Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of the Mechanics and Physics of solids
53 (2005) 1295–1313
JOURNAL OF THE MECHANICS AND PHYSICS OF SOLIDS
www.elsevier.com/locate/jmps

Crack in a material-bond lattice

L.I. Slepyan*

Department of Solid Mechanics, Materials and Systems, Tel Aviv University, Ramat Aviv 69978, Israel

Received 3 September 2004; received in revised form 14 February 2005; accepted 15 February 2005

Abstract

A square-cell lattice is considered consisting of point masses at its knots connected by linearly elastic bonds of nonzero density. Steady-state crack propagation is studied. A general relation between the knot mass and the bond mass is assumed; however, a detailed analytical examination is made for the material-bond lattice with no concentrated masses. It is assumed that the crack divides the bond in half, and the broken bonds remain in the lattice structure. In this model, the fracture energy of the bond is ignored, and hence the local fracture energy of the lattice is zero. The classical formulation in terms of critical stresses is accepted. The macrolevel energy release does exist. The macrolevel energy release rate as a function of the crack speed is found and compared with that for the massless-bond lattice of the same averaged density. While in the main, the dependencies for these two models are similar, there are some essential differences. For the lattice with no concentrated masses this function appears discontinuous. There exists a region where the crack speed is insensitive to the variation of the macrolevel energy release rate. The admissible regions of the crack speeds for the considered two lattice models differ greatly. For the massless-bond lattice this region is rather wide, while for the other it is very narrow. Mathematically, it is of interest that some details of the factorization depend on whether the ratio of the crack speed to the wave speed is rational and, if so, whether it can be represented as a ratio of two odd numbers.

Keywords: Dynamics; Fracture; Square-cell lattice; Integral transforms

*Tel.: +972 3 6406224; fax: +972 3 6407617.
E-mail addresses: leonid@eng.tau.ac.il, l.i.slepyan@rogers.com.

0022-5096/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jmps.2005.02.003

1. Introduction

Analytical solutions for crack dynamics in lattices obtained during last two decades are based on the model where point masses are connected by massless bonds (see Slepyan, 1981, and the following works summarized in Slepyan, 2002). This structure can be considered as related to a regular atomic lattice or as a discrete model of a continuous material with a periodic structure. However, with some of the latter applications in mind, a lattice with material bonds of nonzero density looks more adequate. In particular, in such a model, all the lattice mass can be assumed uniformly distributed as the network mass.

This limiting structure can be interpreted, in particular, as a simplest model of a porous material or a material becoming porous in a vicinity of the growing crack tip. It can also be considered as a model of a fibre-reinforced composite or a fabric. Besides, independently of possible applications, it is of interest to compare these two types of discretization: the lattice of point masses and the network of distributed masses. In this paper, the lattice is considered for a general relation between the knot mass and the bond mass, while detailed examination is made for the lattice with no concentrated masses.

In connection with the fracture energy relations discussed below, we note that for a lattice as a mechanical structure there exist two types of the local fracture energy. We can distinguish the *fracture energy of the bond* as the effective surface energy required for its break, and the *local fracture energy of the lattice* as the energy disappearing with the bond disintegration. The bond length is assumed much greater than its thickness. In this case, for a regular material the effective surface energy of the bond is negligible relative to its limiting strain energy, and the former is ignored.

In this sense, there is an essential difference between the problem formulation for the massless and material-bond lattices. Since the broken massless bonds possess no energy, their strain energy accumulated just before the break can be considered as the local fracture energy of the lattice. The broken bonds do not take part in the lattice dynamics any more, and the local fracture energy is irreparably lost. The global (macrolevel) energy release corresponding to the continuous approximation is spent in part on this breakage. The remaining part is the crack-speed-dependent energy of the structure-associated waves excited by the moving crack. The main goal of the lattice studies was to obtain the local-to-global energy release ratio as a function of the crack speed or, in other words, to obtain the crack-speed-dependent dissipation as the difference between the macrolevel energy release rate and the local fracture energy.

