frac{2\varkappa\omega_0^2\sqrt{\varrho T}(2\sqrt{\varrho T}(\omega_0^2-\omega_{osc}^2)\cos(\omega_0 t) - \varkappa\omega_0\sin(\omega_0 t))}{4\varrho T(\omega_0^2-\omega_{osc}^2)^2+\varkappa^2\omega_0^2}A_0\,,$$
$$u_1(t) \sim \frac{\varkappa\omega_0(2\sqrt{\varrho T}(\omega_0^2-\omega_{osc}^2)\sin(\omega_0 t) + \varkappa\omega_0\cos(\omega_0 t))}{4\varrho T(\omega_0^2-\omega_{osc}^2)^2+\varkappa^2\omega_0^2}A_0\,,$$
$$\frac{\mathrm{d}u_1(t)}{\mathrm{d}t} = \frac{1}{2\sqrt{\varrho T}}P(t)\,,\quad u_1(t,x) = \frac{1}{2\sqrt{\varrho T}}H(ct-|x|)*P(t)\,,\quad c=\sqrt{\frac{T}{\varrho}}\,. \tag{47}$$

Thus, the obstacle induces symmetrically the reflected and refracted waves (as it should). What is defined above, is the wave amplitude, which depends on the waveguide and the oscillator parameters and the incident wave frequency.

5. Conclusion

The paper subject, the structured waveguide under the moving-oscillating load, is considered. In this way, we were trying to avoid calling the structure "microstructure" since the formulation and results do not depend on the structure scale (we discussed this topic in more detail in the first subsection of the Introduction).

Specifically, we studied the role of uniformly distributed and localized dynamic structures (the oscillators) attached to the simplest waveguides: the string and bending beam. Also, we considered the dynamic elastic support as an attached structure. The latter leads to a more reach wave configuration.

While considering the energy flux from the oscillating load to the waveguide, the phase shift was disclosed, which essentially decreases the flux under the same other conditions.

In this brief paper, some points of the general theme, the waves under moving-oscillating loads, were discussed. However, of course, this broad topic remains open. The general two-three dimensional forced waves area remains actual, including the phase shift problem. Note that the wave localization at a line of oscillators in a 3D space was discussed in Mishuris et al. (2020).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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