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Feeding and dissipative waves in fracture and phase transition
II. Phase-transition waves

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Abstract

Discrete and homogeneous models of a structured material are considered to resolve difficulties in the analysis of dynamic phase transition. The discrete model is a chain consisting of particles connected by massless bonds, while the continuous model is represented by a partial differential equation with higher than the second order of coordinate derivatives. The macrolevel constitutive law is represented by a bi-linear stress–strain relation, such that the transition from the first, stiffer phase to the second one is irreversible. Solutions of two types, macrolevel-associated and microlevel, are derived. The first type of solution is characterized by a macrolevel feeding wave (the wave delivering energy to the phase-transition front is of a zero wave number), while the microlevel solutions correspond to a nonzero feeding wave number. Subsonic, intersonic and supersonic phase-transition waves are described. For the homogeneous model it is shown that the contradiction between the limiting stress and energy criteria, inherent for the macrolevel formulation of the problem, is eliminated if and only if the phase transition does not concern the highest-order modulus. Total structure- and speed-dependent dissipation, as the energy carried by microlevel waves away from the phase-transition front, as well as parameters of each dissipative wave are determined. For the fourth-order partial differential equation, the existence of the Maxwell type, dissipation-free, subsonic phase-transition wave is shown. In this case, the microstructure plays the role of a catalyst. Common and distinctive properties of the discrete and homogeneous models are discussed.

Keywords: A. Dynamics; Phase transition; Discrete chain; C. Integral transforms

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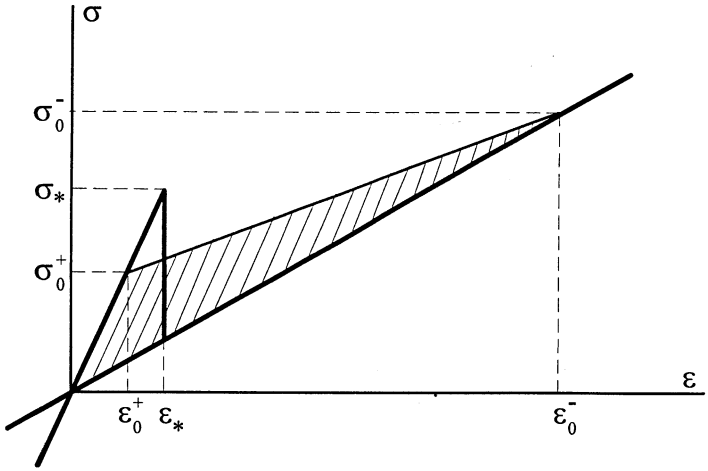

Fig. 1. The bi-linear stress–strain diagram. The lines pass through the origin and the point ε_*, σ_* (the first phase) and the point ε_0^-, σ_0^- (the second phase). The line ε_0^+, $\sigma_0^+ - \varepsilon_0^-$, σ_0^- represents an example of the macrolevel description of the phase transition. For a given point $\varepsilon_0^-, \sigma_0^-$ the starting point $\varepsilon_0^+, \sigma_0^+$ cannot be defined using only macrolevel considerations.

1. Introduction

Consider a two-phase material with the stress–strain diagram shown in Fig. 1. Macrolevel considerations provide no unique answer regarding the transition path from the stiffer branch of this diagram to the softer one. This indefiniteness was reflected in different formulations of the problem of wave propagation in such a material (Galin and Cherepanov, 1966; Grigoryan, 1967; Slepyan, 1968, 1977; Slepyan and Troyankina, 1969). Uniqueness can be achieved in the framework of a structured material model where the total structure- and speed-dependent dissipation as the *wave resistance* to the phase transition can be determined. To describe the related phenomena two types of such models have been studied: a higher-order-derivative (HOD) formulation for an elastic continuum (see Truskinovsky, 1994, 1997; Ngan and Truskinovsky, 1999, and the references therein), and a discrete chain model (Slepyan and Troyankina, 1984, 1988; Slepyan, 2000; Balk et al., 2001a, b, an analysis for a discrete bi-stable chain in statics is represented in Puglisi and Truskinovsky, 2000).

Along with the dissipation as the energy transfers from the macrolevel to the microlevel, the structured models permit *microlevel solutions* which have no analogue on the macrolevel. In this connection, we classify the solutions according to the type of *feeding wave* delivering energy to the propagating phase-transition front.

We call the *macrolevel-associated solution* one that corresponds to a macrolevel feeding wave, that is the wave of a zero wave number. This solution differs from the *macrolevel solution* by the existence of propagating or/and exponentially decreasing microlevel waves of a nonzero wave number. The propagating microlevel waves, carrying energy away from the front, are called here the *dissipative waves*. The dissipative wave is placed ahead (behind) the phase-transition front if its group velocity is higher (lower) than the phase velocity. The macrolevel-associated solution can be considered as an improved macrolevel solution. In contrast, the *microlevel solution* is characterized by a microlevel feeding wave of a nonzero wave number. Such wave is placed ahead (behind) the phase-transition front if its group velocity is lower (higher) than the phase velocity.

Both the discrete chain and the HOD model are considered in the present paper. A comprehensive analysis of the macrolevel-associated and microlevel solutions is represented. The paper is organized as follows.

To show insufficiency of the purely macrolevel formulation of the problem we start with the macrolevel solution. This formulation can be completed with a phase-transition criterion; however, the criterion as a limiting stress in the first phase (see Fig. 1) is in contradiction with the energy criterion following from the same diagram. The only way out of this difficulty is to consider the influence of the microstructure.

Next, we consider a discrete chain consisting of particles connected by massless bonds with the force–strain diagram as in Fig. 1. A steady-state phase-transition wave is considered, that is the strain of any bond is assumed to be the same function of time but with a shift corresponding to a constant time-interval between the phase transition of the neighboring bonds. A general solution is derived using the continuous Fourier transform of this function and the Wiener–Hopf technique.

In terms of the Fourier transform, a long-wave approximation of the solution coincides with that for a homogeneous body, while nonzero real singular points correspond to the microlevel feeding and dissipative waves. In this way, both the macrolevel-associated and microlevel solutions with the corresponding dissipative waves are analyzed. The macrolevel-associated phase-transition wave velocity is bounded by the sound velocity, that is the macrolevel wave velocity, in the second (softer) phase. So, only a subsonic phase-transition wave can exist in this case, while an intersonic wave can exist in the case of a microlevel solution. Note that we use terms *subsonic*, *intersonic* and *supersonic* for the speed lower than the sound velocity in the second phase, between the sound velocities in the first and the second phases and higher than the sound velocity in the first phase, respectively.

Then we return to the macrolevel solution uniqueness of which is achieved with the expression for the speed-dependent total dissipation obtained as a result of the macrolevel-associated formulation. Two examples as the phase transition under an impact and a spontaneous phase transition are considered.

Possible configurations of the feeding and dissipative waves are defined by dispersion dependences for the uniform wave guides corresponding to one and another phase. Such dependences, especially for a fast phase-transition wave, can be approximated by means of higher-order derivatives introduced in the two-phase macrolevel wave equation. It is of interest to compare these, discrete and HOD, approaches. In this connection, the latter is considered in more detail.

We consider a model where the strain energy is represented as a quadratic form including the first and the higher-order derivatives, while the corresponding moduli can be different in the different phases. First, it is shown that the contradiction between the limiting stress and energy criteria is eliminated if and only if the phase transition does not concern the highest-order modulus. This allows one to satisfy interface conditions concerning continuity and discontinuity of generalized strains. The use of the same technique as for the chain leads to a solution in which these conditions are satisfied automatically. A fourth-order partial differential equation for the two-phase continuum is considered in more detail. Macrolevel-associated subsonic and microlevel intersonic and supersonic solutions are derived. In particular, the subsonic solution represents the Maxwell type, dissipation-free phase transition where only exponentially decreasing microlevel waves exist but not the propagating ones. In this case, the microlevel plays the role of a catalyst. It helps to overcome the energy barrier spending no energy.

The results show that the HOD model possesses both the macrolevel-associated and microlevel types of the solutions as well as the discrete chain. It can be expected that an increase in the order of the equation can lead to the conversion of the results for a given nonzero speed. However, there exist some unavoidable distinctions between these models. In particular, in the HOD model, in contrast to the discrete chain, the manifestation of the dynamic amplification factor with its influence on the phase-transition wave speed (Slepyan, 2000) cannot be revealed.

The present paper is a continuation of the previous one (Slepyan, 2001) mainly devoted to the feeding and dissipative waves in dynamic fracture of a square-cell lattice. Although the present paper as part II is independent of it, acquaintance with the first part is desirable for better understanding of the common phenomena and analytical technique. Note that some of the notation used in these two parts is different.

2. Macrolevel solution

At the beginning, we consider a homogeneous, nonstructured, two-phase material. As shown below, equations and other relations following from this model are not sufficient for the determination of a unique solution. The missing condition can be represented as a phase-transition criterion; however, there are difficulties in the formulation of such a criterion in the framework of this model. The situation is similar to that for fracture, but in the case of the phase-transition wave it is more evident: it is clearly seen that such a wave cannot exist without excitation of the microlevel.

Consider a plane wave propagating in a two-phase homogeneous material with the following stress–strain relation (Fig. 1):

$$\sigma = E\varepsilon \quad \text{(the first phase)},$$

$$\sigma = \gamma^2 E\varepsilon \quad \text{(the second phase)}, \tag{1}$$

where E is the elastic modulus in the first phase, $0<\gamma<1$ and the first-to-the-second phase transition occurs when the stress, σ, first reaches the critical value

$$\sigma = \sigma_*. \tag{2}$$

We assume that the phase-transition front, $x = X(t)$, propagates with the speed $\dot{X} = v > 0$ [in general, $v = v(t)$], while the material is in the first (the second) phase ahead (behind) the front. In this model, the displacement, $u(x,t)$, satisfies the one-dimensional linear wave equation (different for $x>X$ and $x<X$):

$$u'' - \frac{1}{c^2}\ddot{u} = 0 \quad (x>X),$$

$$u'' - \frac{1}{\gamma^2 c^2}\ddot{u} = 0 \quad (x<X), \tag{3}$$

where primes and dots denote derivatives with respect to the coordinate, x, and time, t, respectively, $c=\sqrt{E/\rho}$ is the sound velocity in the first phase and ρ is the material density.

A general solution to this problem can be expressed in terms of four arbitrary functions as

$$u = u_1^+(x-ct) + u_2^+(x+ct) \quad (x \geq X),$$

$$u = u_1^-(x-\gamma ct) + u_2^-(x+\gamma ct) \quad (x \leq X). \tag{4}$$

Here and below the superscript '+' ('−') is used for functions with the support at $x>X$ $(x<X)$. If the argument is not shown explicitly this means that $x=X+0$ $(x=X-0)$, for example, $(u')^+ = \lim u'(x,t)$ $(x \to X+0)$. For the determination of the functions in (4) and the speed, v, one has to introduce conditions behind and ahead the front, for example, $[u_1'(x-ct)+u_2'(x+ct)]^- = \text{const}$, $u_2^+(x+ct)=0$. Further, one can use the matter and momentum conservation laws which for small strain lead to the following relations:

$$v = c\sqrt{\frac{\sigma^- - \sigma^+}{\sigma^-/\gamma^2 - \sigma^+}},$$

$$\Delta\dot{u} = \dot{u}^- - \dot{u}^+ = -\frac{\sigma^- - \sigma^+}{\rho v}. \tag{5}$$

A phase-transition criterion can play the role of the last condition; however, the use of the criterion (2) as

$$\sigma^+ = \sigma_* \tag{6}$$

in the framework of the homogeneous material model is questionable.

To see the contradiction, consider the energy release rate, G, at the moving phase-transition front. In this connection, we note that it is independent of a rigid-body velocity, and we can assume the particle velocity ahead the front to be zero. One has

$$G = \frac{(\sigma^+)^2}{2E} - \frac{(\sigma^-)^2}{2E\gamma^2} - \frac{\rho(\Delta\dot{u})^2}{2} - \frac{\sigma^-\Delta\dot{u}}{v}, \tag{7}$$

where the first and the second terms represent the strain energy per unit length ahead and behind the front, respectively, the third term is the kinetic energy per unit length behind the front and the last term is the energy flux as the work of the internal force behind the front during the period $1/v$. It follows that

$$G = \frac{1-\gamma^2}{2}E\varepsilon^+\varepsilon^-, \tag{8}$$

where $\varepsilon = u'$ is strain. In Fig. 1 the corresponding area is shaded.

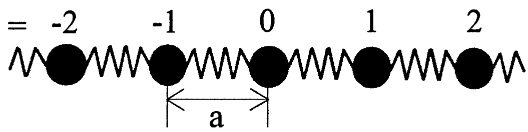

Fig. 2. The discrete chain.