In the case of the material-bond lattice, it is important where the crack cuts the bonds. For simplicity it is assumed here that it divides the bond in half. In the formulation below, it is taken into account that both parts of the broken bond remain in the lattice structure. Thus, since there is no bond disintegration now, there is no local fracture energy of the lattice. This is in contrast to both the massless-bond lattice and a homogeneous-material model. Considering the material-bond lattice we thus return to the classical formulation of fracture in terms of critical stresses. However, the macrolevel energy release remains nonzero in this case as well. Now,

the main goal is to find how the crack speed depends on the macrolevel energy release rate and the critical stresses. In this connection, note that the local fracture energy for the massless-bond lattice can also be expressed in terms of the critical stresses since such a bond, in contrast to the material bond, is uniformly stressed.

To elucidate the difference in the local behavior of these two models, first consider the lattice-network without concentrated masses. After the moment when the bond breaks, relaxation step waves propagate to the knots, and then each part of the bond oscillates with a decreasing amplitude. This, in particular, influences the stress level in the middle of the next bond in the crack line. We now make a thought experiment: we redistribute the mass between the bonds and knots, while the bonds retain their elastic properties. As the ratio of the knot mass to the bond mass increases, the bond mass decreases, the oscillation frequency of the broken bond increases unboundedly, and the concentrated masses at the knots become insensitive to these oscillations. The only fact that remains important is that the broken bond does not connect the knots any more. So, in the massless-bond limit, the bond behaves as though it instantly disappears from the lattice when it breaks. Its limiting strain energy does not belong to the lattice any more. Thus this energy plays the role of the local fracture energy of the lattice. In contrast, in the case of the material bonds, their influence on the lattice dynamics may be essential at least during a period of time. In this latter case, it is reasonable to retain the broken bond in the lattice structure. Doing so and neglecting the surface energy associated with the bond fracture we come to the lattice model possessing no local fracture energy. The difference in fracture of the massless-bond and the material-bond lattices is illustrated in Fig. 1, where the square-cell lattices are shown intended to model anti-plane shearing.

In addition, from the dynamic behavior point of view, the material-bond lattice differs by a finite wave speed. In the case of the lattice with no concentrated masses, this is reflected, in particular, in discontinuity of the dependence of the macrolevel energy release rate on the crack speed. It is of interest that there exists a range of the

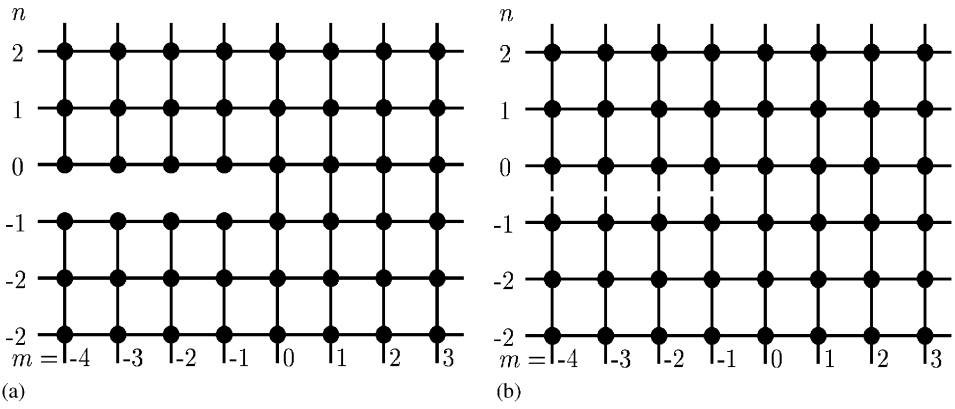

Fig. 1. Two lattice models with the crack at $m<0$. (a) The massless-bond lattice; the broken bonds disappear (they do not influence the lattice dynamics). (b) The material-bond lattice; the broken bonds do not leave the structure.

energy release where the crack speed remains constant as it is insensitive to the variation of the external action.

Mathematically, for the uniform-density network problem it appears that some factorization details depend on the ratio of the crack speed to the wave speed. They depend on whether the ratio is rational and, if so, whether it can be represented as a ratio of odd numbers.