At the same time, the energy barrier, G_0, between the phases (see Fig. 1) is

$$G_0 = \frac{1-\gamma^2}{2} E\varepsilon_*^2 \quad (\varepsilon_* = \frac{\sigma_*}{E}). \tag{9}$$

The difference

$$G - G_0 = \frac{1-\gamma^2}{2} E \quad (\varepsilon^- \varepsilon^+ - \varepsilon_*^2), \tag{10}$$

is the lost energy which is positive if criterion (6) is used. Indeed, as follows from Eqs. (5) and (6) for $V = v/c < \gamma < 1$

$$\varepsilon^- \varepsilon^+ - \varepsilon_*^2 = \frac{1-\gamma^2}{\gamma^2 - V^2} \varepsilon_*^2 > 0. \tag{11}$$

We have no way but to conclude that this energy is spent on the excitation of the microlevel. Thus, in the formulation of the criterion, additional stress σ_M contributed from the microlevel should be taken into account, that is criterion (6) should be rewritten as

$$\sigma^+ + \sigma_M = \sigma_*. \tag{12}$$

However, in the framework of the homogeneous material model, it is impossible to determine the role of the microlevel. Consideration of a structured material model is a way out of this difficulty. Such a model can allow to find a unique macrolevel solution, the corresponding structure-associated dissipation and, in addition, microlevel solutions which have no analogue on the macrolevel. A discrete chain considered below is a simplest example of such a structured material model. In the following, the sum in Eq. (12) is denoted by σ^+, while the macrolevel part of it is denoted by σ_0^+; this also concerns other parameters of the wave, such as the particle velocity and strain.

3. Discrete chain

3.1. Formulation

Consider a chain consisting of point particles of mass M, connected by massless elastic bonds each of the length a, Fig. 2. The force–elongation relation for any intact bond is

$$p_m = K\Lambda_m, \qquad \Lambda_m = u_{m+1} - u_m, \tag{13}$$

where p_m, Λ_m and K are the force, the bond elongation and its stiffness, respectively; the subscript m corresponds to the bond connected the particles numbered by m and

$m+1$. For a long wave the chain corresponds to an elastic rod with the elastic modulus $E=aK/A$ and density $\rho=M/Aa$, where A is the cross-sectional area.

At the moment when the bond elongation, Λ_m, first reaches the critical value

$$\Lambda_m=\Lambda_*, \tag{14}$$

the stiffness K drops and becomes $\gamma^2 K$:

$$p_m=\gamma^2 K\Lambda_m, \quad 0<\gamma<1. \tag{15}$$

Equalities (13) and (15) reflect the two possible phases of the bond state. Note that this transition is assumed to be irreversible. Dynamics of a reversible two-phase chain is considered in Balk et al. (2001a, b).

A phase-transition wave propagating with a constant speed, $v>0$ is studied, that is the time-interval between the phase transition of neighboring bonds, a/v, is assumed to be a constant. At any time t there exists a particle, let its number be m, such that $vt\le am$ and $vt>a(m-1)$. It is assumed that the chain is in the first (the second) phase ahead (behind) this particle. Note that the speed of a long wave ahead the phase-transition front is $c=\sqrt{E/\rho}=\sqrt{Ka^2/M}$, while the speed behind the front is γc. The particle velocities and the bond elongation are assumed to be finite at infinity. Under these conditions the variables can be represented as functions of $\eta=(x-vt)/a=m-vt/a$. Note that such representation does not concern the particle displacement which can also depend on $x\pm ct$ ($\eta>0$) or $x\pm\gamma ct$ ($\eta<0$).

3.2. Derivation of the governing equation

The dynamic equation for a particle is

$$M\frac{\mathrm{d}^2u_m}{\mathrm{d}t^2}=K\{\Lambda_m[1-(1-\gamma^2)H(-\eta)]-\Lambda_{m-1}[1-(1-\gamma^2)H(1-\eta)]\}+q_m^1,$$

$$\Lambda_m=u_{m+1}-u_m=\Lambda(\eta)=u(\eta+1)-u(\eta), \tag{16}$$

where $q_m^1=q^1(\eta)$ is an external force introduced for convenience in an initial stage of the considerations. Comparing the equations for particles m and $m+1$ one obtains the following equation for $\Lambda(\eta)$:

$$V^2\frac{\mathrm{d}^2\Lambda}{\mathrm{d}\eta^2}=\Lambda(\eta+1)[1-(1-\gamma^2)H(-\eta-1)]+\Lambda(\eta-1)[1-(1-\gamma^2)H(-\eta+1)]$$
$$-2\Lambda(\eta)[1-(1-\gamma^2)H(-\eta)]+q(\eta), \tag{17}$$

where $q(\eta)=[q^1(\eta+1)-q^1(\eta)]/K,\ V=v/c$.

Under the causality principle (see Slepyan, 2001) the Fourier transform leads to the equation

$$h(k)\Lambda_+ + g(k)\Lambda_- = q^{\mathrm{F}}(k)=q_+ + q_-,$$

$$h(k)=2(1-\cos k)+(0+\mathrm{i}kV)^2, \quad g(k)=2\gamma^2(1-\cos k)+(0+\mathrm{i}kV)^2, \tag{18}$$

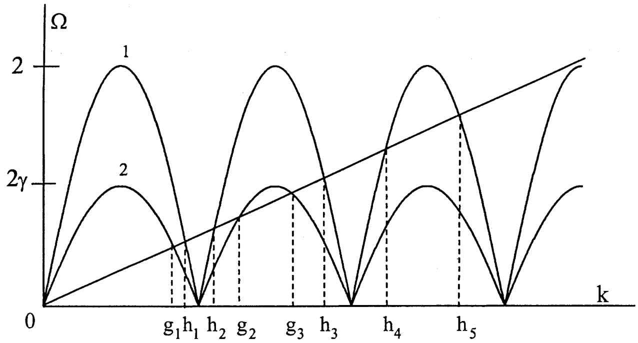

Fig. 3. The dispersion relations: the first (1) and the second (2) phases. The inclined line corresponds to a value of the speed $V = \Omega/k$. Zeros of the functions $h(k)$ and $g(k)$ are shown.

where the subscript '+' ('−') means the right (left) side Fourier transform and the notation $(0 \pm \mathrm{i}kV)$ means the limit as

$$0 \pm \mathrm{i}kV = \lim_{s \to +0} s \pm \mathrm{i}kV. \tag{19}$$

We thus obtain the governing equation as

$$L(k)\Lambda_+ + \Lambda_- = \frac{q^{\mathrm{F}}(k)}{g(k)}, \qquad L(k) = \frac{h(k)}{g(k)}. \tag{20}$$

Functions $h(k)$, $g(k)$ and $L(k)$ have the following asymptotes:

$$\begin{aligned}
&h(k) \sim (1 - V^2)(0 + \mathrm{i}kV)(0 - \mathrm{i}kV) \quad (k \to 0, V < 1),\\
&h(k) \sim (V^2 - 1)(0 + \mathrm{i}kV)^2 \quad (k \to 0, V > 1),\\
&g(k) \sim (\gamma^2 - V^2)(0 + \mathrm{i}kV)(0 - \mathrm{i}kV) \quad (k \to 0, V < \gamma),\\
&g(k) \sim (V^2 - \gamma^2)(0 + \mathrm{i}kV)^2 \quad (k \to 0, V > \gamma),\\
&L(0) = \frac{1 - V^2}{\gamma^2 - V^2} \quad (V < \gamma, V > 1),\\
&L(k) \sim \frac{(1 - V^2)(0 - \mathrm{i}k)}{(V^2 - \gamma^2)(0 + \mathrm{i}k)} \quad (k \to 0, \gamma < V < 1),\\
&L(\pm\infty) = 1.
\end{aligned} \tag{21}$$

The corresponding dispersion relations,

$$\begin{aligned}
&\Omega(k) \equiv kV = \Omega_h = 2|\sin k/2| \quad [h(k) = 0],\\
&\Omega(k) = \Omega_{\mathrm{g}} = 2\gamma|\sin k/2| \quad [g(k) = 0],
\end{aligned} \tag{22}$$

are represented in Fig. 3.

3.3. Zero points of the functions $h(k)$ and $g(k)$

Both the feeding and dissipative waves are associated with zero points of the functions $h(k)$ and $g(k)$ and the determination of these points is a necessary step in the study of the problem. For $V<1$, in addition to the point $k=0$, the function $h(k)$ has one, three or more pairs of simple zeros at $k\neq 0$: $k=\pm h_1, \pm h_2, \ldots, \pm h_{2l+1}$, where number l decreases as V increases. In particular

$$l=0 \quad \text{for } V>V_0\approx 0.2172. \tag{23}$$

There is no such points for $V\geq 1$. The zeros h_v $(v=1,2,\ldots)$ form an increasing sequence $h_1<h_2<\cdots<h_{2l+1}$. Under certain values of the speed, two neighboring zero points can unite; in the following, however, simple zeros are assumed unless otherwise noted. The function $h(k)$ has the following representation in a vicinity of a zero point:

$$h\sim \text{const}[0+\mathrm{i}(k-h_{2v-1})] \quad (k\to h_{2v-1}),$$

$$h\sim \text{const}[0-\mathrm{i}(k-h_{2v})] \quad (k\to h_{2v}). \tag{24}$$

These representations are in agreement with the corresponding relations between the group, v_g, and phase, v, velocities of the waves corresponding to the dispersive relations (22)

$$V_g<V \quad (k=h_{2v-1}), \qquad V_g>V \quad (k=h_{2v}), \tag{25}$$

where

$$V_g=\frac{v_g}{c}=\frac{\mathrm{d}\Omega}{\mathrm{d}k}, \qquad V=\frac{v}{c}=\frac{\Omega}{k}, \qquad \Omega^2=2(1-\cos k). \tag{26}$$

The function $\operatorname{Arg} h(k)$ is a piecewise constant. In accordance with the representations in Eq. (24), while the increasing variable k passes a zero point of h, $\operatorname{Arg} h(k)$ exhibits a jump as

$$\Delta \operatorname{Arg} h=\pi \quad (k=h_{2v-1}), \qquad \Delta \operatorname{Arg} h=-\pi \quad (k=h_{2v}). \tag{27}$$

We can put $\operatorname{Arg} h(0)=0$. Then for $V<1$

$$\operatorname{Arg} h=0 \ \ [h_{2v-2}<k<h_{2v-1}, (h_0=0)], \qquad \operatorname{Arg} h=\pi \quad (h_{2v-1}<k<h_{2v}),$$

$$\operatorname{Arg} h=\pi \quad (k>h_{2l+1}), \qquad \operatorname{Arg} h(-k)=-\operatorname{Arg} h(k), \tag{28}$$

while for $V\geq 1$

$$\operatorname{Arg} h=\pi \quad (k>0), \qquad \operatorname{Arg} h=-\pi \quad (k<0). \tag{29}$$

The function $g(k)$ can be represented as

$$g(k)=g^0(k,V)=\gamma^2 h^0(k,V/\gamma) \quad [h^0(k,V)=h(k)]. \tag{30}$$

This allows one to determine zeros of $g(k)$ in terms of the zeros of $h(k)$. In particular, for $V<\gamma$, in addition to the point $k=0$, the function $g(k)$ has one, three or more couples of simple zeros at $k\neq 0$: $k=\pm g_1, \pm g_2, \ldots, \pm g_{2d+1}$, where number d decreases as V increases. In other respects, similar statements concerning the zeros of $g(k)$, $\operatorname{Arg} g(k)$ and the phase and group velocities are valid as for the function $h(k)$.

3.4. Factorization

One now can see that $\operatorname{Arg} L(k)=0$ for $k^2<g_1^2$ and $k^2>h_{2l+1}^2$, and it is zero or negative in the segment $g_1<k<h_{2l+1}$. Further,

$$\Re L(-k)=\Re L(k), \qquad \Im L(-k)=-\Im L(k) \tag{31}$$

and

$$\operatorname{Ind} L(k)=\frac{1}{2\pi}[\operatorname{Arg} L(+\infty)-\operatorname{Arg} L(-\infty)]=0,$$

$$L(\pm\infty)=1, \quad |\ln L(-k)|=|\ln L(k)|, \quad \operatorname{Arg} L(-k)=-\operatorname{Arg} L(k). \tag{32}$$

This allows the following factorization valid for $V<\gamma$ and $V>1$:

$$L(k)=L_+(k)L_-(k) \tag{33}$$

with

$$L_\pm(k)=\exp\left(\pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln L(\xi)}{\xi-k}\,\mathrm{d}\xi\right), \tag{34}$$

where $\Im k>0$ for the functions marked by the subscript '+' and $\Im k<0$ for the functions marked by the subscript '−'. In this factorization, L_+ has neither singular nor zero points in the upper half-plane k, while L_- has no such points in the lower half-plane [the half-planes include the real axis if in Eq. (19) $\Re s>0$]. This means that $1/L_+$ and $1/L_-$ are regular in the corresponding half-planes as well as L_+ and L_-.

As follows from definition (34) equalities (31) are valid for the functions $L_\pm(k)$ as well as for $L(k)$. The function $L_+(k)$ $[L_-(k)]$ incorporates the zeros of $h(k)$ and $g(k)$ with $V_\mathrm{g}>V[V_\mathrm{g}<V]$ as its zero or singular points, such that

$$L_+(h_{2v})=0, \quad L_+(g_{2v})=\infty,$$

$$L_-(h_{2v-1})=0, \qquad L_-(g_{2v-1})=\infty. \tag{35}$$

The functions $L_\pm(k)$ have the following limits:

$$L_\pm(k)\to 1 \ \ (k\to\pm\,\mathrm{i}\infty), \qquad L_\pm(0)=\sqrt{\frac{1-V^2}{\gamma^2-V^2}}\mathscr{R}^{\pm 1} \tag{36}$$

with

$$\mathscr{R}=\exp\left(\frac{1}{\pi}\int_0^{\infty}\frac{\operatorname{Arg} L(\xi)}{\xi}\,\mathrm{d}\xi\right). \tag{37}$$

Note that for $V>1$, $\operatorname{Arg} L(k)\equiv 0$ and $\mathscr{R}=1$.

Proceeding from Eq. (28) and the corresponding relations for $g(k)$ which, in common, describe piecewise constant $\operatorname{Arg} L(k)$, we can also represent $\mathscr{R}$ as

$$\mathscr{R}=\prod_{v=1}^{l} h_{2v}\prod_{v=1}^{d+1} g_{2v-1}\left(\prod_{v=1}^{l+1} h_{2v-1}\prod_{v=1}^{d} g_{2v}\right)^{-1}\left(\prod_{v=1}^{0}=1\right). \tag{38}$$

In the following, another form of factorization (33) will also be used. We represent the function $L(k)$ as

$$L(k)=\frac{H_+(k)H_-(k)}{G_+(k)G_-(k)}S(k), \tag{39}$$

where for $V<\gamma$

$$H_+(k)=\prod_{v=1}^{l}\left[1+\left(\frac{h_{2v}}{0-\mathrm{i}k}\right)^2\right],\qquad H_-(k)=\prod_{v=1}^{l+1}\left[1+\left(\frac{h_{2v-1}}{0+\mathrm{i}k}\right)^2\right],$$

$$G_+(k)=\prod_{v=1}^{d}\left[1+\left(\frac{g_{2v}}{0-\mathrm{i}k}\right)^2\right],\qquad G_-(k)=\prod_{v=1}^{d+1}\left[1+\left(\frac{g_{2v-1}}{0+\mathrm{i}k}\right)^2\right] \tag{40}$$

for $\gamma\le V<1$

$$H_+(k)=\prod_{v=1}^{l}\left[1+\left(\frac{h_{2v}}{0-\mathrm{i}k}\right)^2\right],\qquad H_-(k)=\prod_{v=1}^{l+1}\left[1+\left(\frac{h_{2v-1}}{0+\mathrm{i}k}\right)^2\right]$$

$$G_+=G_-=1 \tag{41}$$

and for $V\ge 1$

$$H_+=H_-=G_+=G_-=1. \tag{42}$$

Under this definition of the function $S(k)$, it satisfies the conditions listed in Appendix 1 (A.1). This allows one to factorize this function using the Cauchy-type integral (34) with the result

$$S(k)=S_+(k)S_-(k),\qquad S_+(k)=\overline{S_-(\bar{k})} \tag{43}$$

and on the real k-axis

$$S_+(k)=\sqrt{S(k)}\mathrm{e}^{\mathrm{i}\vartheta},\qquad S_-(k)=\sqrt{S(k)}\mathrm{e}^{-\mathrm{i}\vartheta}, \tag{44}$$

where $\vartheta(k)$ is a real function of k [$\vartheta(0)=0$ if $S(0)$ is a nonzero constant]. Thus

$$L_+(k)=\frac{H_+}{G_+}S_+,\qquad L_-(k)=\frac{H_-}{G_-}S_- \tag{45}$$

with

$$S_+(\mathrm{i}\infty)=S_-(-\mathrm{i}\infty)=L_+(\mathrm{i}\infty)=L_-(-\mathrm{i}\infty)=1. \tag{46}$$

It can be seen that representation (45) defines the same asymptotes for $L_\pm(k)$ for $k\to 0$ ($V<\gamma$) as in Eqs. (36)–(38). This type of factorization can be used for the range $\gamma<V<1$ as well as for the determination of waves of a nonzero wave number.

We now can determine $L_\pm$ asymptotes ($k\to 0$) for the intersonic speed, $\gamma<V<1$. Using Eqs. (21) and (41) (also see Appendix 1) one can find

$$S\sim\frac{1-V^2}{V^2-\gamma^2}\left[\prod_{v=1}^{2l+1}h_v^2\right]^{-1}\qquad[(0-\mathrm{i}k)(0+\mathrm{i}k)]^{2l+1},$$

$$S_\pm\sim\sqrt{\frac{1-V^2}{V^2-\gamma^2}}\left[\prod_{v=1}^{2l+1}h_v\right]^{-1}\qquad(0\mp\mathrm{i}k)^{2l+1}\tag{47}$$

and

$$L_+\sim\sqrt{\frac{1-V^2}{V^2-\gamma^2}}\prod_{v=1}^{l}h_{2v}\left[\prod_{v=1}^{l+1}h_{2v-1}\right]^{-1}\qquad(0-\mathrm{i}k),$$

$$L_-\sim\sqrt{\frac{1-V^2}{V^2-\gamma^2}}\prod_{v=1}^{l+1}h_{2v-1}\left[\prod_{v=1}^{l}h_{2v}\right]^{-1}\qquad(0+\mathrm{i}k)^{-1}.\tag{48}$$

3.5. General homogeneous solution

The governing equation (20) can now be expressed in the following form:

$$L_+\Lambda_+ +\frac{\Lambda_-}{L_-}=\Phi,\qquad\Phi=\frac{q^{\mathrm{F}}(k)}{g(k)L_-}.\tag{49}$$

In this equation, $q^{\mathrm{F}}(k)$ is considered as the Fourier transform of a given external force, and for the determination of two unknowns, Λ_+ and Λ_-, only one step remains: to represent the right-hand side of this equation as a sum of terms which can be marked by '+' and '−' separately. In the following, we consider homogeneous solutions which correspond to $q(\eta)=0$. However, Φ must be nonzero; otherwise no nontrivial solution corresponds to Eq. (49). To resolve this conflicting problem we have to depart from the causality principle regarding the product $g(k)L_-$, thus allowing an energy flux from infinity. We come to the equation for Φ:

$$q^{\mathrm{F}}(k)=g(k)L_-\Phi=0.\tag{50}$$

Nontrivial solutions of this equation correspond to zero points of the coefficient $g(k)L_-$, namely, $k=0$, $k=\pm h_{2v-1}$ and $k=\pm g_{2v}$. In a vicinity of such a point

$$g(k)L_-\sim\text{const }k^2\quad(k\to 0, V<\gamma\text{ or }V>1),$$

$$g(k)L_-\sim\text{const }k\quad(k\to 0,\gamma<V<1),$$

$$g(k)L_-\sim\text{const}(h_{2v-1}\mp k)\quad(k\to\pm h_{2v-1}),$$

$$g(k)L_-\sim\text{const}(g_{2v}\mp k)\quad(k\to\pm g_{2v}).\tag{51}$$

Accordingly, a general solution of (50) is

$$\Phi = C_0\delta(k) + C_{00}\delta'(k) + \sum_{v=1}^{l+1}[C^+_{2v-1}\delta(k-h_{2v-1}) + C^-_{2v-1}\delta(k+h_{2v-1})] + \sum_{v=1}^{d}[C^+_{2v}\delta(k-g_{2v}) + C^-_{2v}\delta(k+g_{2v})] \quad (52)$$

for the ranges $V<\gamma$ and $V>1$, while

$$\Phi = C_0\delta(k) + \sum_{v=1}^{l+1}[C^+_{2v-1}\delta(k-h_{2v-1}) + C^-_{2v-1}\delta(k+h_{2v-1})] \quad (53)$$

for the intersonic speed, $\gamma<V<1$. Here the coefficients, $C_0,\ldots,C_{2v}$, are arbitrary constants, δ is the Dirac delta-function and δ' is its derivative.

Further, we can use an analytical representation of the delta-function as

$$\delta(k-k_0) = \delta_+(k-k_0) + \delta_-(k-k_0) \quad (k_0 = \text{const}),$$

$$\delta_+(k-k_0) = \frac{1}{2\pi}\frac{1}{0-\mathrm{i}(k-k_0)}, \qquad \delta_-(k-k_0) = \frac{1}{2\pi}\frac{1}{0+\mathrm{i}(k-k_0)} \quad (54)$$

which gives us the required separation.

A general solution of Eq. (49) which leads to a real result can now be represented as

$$\Lambda_+ = \frac{2\pi}{L_+}\Phi_+, \qquad \Phi_+ = C_0\delta_+(k) + C_{00}\delta'_+(k) + A_+ + B_+,$$

$$A_+ = \sum_{v=1}^{l+1}[C_{2v-1}\delta_+(k-h_{2v-1}) + \overline{C_{2v-1}}\delta_+(k+h_{2v-1})],$$

$$B_+ = \sum_{v=1}^{d}[C_{2v}\delta_+(k-g_{2v}) + \overline{C_{2v}}\delta_+(k+g_{2v})],$$

$$\Lambda_- = 2\pi L_-\Phi_-, \qquad \Phi_- = C_0\delta_-(k) + C_{00}\delta'_-(k) + A_- + B_-,$$

$$A_- = \sum_{v=1}^{l+1}[C_{2v-1}\delta_-(k-h_{2v-1}) + \overline{C_{2v-1}}\delta_-(k+h_{2v-1})],$$

$$B_- = \sum_{v=1}^{d}[C_{2v}\delta_-(k-g_{2v}) + \overline{C_{2v}}\delta_-(k+g_{2v})] \quad (55)$$

for the range $V<\gamma$,

$$\Lambda_+ = \frac{2\pi}{L_+}\Phi_+, \qquad \Phi_+ = C_0\delta_+(k) + A_+,$$

$$A_+ = \sum_{v=1}^{l+1}[C_{2v-1}\delta_+(k-h_{2v-1}) + \overline{C_{2v-1}}\delta_+(k+h_{2v-1})],$$

$$\Lambda_- = 2\pi L_-\Phi_-, \qquad \Phi_- = C_0\delta_-(k) + A_-,$$

$$A_- = \sum_{v=1}^{l+1}[C_{2v-1}\delta_-(k-h_{2v-1}) + \overline{C_{2v-1}}\delta_-(k+h_{2v-1})] \tag{56}$$

for the intersonic speed, $\gamma<V<1$, and

$$\Lambda_+ = \frac{2\pi}{L_+}\Phi_+, \qquad \Phi_+ = C_0\delta_+(k) + C_{00}\delta'_+(k),$$

$$\Lambda_- = 2\pi L_-\Phi_-, \qquad \Phi_- = C_0\delta_-(k) + C_{00}\delta'_-(k) \tag{57}$$

for the supersonic speed, $V>1$.

The 'feeding functions', $\Phi_\pm$, incorporate terms corresponding to (a) zero-wave number waves, that is C_0- and C_{00}-associated terms, (b) nonzero-wave number waves with $v_g<v$ (C_{2v-1}-terms) and (c) nonzero-wave number waves with $v_g>v$ (C_{2v}-terms). The feeding wave associated with the coefficient C_{2v-1} is placed ahead the phase-transition front, while the C_{2v}-wave is placed behind the phase-transition front. This finds confirmation in the above relations. Indeed, for Λ_+ C_{2v}-terms give no wave with g_{2v} as the wave number since $k=g_{2v}$ is a zero point for $1/L_+$. Similarly, for Λ_- C_{2v-1}-terms give no wave with h_{2v-1} as the wave number since $k=h_{2v-1}$ is a zero point for L_-.

3.6. Macrolevel-associated solution

3.6.1. General results

A solution corresponding to a zero feeding wave number, that is the solution associated with the coefficients C_0 and C_{00} is:

$$\Lambda_+ = \frac{1}{L_+}\left[C_0\frac{1}{0-\mathrm{i}k} + C_{00}\frac{\mathrm{i}}{(0-\mathrm{i}k)^2}\right],$$

$$\Lambda_- = L_-\left[C_0\frac{1}{0+\mathrm{i}k} - C_{00}\frac{\mathrm{i}}{(0+\mathrm{i}k)^2}\right], \tag{58}$$

where $C_{00}=0$ for the intersonic speed, $\gamma<V<1$.