2. Force–displacement relations for the bond

In this section, we establish connections between the tensile force and displacements at the bond ends. The results will further be used in the formulation of the dynamic equation for the material-bond lattice. Consider a lattice consisting of point masses, M, at its knots connected by linearly elastic bonds of nonzero density, $\varrho = \text{const} > 0$, as the mass per unit length. The bond length and stiffness are taken as the corresponding units. So, the wave speed in the bond is $c = \sqrt{1/\varrho}$, and the bond obeys the wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} = 0. \tag{1}$$

Here the displacements, u, and the x-axis are directed along the bond. The nondimensional tensile force in the bond $T(x,t) = \varepsilon(x,t)$, where ε is the strain. In terms of the Laplace transform on t with parameter s

$$u^{\mathrm{L}}(x,s) = A\sinh(sx/c) + B\cosh(sx/c). \tag{2}$$

For the bond, $0 < x < 1$, it follows that

$$u^{\mathrm{L}}(x,s) = \left[\cosh(sx/c) - \frac{\cosh(s/c)}{\sinh(s/c)}\sinh(sx/c)\right]u^{\mathrm{L}}(0,s) + \frac{\sinh(sx/c)}{\sinh(s/c)}\,u^{\mathrm{L}}(1,s). \tag{3}$$

The nondimensional tensile forces are

$$T^{\mathrm{L}}(0,s) = \frac{\partial u^{\mathrm{L}}(0,s)}{\partial x} = \frac{s/c}{\sinh(s/c)}\,[u^{\mathrm{L}}(1,s) - \cosh(s/c)u^{\mathrm{L}}(0,s)],$$

$$T^{\mathrm{L}}(1,s) = \frac{\partial u^{\mathrm{L}}(1,s)}{\partial x} = \frac{s/c}{\sinh(s/c)}\,[\cosh(s/c)u^{\mathrm{L}}(1,s) - u^{\mathrm{L}}(0,s)]. \tag{4}$$

We also need a solution of Eq. (1) with the boundary conditions as the displacement at $x = 0$ and the tensile force at $x = -\frac{1}{2}$ (these points correspond to the line of the lattice knots nearest to the crack, and to the crack line, respectively); it is

$$u^{\mathrm{L}}(x,s) = \{\tanh[(1/2)s/c]\sinh(sx/c) + \cosh(sx/c)\}u^{\mathrm{L}}(0,s) + \frac{(c/s)\sinh(sx/c)}{\cosh[(1/2)s/c]}\,T^{\mathrm{L}}(-1/2,s),$$

$$T^{\mathrm{L}}(x,s) = (s/c)\{\tanh[(1/2)s/c]\cosh(sx/c) + \sinh(sx/c)\}u^{\mathrm{L}}(0,s) + \frac{\cosh(sx/c)}{\cosh[(1/2)s/c]}\,T^{\mathrm{L}}(-1/2,s). \tag{5}$$

So

$$T^{\mathrm{L}}(0,s) = (s/c)\,\tanh[(1/2)s/c]u^{\mathrm{L}}(0,s) + \frac{T^{\mathrm{L}}(-1/2,s)}{\cosh[(1/2)s/c]} \tag{6}$$

and

$$u^{\mathrm{L}}(-1/2,s) = \frac{u^{\mathrm{L}}(0,s)}{\cosh[(1/2)s/c)]} - (c/s)\tanh[(1/2)s/c]\,T^{\mathrm{L}}(-1/2,s). \tag{7}$$

3. Equations for the square-cell lattice

The square-cell lattice is intended to model anti-plane shear (mode III); however, a hypothetic plane deformation with only vertical displacements—with the same formulation and results—can also be assumed. The latter viewpoint is used when it is more convenient; for example, when displacements and forces are shown in a plane figure. Note that in the long-wave approximation the lattice represents a continuous elastic medium whose shear modulus is equal to one, the density is equal to $M + 2\varrho$, and the wave speed is $c_0 = c/\sqrt{M_0 + 2}$, where $M_0 = M/\varrho$.

Initial positions of the lattice knots are defined by rectangular coordinates X, Y with $X = m = 0, \pm 1, \ldots$ and $Y = n = 0, \pm 1, \ldots$. The crack line is at $Y = -\frac{1}{2}$, Fig. 1. Displacements of the knots are denoted by $w_{m,n}(t)$.