For $V < \gamma$ the contribution of the point $k = 0$ (see Eq. (36)) is a piecewise linear solution as

$$\Lambda_0^+(\eta) = \frac{1}{\mathscr{R}}\sqrt{\frac{\gamma^2 - V^2}{1 - V^2}}(C_0 + \mathrm{i}C_{00}\,\eta) \quad (\eta > 0),$$

$$\Lambda_0^-(\eta) = \frac{1}{\mathscr{R}}\sqrt{\frac{1 - V^2}{\gamma^2 - V^2}}(C_0 + \mathrm{i}C_{00}\,\eta) \quad (\eta < 0). \tag{59}$$

In this solution, to satisfy the requirement of limited elongation we put $C_{00} = 0$. Since $\Lambda(\eta)$ is continuous, the constant C_0 can be determined using the phase-transition criterion (14). One has (see Eqs. (58) and (36))

$$\Lambda(0) = \lim_{k \to \mathrm{i}\infty} [-\mathrm{i}k\Lambda_+(k)] = C_0 = \Lambda_*. \tag{60}$$

Thus

$$\Lambda_0^+ = \frac{1}{\mathscr{R}}\sqrt{\frac{\gamma^2 - V^2}{1 - V^2}}\Lambda_*, \qquad \Lambda_0^- = \frac{1}{\mathscr{R}}\sqrt{\frac{1 - V^2}{\gamma^2 - V^2}}\Lambda_*. \tag{61}$$

3.6.2. Macrolevel energy release and total dissipation

In addition to this piecewise uniform moving state, there exist sinusoidal and exponential decreasing waves; nonzero real and complex singular points of $\Lambda_\pm$ (58) are their wave numbers, respectively. However, the jump in the macrolevel piecewise uniform component of the particle velocity, associated with the zero wave number, can be calculated without paying regard to these waves since the total dissipation is already taken into account in expressions (61). As follows from the momentum conservation law and (61) the jump is

$$\Delta\dot{u}_0 = \dot{u}_0^- - \dot{u}_0^+ = \frac{a(p_0^+ - p_0^-)}{Mv} = -\frac{\Lambda_* Vc}{\mathscr{R}}\frac{1 - \gamma^2}{\sqrt{(1 - V^2)(\gamma^2 - V^2)}}. \tag{62}$$

We now can determine the macrolevel energy release rate due to the phase transition. The energy release is independent of a rigid-body velocity, and we can assume the particle velocity ahead the front to be zero. The energy release per cell is then

$$G = \frac{1}{2}K(\Lambda_0^+)^2 - \frac{1}{2}K(\Lambda_0^-)^2 - \frac{1}{2}M(\Delta\dot{u}_0)^2 - \frac{a}{v}\gamma^2 K\Lambda_0^-\Delta\dot{u}_0, \tag{63}$$

where the first and the second terms are the strain energy per cell ahead and behind the front, respectively, the third term is the kinetic energy per cell behind the front and the last term is the energy flux as the work of the internal force behind the front during the period a/v. Using relations (61) and (62) one can find

$$G = \frac{K\Lambda_*^2(1 - \gamma^2)}{2\mathscr{R}^2}. \tag{64}$$

At the same time, the energy barrier, G_0, between the phases is

$$G_0 = \tfrac{1}{2}K\Lambda_*^2(1 - \gamma^2). \tag{65}$$

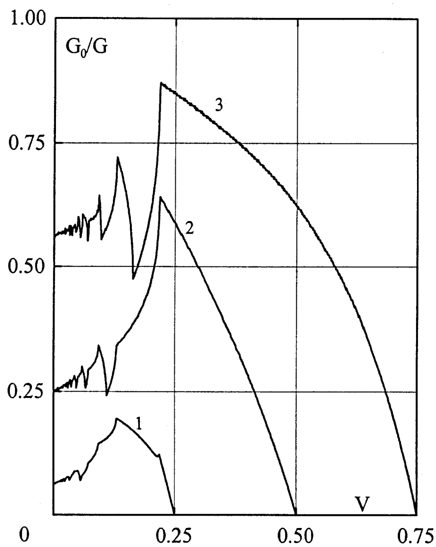

Fig. 4. The energy release ratios for $\gamma = 1/4$ (1), $\gamma = 1/2$ (2) and $\gamma = 3/4$ (3).

The difference is the wave resistance, or total dissipation as the energy carried away from the front by the dissipative waves. Thus $\mathscr{R}^2$ is the ratio of these energies

$$\mathscr{R}^2 = \frac{G_0}{G}. \tag{66}$$

Note that this relation has the same form as the corresponding relation for the crack in a lattice (Slepyan, 2001).

The *dissipative function* $\mathscr{R} = \mathscr{R}(V)$ can be calculated using its integral (37) or product (38) representations. These methods, however, become inconvenient for $V \to 0$. At the same time, one can find $\mathscr{R}(0)$ in a direct way. Indeed, if $V = 0$ then $\Lambda_0(+0) = \Lambda_0^+ = \Lambda_*$, $p_0^- = p_0^+$ (5) and hence

$$G = \frac{K\Lambda_*^2(1-\gamma^2)}{2\gamma^2}, \qquad \mathscr{R}(0) = \gamma. \tag{67}$$

The ratio $G_0/G = \mathscr{R}^2(V)$ for some values of γ is represented in Fig. 4.

Such a macrolevel-associated solution does not exist for $V \geq \gamma$. For $\gamma < V < 1$ this conclusion follows from Eqs. (58) and (48). In this case, the contribution of the singular point $k = 0$ is unlimited when $\eta \to \pm\infty$. For $V \geq 1$ such a solution does not satisfy the phase-transition criterion, namely, if $\Lambda(+0) = \Lambda_*$ then, in contradiction to the criterion, $\Lambda(+\infty) = \Lambda_* \sqrt{(V^2-1)/(V^2-\gamma^2)} > \Lambda_*$ (see Eq. (36)).

3.7. Chain-based macrolevel solution

The macrolevel formulation (4), (5) can now be completed with a phase-transition criterion based on the energy release rate (64) for the chain. In the formulation of the

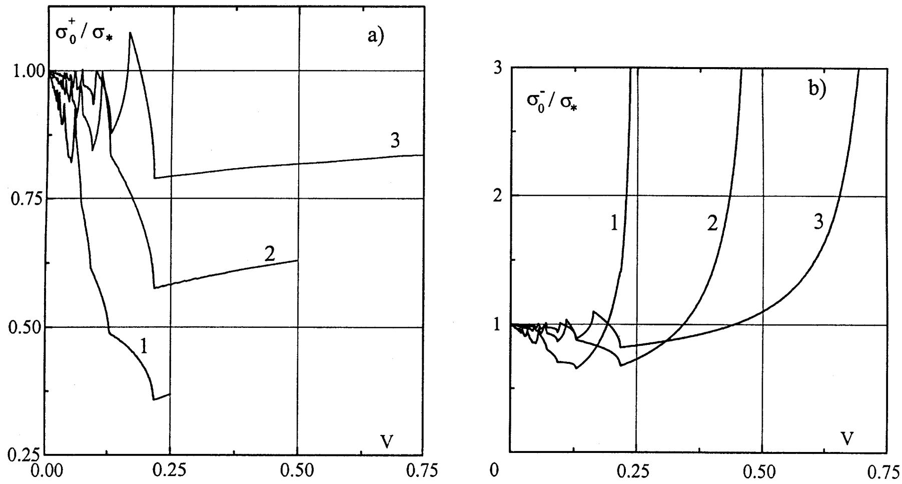

Fig. 5. The stress–speed dependences for $\gamma=1/4$ (1), $\gamma=1/2$ (2) and $\gamma=3/4$ (3): σ_0^+/σ_* (a) and σ_0^-/σ_* (b).

criterion, we take into account the fact that the solution for the homogeneous material is none other than a long-wave approximation of that for the structured material, and the total dissipation should be taken into account in the homogeneous-material solution. Noting that relation (8) is written for the energy release per unit cross-section area and unit length, while expression (64) is the energy release per cell of the length a, one has to substitute $E\varepsilon_*^2$ ($\varepsilon = \Lambda/a$) for $K\Lambda_*^2$ in Eq. (64). After this, equating these expressions for the energy release rate one obtains

$$\sigma_0^+\sigma_0^- = \sigma_*^2\gamma^2\mathscr{R}^{-2}. \tag{68}$$

With the use of this relation, the phase-transition wave, both steady-state and transient, can be considered in the framework of the homogeneous material model. Note, however, that in a transient problem, this model is valid if the speed, v, and the fields ahead and behind the front are slow-varying functions of x/a and ct/a.

Relations (68) and (5) lead to the following expressions for $\sigma_0^\pm$ [compare with (61)]:

$$\sigma_0^+ = \frac{1}{\mathscr{R}}\sqrt{\frac{\gamma^2 - V^2}{1 - V^2}}\sigma_*, \qquad \sigma_0^- = \frac{\gamma^2}{\mathscr{R}}\sqrt{\frac{1 - V^2}{\gamma^2 - V^2}}\sigma_*. \tag{69}$$

These dependences for several values of γ are shown in Fig. 5. It can be seen that not only σ_0^+ but even σ_0^- can be lower than σ_*.

So, there are three relations, two for stresses (69) and one for the jump of the particle velocity (5). The final relations required for completion of the problem formulation can be represented by initial and boundary conditions. Two examples are shown below.

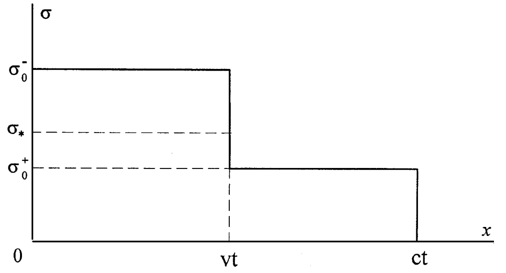

Fig. 6. The macrolevel phase-transition wave. The material is in the first (the second) phase at $x > vt (x < vt)$.

3.7.1. Phase transition under an impact

Consider the homogeneous-material problem for a half-space $x > 0$ with boundary and initial conditions as

$$\dot{u}_0 = v_0 = \text{const} \quad (x = 0, t > 0), \qquad \dot{u}_0 = \varepsilon_0 = 0 \quad (t = 0, 0 < x < \infty) \tag{70}$$

and assume that the material can show the two-phase behavior under extension ($v_0 < 0$, $\sigma_* > 0$), or under compression ($v_0 > 0, \sigma_* < 0$). If the impact velocity, v_0, is high enough the phase-transition wave can arise (Fig. 6). From Eqs. (5) it follows that

$$\dot{u}_0^+ = v_0 \pm \frac{c}{E}\sqrt{(\sigma_0^-/\gamma^2 - \sigma_0^+)(\sigma_0^- - \sigma_0^+)} \tag{71}$$

for the extension and compression, respectively. At the same time, for the wave propagating to the right ahead of the phase-transition front, the relation

$$\dot{u}_0^+ = -\sigma_0^+/(\rho c) \tag{72}$$

is valid. It follows that

$$\sigma_0^+ \pm \sqrt{(\sigma_0^-/\gamma^2 - \sigma_0^+)(\sigma_0^- - \sigma_0^+)} = \sigma^0, \quad \sigma^0 = -\rho v_0 c \tag{73}$$

and using Eq. (69) we obtain the following equation for the phase-transition front speed:

$$\frac{\gamma^2 - V^2 + (1-\gamma^2)V}{\mathcal{R}\sqrt{(1-V^2)(\gamma^2 - V^2)}} = \frac{\sigma^0}{\sigma_*}. \tag{74}$$

This dependence for some values of γ is represented in Fig. 7. Note that the 'impact stress', σ^0, is the true stress if no phase transition arises under the impact.

From Eqs. (74) and (69) it follows that

$$\frac{\sigma^0}{\sigma_*} = \frac{\sigma_0^-}{\sigma_*} + \frac{(1-\gamma^2)V(1-V)}{\mathcal{R}\sqrt{(1-V^2)(\gamma^2 - V^2)}}. \tag{75}$$

Thus, for any positive subsonic speed, $|\sigma_0^-| < |\sigma^0|$ and the wave can propagate under the condition $\sigma_0^- > \sigma_{\min}^-(\gamma)$, where $\sigma_{\min}^-(\gamma) < \sigma_*$ (see Fig. 5b).

In this process, the energy, delivered by the feeding wave propagating behind the phase-transition front, is spent in the phase transition itself, in the energy flux associated with the wave propagating ahead of the front and finally in the energy radiated by dissipative waves which are defined by the solution for the chain.

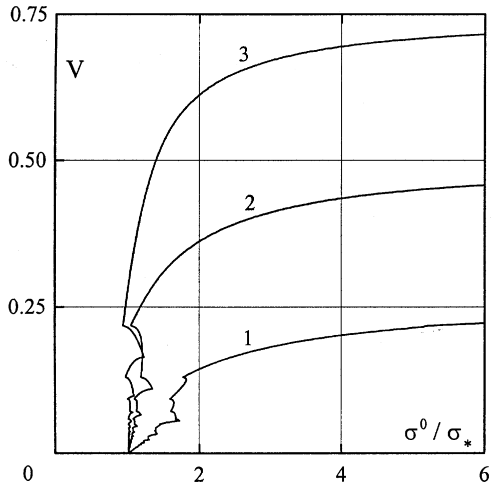

Fig. 7. The 'impact stress'–phase-transition speed dependences for $\gamma = 1/4$ (1), $\gamma = 1/2$ (2) and $\gamma = 3/4$ (3).

3.7.2. Spontaneous phase transition

Now consider the problem for an initially stressed body with

$$\sigma_0 = \sigma^0, \qquad \dot{u}_0 = 0 \quad (-\infty < x < \infty, t = 0), \tag{76}$$

where σ^0 does not reach σ_*. The material is assumed to be initially in the first phase.

We suggest that under a disturbance the phase transition occurs at a point (at a cell in the case of the chain), say, $x = 0$. If the initial stress is high enough, one can expect that this disturbance excites two phase-transition waves propagating to the right and to the left. In this symmetrical process, $\dot{u}_0 = 0$ in the second-phase region, $|x| < vt$.

Consider the wave propagating to the right. The stresses behind the phase-transition front, σ_0^-, is defined by (69), while an unloading step wave with the stress σ_1 should propagate ahead the front provided the sum,

$$\sigma^0 + \sigma_1 = \sigma_0^+, \tag{77}$$

is in accordance with Eq. (69) as well. In the step wave,

$$\dot{u}_0^+ = -\sigma_1/(\rho c), \tag{78}$$

while $\dot{u}_0^- = 0$. Using Eq. (5) we obtain σ_1,

$$\sigma_1 = -\frac{\sigma_0^- - \sigma_0^+}{V} \tag{79}$$

and the same results as for the impact: relation (73) and further — Eq. (74) respective V.

This coincidence follows from the fact that the conditions ahead and behind the phase-transition front in this problem can be satisfied by the previous solution with the particle velocity added by the rigid-body speed $-\dot{u}_0^-$. However, in contrast to the impact problem, the subcritical initial stress condition must be satisfied here, and this requirement looks to be in contradiction with the results shown in Fig. 7. The contradiction does not exist in the case of the so-called Maxwell transition where no energy is transferred to the microlevel. In this case, no dissipative wave exists, $\mathscr{R}=1$, and the condition can be satisfied for a finite range $\sigma^-_{\min}(\gamma)<\sigma^0<\sigma_*$. As can be seen below the case $\mathscr{R}=1$ is realized in a homogeneous model described by an equation of the fourth order (see Fig. 12).

3.8. Dissipative waves

We now return to Eq. (58) with $C_0=\Lambda_*$, $C_{00}=0$. Dissipative waves related to the macrolevel-associated solution are defined as residuals at zero points $k=h_{2v}$ of $L_+(k)$ and poles $k=g_{2v-1}$ of $L_-(k)$ for $k\neq 0$. The inverse Fourier transform leads to the following results:

$$\Lambda^+=\Lambda_*\sum_{v=1}^{l}\frac{G_+(h_{2v})}{\sqrt{S(h_{2v})}}\prod_{\mu\neq v}^{l}\left[1-\left(\frac{h_{2\mu}}{h_{2v}}\right)^2\right]^{-1}\cos[h_{2v}\eta+\vartheta(h_{2v})]H(\eta),$$

$$\Lambda^- = \Lambda_*\sum_{v=1}^{d+1}H_-(g_{2v-1})\sqrt{S(g_{2v-1})}\prod_{\mu\neq v}^{d+1}\left[1-\left(\frac{g_{2\mu-1}}{g_{2v-1}}\right)^2\right]^{-1}$$
$$\times\cos[g_{2v-1}\eta+\vartheta(g_{2v-1})]H(-\eta), \tag{80}$$

where the positive function S is represented in Appendix 2 and H is the unit step function. The total dissipation rate related to these waves as the energy lost on the macrolevel per unit time is (see Eqs. (64) and (65))

$$\frac{v}{a}(G-G_0)=\frac{vK}{2a}\Lambda_*^2(1-\gamma^2)(\mathscr{R}^{-2}-1). \tag{81}$$

3.9. Microlevel solutions

We now consider the general solution (55)–(57). The feeding functions $\Phi_\pm$ lead to the feeding waves of nonzero wave number by means of the term A_+ for Λ_+ and by means of the term B_- for Λ_-. It can be seen that a microlevel (nonzero wave number) feeding wave is placed ahead the phase-transition front if its group velocity is lower than the phase velocity ($k=h_{2v-1}$), while the wave is placed behind the front in the case $V_{\mathrm{g}}>V$ ($k=g_{2v}$). In the first case, both the subsonic and intersonic regimes of the phase transition are possible, while only the subsonic regime can exist in the second case.

We assume that the phase-transition process is going on under one of the possible feeding waves. A nonzero-wave number feeding wave is defined as the residual at the point $k=\pm h_{2v-1}$ or $k=\pm g_{2v}$ in the inverse Fourier transform. One can find for any

v that the feeding wave is

$$\Lambda^+(\eta)=\frac{2}{L_+(h_{2v-1})}\Re[C_{2v-1}\exp(-\mathrm{i}h_{2v-1}\eta)]H(\eta)\quad(0<V<1),$$

$$\text{or } \Lambda^-(\eta)=2L_-(g_{2v})\Re[C_{2v}\exp(-\mathrm{i}g_{2v}\eta)]H(-\eta)\quad(0<V<\gamma). \tag{82}$$

At the same time, the elongation at $\eta=0$ is

$$\Lambda(0)=\lim_{k\to\pm i\infty}(\mp \mathrm{i}k)\Lambda_\pm(k)=\Lambda_* \tag{83}$$

and hence as follows from Eqs. (55) and (56)

$$\Re C_{2v-1}=\tfrac{1}{2}\Lambda_* \text{ or } \Re C_{2v}=\tfrac{1}{2}\Lambda_*. \tag{84}$$

Thus the amplitude of the wave is

$$\mathscr{L}_v^+=\frac{1}{|L_+(h_{2v-1})|}\sqrt{\Lambda_*^2+(\Im C_{2v-1})^2} \text{ or } \mathscr{L}_v^-=|L_-(g_{2v})|\sqrt{\Lambda_*^2+(\Im C_{2v})^2}. \tag{85}$$

3.9.1. High-speed solution

Consider the range $V_0<V<1$ (23). In this case, $l=d=0$, only two couples of zeros remain, $k=\pm h_1$ and $k=\pm g_1$ ($g_1=0$ if $V>\gamma$), and the only feeding wave can be represented by the first pair. For $k\to\pm h_1$

$$H_-\sim\frac{2}{h_1}(k-h_1),\quad G_-\to 1-\frac{g_1^2}{h_1^2},\quad H_+=G_+=1,$$

$$L_+=\sqrt{S(h_1)}\mathrm{e}^{\mathrm{i}\vartheta},\quad S(h_1)\to\frac{(V-V_\mathrm{g})(h_1^2-g_1^2)}{(1-\gamma^2)h_1^2V}. \tag{86}$$

The feeding wave can now be represented as

$$\Lambda^+(\eta)=\mathscr{L}\cos(h_1\eta+\phi)H(\eta),\quad \mathscr{L}=\sqrt{\frac{\Lambda_*^2+(\Im C_1)^2}{S(h_1)}}, \tag{87}$$

where ϕ are real constant. This equality evidences that the amplitude of the feeding wave is bounded from below. At the same time, the necessary condition of the existence of such wave requires the wave amplitude to be less than the corresponding critical value, Λ_*. So, the amplitude must satisfy the inequalities as

$$\frac{\Lambda_*}{\sqrt{S(h_1)}}\le\mathscr{L}<\Lambda_*. \tag{88}$$

It requires

$$S(h_1)>1, \tag{89}$$

that is

$$\left(1-\frac{g_1^2}{h_1^2}\right)\left(1-\frac{V_\mathrm{g}}{V}\right)>1-\gamma^2, \tag{90}$$

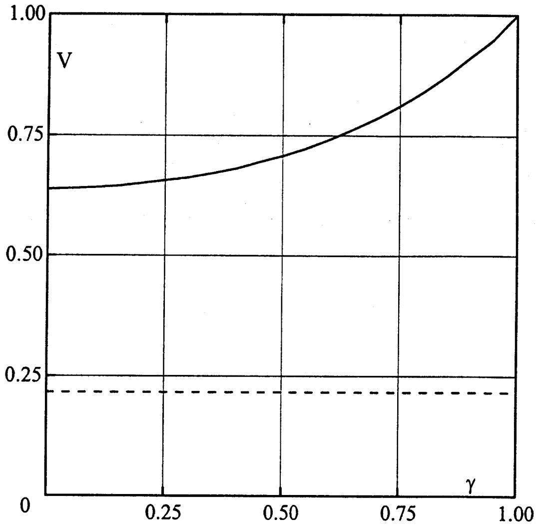

Fig. 8. The upper bound for the validity of the high-speed solution (90). The lower bound lies below the dotted line.

where

$$2\sin(h_1/2)/h_1 = V, \quad 2\gamma\sin(g_1/2)/g_1 = V,$$

$$V_g = \cos(h_1/2), \quad g_1 = 0 \quad \text{for } V \geq \gamma. \tag{91}$$

Inequality (90) is satisfied for a range of V as

$$V_{\min}(\gamma) < V < V_{\max}(\gamma). \tag{92}$$

The upper bound is shown in Fig. 8, while the lower bound lies below the dotted line, $V = V_0$, i.e. $V_{\min}(\gamma) < V_0$.

Thus, the phase-transition front can propagate with an intersonic speed, that is faster than any wave in the second phase. In this case, the energy required for the phase transition is delivered by the feeding wave propagating ahead the front. It does deliver the energy since its group velocity is less than the phase velocity and hence energy flux relative to the front is directed toward the front.

3.9.2. Dissipative waves

Dissipative waves for such a microlevel solution are defined as residuals at zero points $k = \pm h_{2\mu}$ of $L_+(k)$ and poles $k = \pm g_{2\mu-1}$ of $L_-(k)$. One can find

$$\Lambda^+ = \sum_{\mu=1}^{l} \frac{G_+(h_{2\mu})}{\sqrt{S(h_{2\nu})}} \prod_{\alpha\neq\mu}^{l} \left[1 - \left(\frac{h_{2\alpha}}{h_{2\mu}}\right)^2\right]^{-1} \frac{h_{2\mu}^2}{h_{2\mu}^2 - h_{2\nu-1}^2} \Lambda_\mu^+ H(\eta)$$

$$\Lambda_\mu^+ = \Lambda_* \cos(h_{2\mu}\eta + \vartheta) + \frac{2h_{2\nu-1}}{h_{2\mu}} \Im C_{2\nu-1} \sin(h_{2\mu}\eta + \vartheta),$$

$$\Lambda^- = \sum_{\mu=1}^{d+1} \frac{H_-(g_{2\mu-1})\sqrt{S(g_{2\mu-1})}g_{2\mu-1}^2}{g_{2\mu-1}^2 - h_{2\nu-1}^2} \prod_{\alpha\neq\mu}^{d+1}\left[1-\left(\frac{g_{2\alpha-1}}{g_{2\mu-1}}\right)^2\right]^{-1} \Lambda_\mu^- H(-\eta)$$

$$\Lambda_\mu^- = \Lambda_* \cos(g_{2\mu-1}\eta + \vartheta) + \frac{2h_{2\nu-1}}{g_{2\mu-1}}\Im C_{2\nu-1}\sin(h_{2\mu}\eta+\vartheta), \tag{93}$$

where $h_{2\nu_0-1}$ is the feeding wave number.

Similarly for the feeding wave number $g_{2\nu}$

$$\Lambda^+ = \sum_{\mu=1}^{l} \frac{G_+(h_{2\mu})}{\sqrt{S(h_{2\nu})}} \prod_{\alpha\neq\mu}^{l}\left[1-\left(\frac{h_{2\alpha}}{h_{2\mu}}\right)^2\right]^{-1} \frac{h_{2\mu}^2}{h_{2\mu}^2 - g_{2\nu}^2}\Lambda_\mu^+ H(\eta)$$

$$\Lambda_\mu^+ = \Lambda_* \cos(h_{2\mu}\eta+\vartheta) + \frac{2g_{2\nu}}{h_{2\mu}}\Im C_{2\nu}\sin(h_{2\mu}\eta+\vartheta),$$

$$\Lambda^- = \sum_{\mu=1}^{d+1} \frac{H_-(g_{2\mu-1})\sqrt{S(g_{2\mu-1})}g_{2\mu-1}^2}{g_{2\mu-1}^2 - g_{2\nu}^2} \prod_{\alpha\neq\mu}^{d+1}\left[1-\left(\frac{g_{2\alpha-1}}{g_{2\mu-1}}\right)^2\right]^{-1} \Lambda_\mu^- H(-\eta)$$

$$\Lambda_\mu^- = \Lambda_* \cos(g_{2\mu-1}\eta+\vartheta) + \frac{2g_{2\nu}}{g_{2\mu-1}}\Im C_{2\nu}\sin(h_{2\mu}\eta+\vartheta). \tag{94}$$

In addition to this, in the case of the intersonic range of the speed, the functions $1/L_+$ and L_- have simple poles at $k=0$ and this leads to 'dissipative' waves of zero wave number. Such waves associated with the feeding wave number $h_{2\nu-1}$ (the function $g(k)$ has no zeros at $k\neq 0$ for this range of the speed) have the following expressions [see Eqs. (48) and (56)]:

$$\Lambda^+ = \sqrt{\frac{V^2-\gamma^2}{1-V^2}}\prod_{\mu=1}^{l+1} h_{2\mu-1}\left[\prod_{\mu=1}^{l} h_{2\mu}\right]^{-1}\frac{2\Im C_{2\nu-1}}{h_{2\nu-1}},$$

$$\Lambda^- = -\sqrt{\frac{1-V^2}{V^2-\gamma^2}}\prod_{\mu=1}^{l+1} h_{2\mu-1}\left[\prod_{\mu=1}^{l} h_{2\mu}\right]^{-1}\frac{2\Im C_{2\nu-1}}{h_{2\nu-1}}. \tag{95}$$

In particular, for $V>V_0$ [see Eq. (23)]

$$\Lambda^+ = 2\Im C_{2\nu-1}\sqrt{\frac{V^2-\gamma^2}{1-V^2}},\quad \Lambda^- = -2\Im C_{2\nu-1}\sqrt{\frac{1-V^2}{V^2-\gamma^2}}. \tag{96}$$

Note that these waves exist if the feeding wave amplitude is greater than the minimal value required for the phase transition (the frequency of the feeding wave which dictates the phase-transition wave speed is considered as given).

4. Higher-order-derivative model

To resolve the difficulty in the formulation of the phase-transition problem for a homogeneous model we have considered the discrete chain. This has led to the determination of both the energy criterion, as the missing condition in the homogeneous-model formulation, and the microlevel solutions which have no analogue on the macrolevel. These results, however, follow from the existence of the wave dispersion associated with the discrete structure rather than the structure itself. This suggests an alternative way as the use of a higher-order-derivative (HOD) homogeneous model which also can represent a wave guide for waves with dispersion. We now consider such a model.