Consider the dynamic equilibrium of a knot

$$M\,\mathrm{d}^2 w_{m,n}/\,\mathrm{d}t^2 = T_{m+,n} + T_{m-,n} + T_{m,n+} + T_{m,n-}, \tag{8}$$

where $T_{m+,n}, \ldots, T_{m,n-}$ are the forces acting on the considered mass from the right, from the left, from above and from below, respectively. Note that such a force, say $T_{m+,n}$, is formed by the displacements of the considered knot, m, n, and the next one, $m+1, n$. The influence of the latter on the former comes with a delay corresponding to the period of the wave propagation along the bond. Mathematically, it directly follows from the dependencies obtained in the previous section.

Using relations (4), where x is now considered as the local coordinate, we obtain ($n > 0$ or $n < -1$)

$$M_0(s/c)^2 w^{\mathrm{L}}_{m,n}(s) = \frac{s/c}{\sinh(s/c)}\,[w^{\mathrm{L}}_{m+1,n}(s) + w^{\mathrm{L}}_{m-1,n}(s) + w^{\mathrm{L}}_{m,n+1}(s) + w^{\mathrm{L}}_{m,n-1}(s) - 4w^{\mathrm{L}}_{m,n}(s)\cosh(s/c)], \tag{9}$$

where $\Re s > 0$. In terms of the discrete Fourier transform on m, this equation becomes

$$M_0(s/c)^2 w^{\mathrm{LF}}_{n}(s,k) = \frac{s/c}{\sinh(s/c)}\,\{w^{\mathrm{LF}}_{n+1}(s,k) + w^{\mathrm{LF}}_{n-1}(s,k) - w^{\mathrm{LF}}_{n}(s,k)[4\cosh(s/c) - 2\cos k]\}. \tag{10}$$

Denote

$$S = M_0(s/c)\sinh(s/c) + 4\cosh(s/c) - 2\cos k \tag{11}$$

and represent for $n \geqslant 0$

$$w_n^{\mathrm{LF}}(s,k) = A\lambda^n. \tag{12}$$

From Eq. (10) it can now be found that

$$\lambda = (1/2)[S - \sqrt{S^2-4}] = \frac{2}{S + \sqrt{S^2-4}}. \tag{13}$$

In this expression, the square root is assumed positive for $S>2$. In this case, $0<\lambda<1$ and $w_n^{\mathrm{LF}} \to 0$ when $n \to \infty$. For $S^2<4$ this function is complex with $|\lambda| = 1$. Analytic continuation of the square root function from $S>2$ to $S<-2$ leads to change of its sign; hence $-1<\lambda<0$ for $S<-2$ and $w_n^{\mathrm{LF}} \to 0$ $(n \to \infty)$ in this case as well.

Further consider the equation with respect to $w_0^{\mathrm{LF}}(s,k)$. It follows from Eq. (4) that

$$w_0^{\mathrm{LF}}\cosh(s/c) - w_{-1}^{\mathrm{LF}} = \frac{\sinh(s/c)}{s/c}\, T_0^{\mathrm{LF}}, \tag{14}$$

where T_0 is the tensile force in the bond $-1<Y<0$ at the knot $n=0$ $(Y=0)$, Fig. 2.

Referring to Eqs. (6) and (14) we now can represent Eq. (10) in the form

$$M_0(s/c)^2 w_0^{\mathrm{LF}}(s,k) = \frac{s/c}{\sinh(s/c)}\{w_1^{\mathrm{LF}}(s,k) - w_0^{\mathrm{LF}}(s,k)[3\cosh(s/c) - 2\cos k]\} - (s/c)w_0^{\mathrm{LF}}(s,k)\tanh[(1/2)s/c] - \frac{T_{1/2}^{\mathrm{LF}}(s,k)}{\cosh[(1/2)s/c]}. \tag{15}$$