4.1. Some general considerations

Let us introduce the kinetic and strain energies per unit volume as

$$T=\frac{\rho\dot{u}^2}{2},$$

$$W=\frac{1}{2}[a_1(u')^2+a_2(u'')^2+\cdots+a_n(u^{(n)})^2]\quad\left(u^{(n)}\equiv\frac{\mathrm{d}^n u}{\mathrm{d}x^n}\right). \tag{97}$$

The terms which form the strain energy reflect different modes of generalized strain existing in the model. It is assumed that each of the moduli, $a_1,\dots,a_n$, can be different in different phases

$$a_v=E_v\quad\text{(the first phase)},\quad a_v=\gamma_v^2E_v\quad\text{(the second phase)},$$

$$0<\gamma_v\le 1,\quad v=1,\dots,n. \tag{98}$$

Note that stability considerations result in the inequality

$$\sum_{m=1}^{n}a_mk^{2m-2}>0 \tag{99}$$

for any nonzero real k. In particular, it follows that the first and the last moduli, a_1 and a_n, must be positive, and we have to submit to terms that for any $n>1$ the phase and group velocities (v and v_g, respectively) of a sinusoidal wave, $\exp[\mathrm{i}(\omega t-kx)]$, existing in this model, have no upper bounds. These velocities

$$v=\frac{\omega}{k}=\left(\sum_{m=1}^{n}a_mk^{2m-2}\right)^{1/2},\quad v_g=\frac{\mathrm{d}\omega}{\mathrm{d}k}=\sum_{m=2}^{n}(m-1)a_mk^{2m-2}$$
$$\times\left(\sum_{m=1}^{n}a_mk^{2m-2}\right)^{-1/2}, \tag{100}$$

increase unboundedly together with the wave number. This is in contrast to the chain model where the velocities of such microlevel waves are bounded by the macrolevel wave velocity, c.

We now intend to find necessary and sufficient conditions for the energy conservation law to hold on the microlevel under the phase-transition criterion (2) for a mode of generalized strain. Let us discuss some points concerning macrolevel and microlevel relations.

4.1.1. Stresses and the particle velocity jumps on the macro and microlevels

The microlevel dynamic equation

$$\sum_{v=1}^{n}(-1)^{v}[a_v u^{(v)}]^{(v)} + \rho\ddot{u} = 0 \tag{101}$$

resulting from Eq. (97) leads to the same expression for the momentum conservation law on the microlevel as in Eq. (5) where the stress is now

$$\sigma = \sum_{v=1}^{n}(-1)^{v-1}[a_v u^{(v)}]^{(v-1)}, \tag{102}$$

such that

$$\rho\ddot{u} = \sigma'. \tag{103}$$

At the same time, expression (5) with $\sigma_0 = a_1\varepsilon_0$ is still valid for the macrolevel. This means that the jump in the particle velocity can be different for these two levels.

Along with the stresses σ and σ_0 there exist partial generalized stresses related to each of the modes forming the total strain energy (97)

$$\sigma_v = a_v\varepsilon_v, \quad \varepsilon_v = u^{(v)}, \tag{104}$$

where ε_v is the corresponding partial generalized strain. Note that $v = 1$ corresponds to the macrolevel and the subscript '1' denotes here the same as the subscript '0' used for the macrolevel values.

Further, the matter conservation law results in the relation

$$\Delta\dot{u} = \dot{u}^{-} - \dot{u}^{+} = v(\varepsilon^{+} - \varepsilon^{-}) \tag{105}$$

and, since v is the same for both levels, the ratio, $\Delta\dot{u}/\Delta\varepsilon$, is the same as well.

The energy lost in the phase transition for each mode is defined by the same way as for macrolevel (10), namely

$$(G - G_0)_v = \frac{1-\gamma_v^2}{2}E_v(\varepsilon_v^{-}\varepsilon_v^{+} - \varepsilon_{v*}^2). \tag{106}$$

This follows directly from the stress–strain relation (98).

Finally, we note that the microlevel dynamic equation (101) can also be represented as

$$\sum_{v=1}^{n}(-1)^{v}[a_v\varepsilon^{(v-1)}]^{(v+1)} + \rho\ddot{\varepsilon} = 0 \quad (\varepsilon = u'). \tag{107}$$

4.1.2. Theorem on the highest modulus

The necessary and sufficient condition for energy conservation on the microlevel under the phase-transition criterion (2) *is that the phase transition does not concern the highest order derivative, that is the modulus $a_n > 0$ is the same in both phases.*

At first, we note that energy is not lost in the phase transformation only if each partial generalized strain, ε_ν, is continuous. This follows directly from Eq. (106) where, in accordance with the phase-transition criterion, $\sigma_\nu(+0) = E_\nu\varepsilon_\nu(+0) = \sigma_{\nu*}$, $\varepsilon_\nu^+ = \varepsilon_{\nu*}$.

Suppose that $\gamma_n < 1$. In this case, the last term in Eq. (107) is

$$(-1)^n[a_n\varepsilon^{(n-1)}]^{(n+1)} = (-1)^n(1-\gamma_n^2)E_n\delta^{(n+1)}(\eta) + \text{smaller-order singularities} \tag{108}$$

and this is a noncompensated singular term since the other terms cannot be so singular [the highest-order singularity which can be introduced by the ν-term is $\delta^{(\nu+1)}(\eta)$]. As can be seen in Eq. (107) this leads to the same singularity in $\ddot{\varepsilon} = v^2\varepsilon''$ which is in contradiction with continuity of $\varepsilon = \varepsilon_1$. Thus such a solution does not exist and the continuity of the highest modulus as a necessary condition is proved.

Now consider Eq. (107) with

$$\gamma_n = 1, \quad \varepsilon(+0) - \varepsilon(-0) = \varepsilon'(+0) - \varepsilon'(-0)$$
$$= \cdots = \varepsilon^{(n-1)}(+0) - \varepsilon^{(n-1)}(-0) = 0. \tag{109}$$

In general, the left part of the equation contains a linear form of generalized functions $\delta(\eta), \delta'(\eta), \ldots, \delta^{(n-1)}(\eta)$ introduced with the differentiation of the discontinuities of both the moduli, $a_\nu, \nu = 1, 2, \ldots, n-1$, and derivatives of the strain, $\varepsilon^{(n)}(\pm 0), \varepsilon^{(n+1)}(\pm 0), \ldots,$ $\varepsilon^{(2n-1)}(\pm 0)$. For example, for $n = 2$

$$(a_1\varepsilon)'' = (1-\gamma_1^2)E_1[\varepsilon(0)\delta'(\eta) + \varepsilon'(0)\delta(\eta)],$$

$$a_2\varepsilon^{\mathrm{IV}} = E_2[(\varepsilon''(+0) - \varepsilon''(-0))\delta'(\eta) + (\varepsilon'''(+0) - \varepsilon'''(-0))\delta(\eta)]. \tag{110}$$

These singularities must be eliminated (since the left part of the equation must be equal to zero) and this imposes n conditions on the interface discontinuities of the strain and its derivatives. However, the condition concerning $\delta(\eta)$ (but not for derivatives of it) is satisfied automatically as follows directly from Eq. (107) and the condition $\varepsilon'(+0) = \varepsilon'(-0)$.

Thus the solution must satisfy n continuity conditions (109) and $n-1$ discontinuity conditions. In addition, there is the phase-transition criterion. So, there are $2n$ conditions in total. For example, for $n = 2$ these conditions are

$$\varepsilon(+0) - \varepsilon(-0) = \varepsilon'(+0) - \varepsilon'(0) = 0,$$

$$\varepsilon''(+0) - \varepsilon''(-0) = -(1-\gamma_1^2)\frac{E_1}{E_2}\varepsilon(0), \quad \sigma(0) = \sigma_*, \tag{111}$$

while the condition

$$\varepsilon'''(+0) - \varepsilon'''(-0) = -(1-\gamma_1^2)\frac{E_1}{E_2}\varepsilon'(0) \tag{112}$$

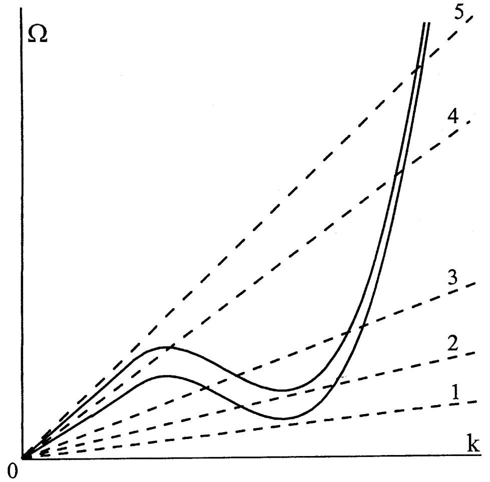

Fig. 9. Dispersion curves for the HOD model with $n>2$. The dotted lines correspond to different values of the phase transition speed: the subsonic (1–3), intersonic (4) and supersonic (5) regimes. Two nonzero feeding wave numbers correspond to the lines 3 and 4. The low-speed case (1) corresponds to a dissipation-free macrolevel-associated wave.

is satisfied automatically. As shown below a nongrowing general solution with $\gamma_n = 1$ represents a sufficient number of arbitrary constants to satisfy these $2n$ conditions and hence the condition of the theorem is also sufficient.

General solutions to Eq. (107) for $\eta>0$ and $\eta<0$ contain $2n$ linearly independent functions each. For a macrolevel-associated nongrowing solution only $2n$ functions remain, namely, macrolevel constants, ε_0^- and ε_0^+, and $2(n-1)$ functions of nonzero (real or complex) wave number each. In connection with these microlevel functions, we note that for a *subsonic speed* there are even real wave numbers for each solution (since $a_n>0$) with $v<v_g$ for a half of them and $v>v_g$ for the rest part. Only the low-group-velocity solutions, $v_g<v$, can be considered as dissipative waves for $\eta<0$, while only the high-group-velocity solutions, $v_g>v$, — for $\eta>0$. Similar conclusion concerning to the complex-wave number functions is valid: only a half of such functions can be used, namely, only the functions which tend to zero when $\eta\to\pm\infty$ are acceptable. Thus, the complete solution contains two macrolevel constant and $2(n-1)$ microlevel coefficients of the microlevel functions, that is $2n$ arbitrary constants in total.

For a macrolevel-associated solution, one of the constants, ε_0^+ or ε_0^-, can be considered as given. So, there are $2n-1$ arbitrary constants and, in addition, the speed, v. Thus, there is the same number of the arbitrary constants as the interface conditions together with the phase-transition criterion.

For a microlevel solution, there are $2n$ arbitrary constants corresponding to dissipative waves and this is sufficient for the $2n$ conditions to be satisfied. A feeding wave should be introduced by its amplitude and frequency, ω. This defines the speed, $v=\omega/k$. However, in a general case, for $n>2$ the corresponding wave number, k, is not defined uniquely (see Fig. 9). Additional considerations can also be required for

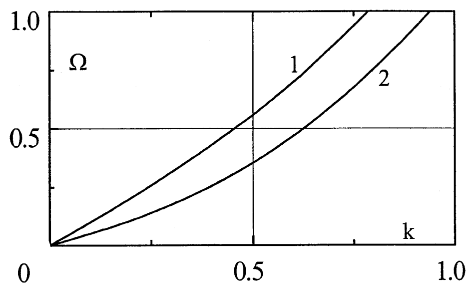

Fig. 10. Dispersion curves for the HOD model with $n = 2$.

the determination of an unknown phase of the feeding wave for $V < \gamma$. For example, the phase can be found using a given relation between the energies delivered by the macrolevel and microlevel feeding waves which can coexist.

For *intersonic and supersonic* cases, the number of admissible functions decreases; it is $2n - 1$. So, in this case, the phase of the feeding wave as an additional unknown is defined uniquely.

Thus, the condition of continuity of the highest modulus is also sufficient. Along with this, there can be more freedom in the subsonic microlevel phase-transition solution for $n > 2$.

4.2. Equation of the fourth order

Consider the simplest version of the HOD model with $n = 2$. The steady-state equations for $\eta > 0$ and $\eta < 0$ are

$$(1 - V^2)u'' - E_2 u^{\mathrm{IV}} = 0 \quad (x > vt),$$

$$(\gamma^2 - V^2)u'' - E_2 u^{\mathrm{IV}} = 0 \quad (x < vt), \quad V = v/c. \tag{113}$$

The corresponding dispersive relations

$$\Omega = \Omega^+ = \sqrt{k^2 + a_2 k^4} \quad \text{(the first phase)},$$

$$\Omega = \Omega^+ = \sqrt{\gamma^2 k^2 + a_2 k^4} \quad \text{(the second phase)}, \tag{114}$$

are shown in Fig. 10.

4.2.1. Governing equation

We use here the Wiener–Hopf technique as in the case of the discrete chain. This allows one to satisfy the interface conditions automatically by equating out-of-integral terms in the Fourier transform to zero. (These terms reflecting the above-mentioned singular functions are not present if the singularities are compensated as discussed above.)

We begin from the steady-state equation

$$D^2 \varepsilon^{\mathrm{IV}} - [(\chi - V^2)\varepsilon]'' = q, \quad [\chi = 1\ (\eta > 0),\ \chi = \gamma^2\ (\eta < 0)], \tag{115}$$

where the coefficient $D^2 = E_2$ and q is an external force introduced for convenience in an initial stage of the considerations.

The Fourier transform with no out-of-integral terms is

$$L_1\varepsilon_+ + L_2\varepsilon_- = q^{\mathrm{F}} \tag{116}$$

with

$$\begin{aligned}
&L_1 = (0+\mathrm{i}k)(0-\mathrm{i}k)[1 - V^2 + D^2k^2] \quad (V<1),\\
&L_1 = (0+\mathrm{i}k)^2[0-\mathrm{i}(Dk - \sqrt{V^2-1})][0-\mathrm{i}(Dk + \sqrt{V^2-1})] \quad (V>1),\\
&L_2 = (0+\mathrm{i}k)(0-\mathrm{i}k)[\gamma^2 - V^2 + D^2k^2] \quad (V<\gamma),\\
&L_2 = (0+\mathrm{i}k)^2[0-\mathrm{i}(Dk - \sqrt{V^2-\gamma^2})][0-\mathrm{i}(Dk + \sqrt{V^2-\gamma^2})] \quad (V>\gamma),
\end{aligned} \tag{117}$$

where the causality principle is taken into account.

We now can represent

$$\frac{L_1}{L_2} = L = L_+L_-, \tag{118}$$

where

$$\begin{aligned}
&L_+ = \frac{\sqrt{1-V^2} - \mathrm{i}Dk}{\sqrt{\gamma^2 - V^2} - \mathrm{i}Dk} \quad (V<\gamma),\\
&L_+ = \frac{D(0-\mathrm{i}k)(\sqrt{1-V^2} - \mathrm{i}Dk)}{[0-\mathrm{i}(Dk - \sqrt{V^2-\gamma^2})][0-\mathrm{i}(Dk + \sqrt{V^2-\gamma^2})]} \quad (\gamma<V<1),\\
&L_+ = \frac{[0-\mathrm{i}(Dk - \sqrt{V^2-1})][0-\mathrm{i}(Dk + \sqrt{V^2-1})]}{[0-\mathrm{i}(Dk - \sqrt{V^2-\gamma^2})][0-\mathrm{i}(Dk + \sqrt{V^2-\gamma^2})]} \quad (V>1)
\end{aligned} \tag{119}$$

and

$$\begin{aligned}
&L_- = \frac{\sqrt{1-V^2} - \mathrm{i}Dk}{\sqrt{\gamma^2 - V^2} - \mathrm{i}Dk} \quad (V<\gamma),\\
&L_- = \frac{\sqrt{1-V^2} + \mathrm{i}Dk}{D(0+\mathrm{i}k)} \quad (\gamma<V<1),\\
&L_- = 1 \quad (V>1).
\end{aligned} \tag{120}$$

Eq. (115) can now be rewritten in the form similar to Eq. (49)

$$L_+\varepsilon_+ + \frac{\varepsilon_-}{L_-} = \frac{q^{\mathrm{F}}}{L_2L_-}. \tag{121}$$

The following considerations are also similar to that used in the case of the discrete chain. In the following, we consider three possible ranges of the phase-transition wave speed, $V<\gamma$, $\gamma<V<1$ and $V>1$.

4.3. Subsonic speed

In the case $V<\gamma$, the product L_2L_- turns into zero on the real k-axis only at $k=0$. Thus, in this case, only a macrolevel-associated solution can exist. For such a bounded homogeneous solution we can write

$$L_+\varepsilon_+ + \frac{\varepsilon_-}{L_-} = \frac{C_0}{0+\mathrm{i}k} + \frac{C_0}{0-\mathrm{i}k} \tag{122}$$

with the solution as

$$\varepsilon_+ = \frac{C_0}{L_+(0-\mathrm{i}k)}, \qquad \varepsilon_- = \frac{C_0L_-}{0+\mathrm{i}k}. \tag{123}$$

The unknown constant C_0 can be determined using the phase-transition criterion. Since in the HOD model strain is continuous (as well as its first-order derivative), the relations are valid as

$$\varepsilon(+0)=\varepsilon(-0)=\lim_{k\to\pm\mathrm{i}\infty}(\mp\mathrm{i}k)\varepsilon_\pm(k)=\varepsilon_*. \tag{124}$$

From this and Eq. (119) it follows that $C_0=\varepsilon_*$. At the same time, the macrolevel waves, the feeding wave at $\eta<0$ and the 'dissipative' wave at $\eta>0$, are defined as follows:

$$\varepsilon(\eta)=\varepsilon_0^+ = \frac{\varepsilon_*}{L_+(0)} = \varepsilon_*\sqrt{\frac{\gamma^2-V^2}{1-V^2}} \quad (\eta>0),$$

$$\varepsilon(\eta)=\varepsilon_0^- = \frac{\varepsilon_*}{L_+(0)} = \varepsilon_*\sqrt{\frac{\gamma^2-V^2}{1-V^2}} \quad (\eta<0). \tag{125}$$

Note that these relations correspond to that in Eq. (69) with $\mathscr{R}=1$.

The feeding wave should be considered as given and this allows one to find the phase-transition wave speed as

$$V=\sqrt{\frac{\gamma^2-(\varepsilon_*/\varepsilon_0^-)^2}{1-(\varepsilon_*/\varepsilon_0^-)^2}}. \tag{126}$$

In these terms,

$$\varepsilon_0^+ = \frac{\varepsilon_*^2}{\varepsilon_0^-}. \tag{127}$$

In addition to the macrolevel waves (125), exponentially decreasing waves exist in a vicinity of the phase-transition front. These wave can be found directly from expressions (123) as the contribution of the poles $k=-\mathrm{i}\sqrt{1-V^2}/D$ for ε_+ and $k=-\mathrm{i}\sqrt{\gamma^2-V^2}/D$ for ε_-

$$\varepsilon=\varepsilon_{(1)}^+ = \varepsilon_*\left(1-\sqrt{\frac{\gamma^2-V^2}{1-V^2}}\right)\exp[-(\sqrt{1-V^2}/D)\eta] \quad (\eta>0),$$

$$\varepsilon=\varepsilon_{(1)}^- = -\varepsilon_*\left(\sqrt{\frac{1-V^2}{\gamma^2-V^2}}-1\right)\exp[(\sqrt{\gamma^2-V^2}/D)\eta] \quad (\eta<0). \tag{128}$$

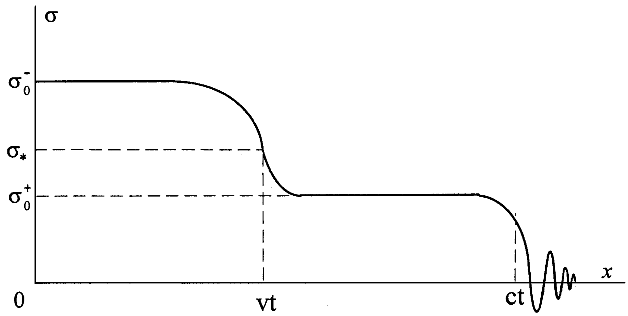

Fig. 11. The dissipation-free phase-transition wave in the fourth-order-derivative model.

Thus the complete solution

$$\varepsilon^{\pm}(\eta) = \varepsilon_0^{\pm} + \varepsilon_{(1)}^{\pm} \tag{129}$$

represents the strain monotonically increasing from ε_0^+ $(\eta \to \infty)$ to ε_* $(\eta = 0)$ and then to ε_0^- $(\eta \to -\infty)$. In contrast to the macrolevel formulation (3) both the strain, ε, and its derivative, ε', are continuous and there is no loss in energy here. Indeed (see Eq. (10)),

$$G - G_0 = \frac{1-\gamma^2}{2} E(\varepsilon_0^- \varepsilon_0^+ - \varepsilon_*^2) = 0. \tag{130}$$

In this case, there are no microlevel dissipative waves and hence there is no energy transfer to the microlevel. At the same time, the microlevel helps to overcome the energy barrier ($\sigma_0^+ < \sigma_*$). Thus the role of the microlevel introduced by the fourth-order derivative is similar to the role of a catalyst.

An example of a wave is represented in Fig. 11. Note that under the considered equation of the fourth order, in a vicinity of the moving point $x = ct$, there exists a quasi-front where stresses and particle velocities are continuous.

The two examples as the phase transition under an impact and a spontaneous phase transition considered above have the same solutions in the case of the fourth-order differential equation, however, with $\mathscr{R} = 1$. In this latter case, the spontaneous phase transition is possible in a range

$$\sigma_{\min}^-(\gamma) < \sigma^0 < \sigma_*, \tag{131}$$

where $\sigma_{\min}^-$ for several values of γ can be found in Fig. 12. The curves correspond to relation (74) with $\mathscr{R} = 1$. Note that in the considered case, the feeding wave is represented by the unloading wave propagating ahead the phase-transition front. The wave configuration for $\gamma = 3/4$, $\sigma^0/\sigma_* = 0.890$ is represented in Fig. 13.

4.4. Intersonic speed

In the case, $\gamma < V < 1$, the product $L_2 L_-$ has the following real zeros: $k = 0$ and $\pm k_1 - \mathrm{i}0$, $k_1 = \sqrt{V^2 - \gamma^2/D}$. Also we denote $k_2 = \sqrt{1 - V^2/D}$. We come to the

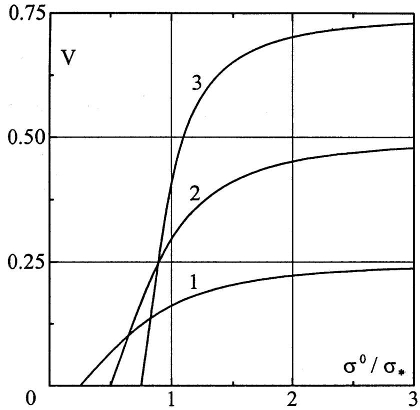

Fig. 12. The initial stress–phase-transition speed dependences for $\gamma = 1/4$ (1), $\gamma = 1/2$ (2) and $\gamma = 3/4$ (3) for the Maxwell-type, dissipation-free phase transition [see Eq. (74) with $\mathscr{R} = 1$].

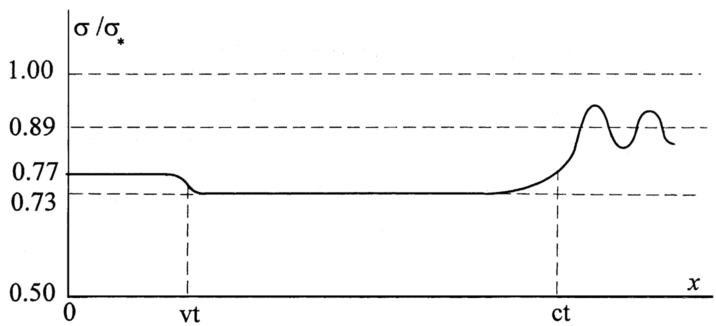

Fig. 13. The spontaneous Maxwell-type phase-transition wave for $\gamma = 0.75$ and $\sigma^0/\sigma_* = 0.890$: $\sigma_0^+ = 0.730$, $\sigma_0^- = 0.770$, $V = 0.250$.

equation:

$$L_+\varepsilon_+ + \frac{\varepsilon_-}{L_-} = \frac{C_0}{0 + \mathrm{i}k} + \frac{C_0}{0 - \mathrm{i}k} + \frac{C_1}{0 + \mathrm{i}(k - k_1)} + \frac{C_1}{0 - \mathrm{i}(k - k_1)} + \frac{\overline{C_1}}{0 + \mathrm{i}(k + k_1)} + \frac{\overline{C_1}}{0 - \mathrm{i}(k + k_1)}. \tag{132}$$

Note that $L_+(0) = 1/L_-(0) = 0$ and hence no bounded feeding wave of zero wave number can exist. We have to put $C_0 = 0$. Thus, there is no macrolevel-associated solution valid for this range of the speed. The rest solution is a microlevel one; it corresponds to the feeding wave number $k = k_1$:

$$\varepsilon_+ = \frac{[0 - \mathrm{i}(k - k_1)][0 - \mathrm{i}(k + k_1)]}{(0 - \mathrm{i}k)(k_2 - \mathrm{i}k)} \left[\frac{C_1}{0 - \mathrm{i}(k - k_1)} + \frac{\overline{C_1}}{0 - \mathrm{i}(k + k_1)}\right],$$

$$\varepsilon_- = \frac{k_2 + \mathrm{i}k}{0 + \mathrm{i}k} \left[\frac{C_1}{0 + \mathrm{i}(k - k_1)} + \frac{\overline{C_1}}{0 + \mathrm{i}(k + k_1)}\right]. \tag{133}$$

The inverse Fourier transform leads to the following solution:

$$\varepsilon(\eta) = \left(\varepsilon_* - \frac{k_1}{k_2} 2\Im C_1\right) \exp(-k_2\eta) + \frac{k_1}{k_2} 2\Im C_1 \quad (\eta > 0),$$

$$\varepsilon(\eta) = \varepsilon_* \left(\cos k_1\eta - \frac{k_2}{k_1}\sin k_1\eta\right) + 2\Im C_1 \left[\sin k_1\eta - \frac{k_2}{k_1}(1 - \cos k_1\eta)\right] \quad (\eta < 0), \tag{134}$$

where in accordance with criterion (124), $2\Re C_1 = \varepsilon_*$.

In this solution, the feeding wave is that with the wave number k_1, while the rest terms represent the dissipative waves. The feeding wave amplitude

$$\mathscr{L} = \sqrt{\frac{1 - \gamma^2}{V^2 - 1}} \sqrt{\varepsilon_*^2 + (2\Im C_1)^2} \tag{135}$$

as well as its frequency, ω, can be considered as given. This can be used for the determination of $\Im C_1$ and the phase-transition front speed. The latter is connected with the frequency by the known relation as

$$\omega = Vk_1 = Vc\sqrt{V^2 - \gamma^2}/D \tag{136}$$

and hence

$$V = \sqrt{\sqrt{\frac{D^2\omega^2}{c^2} + \frac{\gamma^4}{4}} + \frac{\gamma^2}{2}}. \tag{137}$$

This equality allows the feeding wave amplitude to be rewritten as

$$\mathscr{L} = \sqrt{1 - \gamma^2}\frac{v}{D\omega}\sqrt{\varepsilon_*^2 + (2\Im C_1)^2}. \tag{138}$$

In addition, we note that the requirement $\mathrm{d}\varepsilon/\mathrm{d}\eta < 0$ $(\eta = 0)$ leads to the inequality as

$$2\Im C_1 < \varepsilon_*\sqrt{\frac{1 - V^2}{V^2 - \gamma^2}}. \tag{139}$$

The same inequality follows from the requirement $\varepsilon_0^+ = \varepsilon(\infty) < \varepsilon_*$. So, the solution is valid if

$$\varepsilon_* < 2|C_1| < \varepsilon_*\sqrt{\frac{1 - \gamma^2}{V^2 - \gamma^2}}. \tag{140}$$

Thus, for the intersonic case there exists a solution corresponding to the phase-transition process excited by a sinusoidal (microlevel) feeding wave. In this solution, the strain monotonically increases from $2\Im C_1 k_1/k_2$ ($\eta=\infty$) to ε_* ($\eta=0$). For given frequency, $\omega>0$, the lower bound of the feeding wave amplitude is

$$\mathscr{L}=\sqrt{1-\gamma^2}\frac{\varepsilon_* v}{D\omega}. \tag{141}$$

With an increase of the amplitude, the phase-transition front varies its position relatively the feeding wave in such a way that the phase-transition criterion is still satisfied.

4.5. Supersonic speed

As can be seen in Eq. (136), the phase-transition front speed increases with the frequency of the feeding wave and when the frequency passes a critical value, $\omega=\omega_1=c\sqrt{1-\gamma^2}/D$, the phase-transition wave becomes supersonic ($V>1$). In this case, k_1 remains to be the feeding wave number (the wave is placed at $\eta<0$ and $V_g>V$), while a new nonzero wave number, $k=\pm k_2$, $k_2=\sqrt{V^2-1}/D$, is the dissipative one (the wave is placed at $\eta>0$ and $V_g>V$).

To derive a solution for this case, one has to return to expressions (117)–(120). As can be seen on the real k-axis the product L_2L_- has zero points at $k=0$ and $\pm k_1$ as in the intersonic case and hence representation (132) is valid in the present case as well. The Fourier transforms are

$$\varepsilon_+=\frac{[0-\mathrm{i}(k-k_1)][0-\mathrm{i}(k+k_1)]}{[0-\mathrm{i}(k-k_2)][0-\mathrm{i}(k+k_2)]}\Phi_+,$$

$$\Phi_+=\frac{C_0}{0-\mathrm{i}k}+\frac{C_1}{0-\mathrm{i}(k-k_1)}+\frac{\overline{C_1}}{0-\mathrm{i}(k+k_1)},$$

$$\varepsilon_-=\Phi_-,\quad \Phi_-=\frac{C_0}{0+\mathrm{i}k}+\frac{C_1}{0+\mathrm{i}(k-k_1)}+\frac{\overline{C_1}}{0+\mathrm{i}(k+k_1)}. \tag{142}$$

The inverse Fourier transform and condition (124) lead to the following solution:

$$\varepsilon(\eta)=(2\Re C_1+C_0)\cos k_2\eta+2\Im C_1\frac{k_1}{k_2}\sin k_2\eta+C_0\frac{k_1^2}{k_2^2}(1-\cos k_2\eta)\quad(\eta>0),$$

$$\varepsilon(\eta)=2\Re C_1\cos k_1\eta+2\Im C_1\sin k_1\eta+C_0\quad(\eta<0),$$

$$\varepsilon(0)=2\Re C_1+C_0=\varepsilon_*. \tag{143}$$

Thus, if C_1 is given another constant, C_0, is defined by the last relation.

This solution can receive only conditional acceptance. Indeed, the condition $\varepsilon(+0)=\varepsilon_*\geq\varepsilon(\eta)$ ($\eta>0$) is satisfied if we put $\Im C_1=0$. However, the strain is represented here as a periodic function of η and this shows that it periodically reaches the critical value ahead the front. To avoid this drawback one can assume a low dissipation on the microlevel which can lead to a decrease in the wave amplitude with the distance from the front.

5. Conclusions

1. On the macrolevel, that is in the framework of the classical homogeneous model, the material can be in one or another phase, but, without invoking the microlevel, one cannot retrace the transformation process itself. This is the reason of the above-mentioned indefiniteness. The discrete chain and the HOD model give one a possibility to describe the process without any jump in the state. This results in uniqueness of the solution. In such a model, phase transition is accompanied or caused by high-frequency (microlevel) waves, and there exists exchange of energy between macro and micro levels. One can say that the microlevel waves are associated with 'internal' degrees of freedom since these waves do not result in macrolevel displacements.

2. The general strain–speed (61) or stress–speed (69) relations related to the macrolevel-associated solution are obtained. These relations are based on both the matter and momentum conservation laws, expressed in terms of the macrolevel, and the total dissipation found by means of the microlevel considerations. The speed-dependent 'dissipation function' $\mathscr{R}$ also depends on the microstructure. At the same time, expressions (69) are still valid in a general case where the microstructure influence is reflected by this function. Note, however, that such structure of the stress–speed relations is characteristic only for the bi-linear macrolevel stress–strain diagram (1). In the case of a general diagram, the incline of the phase-transition path on the stress–strain plane defines the speed just as in the first equality in (5), while the position of this path is defined by the dissipation rate.

3. In a homogeneous material model described by the equation of the fourth order (113), the Maxwell-type, dissipation-free phase transition is shown to exist. In this model, relations (69) are valid with $\mathscr{R}=1$ and the microlevel plays the role of a catalyst. In this case, a spontaneous phase transition in an initially stressed material can arise.

4. Microlevel solutions with a microlevel sinusoidal feeding wave can exist in both the discrete chain and the HOD model. In such solutions, the phase-transition front speed can exceed the sound speed in the softer phase.

5. Possible configurations of feeding and dissipative waves are defined by wave dispersion in both phases. The wave dispersion can be introduced by both the discrete chain and a HOD model of a homogeneous two-phase material. However, these models possess some distinguish features.

First, any dispersive relation related to the discrete chain or other periodic structure is a periodic function of the wave number and the corresponding group velocity is bounded. At the same time, in the HOD model considered in this paper, such relations cannot be periodic and the group velocity is unbounded: it tends to infinity together with the wave number. Note, however, that the bounded group velocity can be obtained if a higher-order derivative respective time is introduced, such that the HOD equation is still the wave equation.

Next, for a given frequency of the wave in the discrete chain, each wave number satisfying the dispersive relation corresponds to the same motion of the particles. This, however, is not valid for the homogeneous model where different waves can be excited by a sinusoidal external action. These differences can manifest themselves mainly in microlevel solutions.

We also note that in the HOD model, in contrast to the discrete chain, the manifestation of the dynamic amplification factor with the related phenomena cannot be revealed.

At last, a questionable point in the formulation of a HOD homogeneous model is that the physical grounds as relations between the model and a possible structure are usually shaded.

6. In the present paper, possible *steady-state* solutions for dynamic phase transition in some structured models were derived. For the discrete chain it was assumed that the motion of each particle is the same (but with the corresponding shift in time). However, the existence of other types of ordered processes cannot be excluded in advance.

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Appendix 1. Factorization of a positive function

Consider a function, $S(k)$, such that on the real k-axis

$$\operatorname{Arg} S(k) \equiv 0, \quad 0 < S(k) = S(-k) < \infty \quad (k \neq 0), \quad \lim_{k \to \pm\infty} S(k) = 1,$$

$$S(k) \sim C[(0 + \mathrm{i}k)(0 - \mathrm{i}k)]^{\nu} \quad (k \to 0,\ C = \text{const} > 0,\ \nu = \text{const}). \tag{A.1}$$

This function can be represented as a product

$$S = S_+ S_- \tag{A.2}$$

with

$$S_\pm(k) = \exp\left[\pm \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\ln S(\xi)}{\xi - k} \mathrm{d}\xi\right], \tag{A.3}$$

where $\Im k > 0$ for S_+ and $\Im k < 0$ for S_-. Note that if $S(k)$ is an analytical function of the complex variable k equality (A.2) implies an analytical continuation of the functions $S_\pm(k)$ defined on different half-planes of k. Otherwise, for $S(k)$ as a function of the real variable this equality implies a limit of $S_+(k)S_-(k)$ ($\Im k \to 0$).

For any real k where $0 < S(k) < \infty$, that is, at least, for $k^2 > 0$, it follows directly from Eq. (A.3) that

$$S_\pm(k) = \sqrt{S(k)}\mathrm{e}^{\pm \mathrm{i}\vartheta},$$

$$\vartheta = \vartheta(k) = -\frac{1}{2\pi} V.P. \int_{-\infty}^{\infty} \frac{\ln S(\xi)}{\xi - k} \mathrm{d}\xi. \tag{A.4}$$

Since S is an even function of k, one can conclude that $\vartheta(0) = 0$ if $\nu = 0$.

To find asymptotes of these functions for $k \to 0$ substitute in (A.3) $\xi = sx$ and $k = \pm \mathrm{i}s$, $s > 0$, for $L_\pm$, respectively. One has

$$\pm\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty}\frac{\ln S(\xi)}{\xi - k}\mathrm{d}\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\ln S(sx)}{x^2+1}\,\mathrm{d}x$$

$$\sim \frac{\ln C}{2} + \frac{v}{2\pi}\int_{-\infty}^{\infty}\frac{\ln(0+\mathrm{i}sx)}{x^2+1}\,\mathrm{d}x + \frac{v}{2\pi}\int_{-\infty}^{\infty}\frac{\ln(0-\mathrm{i}sx)}{x^2+1}\,\mathrm{d}x \quad (s \to 0). \tag{A.5}$$

The first integral can be calculated as a residual at $x = -\mathrm{i}$, while the second one — as a residual at $x = \mathrm{i}$. As a result, one finds

$$S_\pm(k) \sim \sqrt{C}(0+s)^v, \quad S_+(k) \sim \sqrt{C}(0-\mathrm{i}k)^v, \quad S_-(k) \sim \sqrt{C}(0+\mathrm{i}k)^v. \tag{A.6}$$

At the same time,

$$S_\pm = 1 \quad (|k| = \infty). \tag{A.7}$$

Appendix 2. The function $S(k)$ at zero points of $h(k)$ and $g(k)$

The function $S(k)$ defined by Eq. (39) is

$$S(k) = \frac{G_+(k)G_-(k)h(k)}{H_+(k)H_-(k)g(k)}. \tag{B.1}$$

Both the numerator and denominator of this fraction are equal to zero if $h(k) = 0$ $[g(k) = -2(1-\gamma^2)(1-\cos k) < 0]$ or $g(k) = 0$ $[h = 2(1-\gamma^2)(1-\cos k) > 0]$. To find the ratio, one can take into account the following relations ($k_v \neq 0$):

$$g(h_v) < 0, \quad h(g_v) > 0,$$

$$G_+(h_v)G_-(h_v) > 0, \quad H_+(g_v)H_-(g_v) < 0,$$

$$\lim_{k \to h_v}\frac{h(k)}{H_+(k)H_-(k)} = h_v^2 V(V_\mathrm{g} - V)\prod_{\mu \neq v}^{2l+1}\left[1 - \left(\frac{h_\mu}{h_v}\right)^2\right]^{-1} < 0,$$

$$\lim_{k \to g_v}\frac{g(k)}{G_+(k)G_-(k)} = g_v^2 V(V_\mathrm{g} - V)\prod_{\mu \neq v}^{2d+1}\left[1 - \left(\frac{g_\mu}{g_v}\right)^2\right]^{-1} < 0, \tag{B.2}$$

where the nondimensional group velocity, $V_\mathrm{g} = v_\mathrm{g}/c$ is

$$V_\mathrm{g} = \frac{\mathrm{d}\Omega}{\mathrm{d}k}, \quad V_\mathrm{g} < V \ (v = 2\mu - 1), \quad V_\mathrm{g} > V \ (v = 2\mu),$$

$$\Omega^2 = 2(1-\cos k) \quad \text{for } k = h_v, \quad \Omega^2 = 2\gamma^2(1-\cos k) \quad \text{for } k = g_v. \tag{B.3}$$

It follows

$$S(h_v) = \frac{G_+(h_v)G_-(h_v)h_v^2 V(V_g - V)}{g(h_v)} \prod_{\mu \neq v}^{2l+1} \left[1 - \left(\frac{h_\mu}{h_v}\right)^2\right]^{-1} > 0,$$

$$S(g_v) = \frac{h(g_v)}{H_+(g_v)H_-(g_v)g_v^2 V(V_g - V)} \prod_{\mu \neq v}^{2d+1} \left[1 - \left(\frac{g_\mu}{g_v}\right)^2\right] > 0. \tag{B.4}$$

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