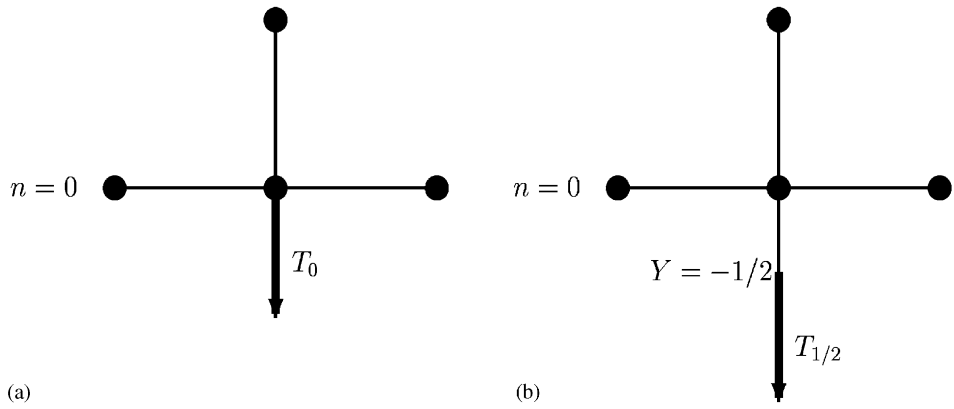

Fig. 2. The forces. (a) The tensile force in the lower bond at $Y = n = 0$, T_0, acting on the lattice knot $n = 0$ from below. (b) The tensile force in the bond at $Y = -\frac{1}{2}$, $T_{1/2}$, acting on the upper half of the lower bond from below.

This and the relation $w_1^{\mathrm{LF}} = \lambda w_0^{\mathrm{LF}}$ yield

$$w_0^{\mathrm{LF}}(s,k) = -T_{1/2}^{\mathrm{LF}}(s,k)\,\frac{c}{s}\sinh[(1/2)s/c]\left(\sqrt{\frac{r}{h}}-1\right) \tag{16}$$

with

$$h(s,k) = S-2 = M_0(s/c)\sinh(s/c) + 4[\cosh(s/c)-1] + 2(1-\cos k),$$

$$r(s,k) = S+2 = M_0(s/c)\sinh(s/c) + 4[\cosh(s/c)+1] - 2(1+\cos k). \tag{17}$$

Recall that $T_{1/2}$ is the tension force at $y=-\frac{1}{2}$ [in terms of the previous section, it is equal to $T(-\frac{1}{2},t)$], while w_0 is the displacement at $y=0$. The use of Eq. (7) allows w_0^{LF} to be expressed through the displacement at $y=-\frac{1}{2}$; we call the latter w_- since the corresponding original function is zero on the crack continuation. At the same time, $T_{1/2}^{\mathrm{LF}}$ is assumed to be zero at the crack; we call it T_+. In the corresponding conversions [with the use of Eq. (7)], the equation is neither multiplied nor divided by zero at $k=s=0$; this is important to justify the validity of the homogeneous solution obtained below. The following governing equation is obtained:

$$T_+ + L(s,k)w_- = 0 \tag{18}$$

with

$$L(s,k) = \frac{s}{c}\coth[(1/2)s/c]\sqrt{\frac{h(s,k)}{r(s,k)}}. \tag{19}$$

Note that $M_0(s/c)^2 = Ms^2$ and in the limit, $c\to\infty$ ($\varrho\to 0$), we obtain the corresponding relations for the lattice with massless bonds. In this case, Eq. (18) with

$$L(s,k) = 2\sqrt{\frac{h(s,k)}{r(s,k)}},\quad h(s,k) = Ms^2 + 2(1-\cos k),\quad r(s,k) = h+4\ \ (c=\infty) \tag{20}$$

is valid for both $w_-(s,k)$ for $y=0$ and $w_-(s,k)$ for $y=-\frac{1}{2}$. The only difference exists with respect to w_+ : $w_+\neq 0$ ($Y=0$), $w_+=0$ ($Y=-\frac{1}{2}$).

4. The steady-state problem

Further consider the corresponding steady-state problem where the tensile force and displacement at $Y=-\frac{1}{2}$ depend on $\eta = m-vt$ only. Here $v=\mathrm{const}$ is the crack speed, and the crack is at $\eta<0$. The solution to this problem is considered under the causality principle (see Slepyan, 2002), that is, it is considered as the limit of the solution to the corresponding transient problem with zero initial conditions. In the latter problem, the variables depend on η and t, where $t\to\infty$. In the following, a homogeneous problem for $Y\geqslant -\frac{1}{2}$ is studied with the boundary conditions at $Y=-\frac{1}{2}$

$$w=0\quad(\eta>0),\qquad T=0\quad(\eta<0). \tag{21}$$

Note that the first equality is a consequence of the assumed symmetry: $w(\eta,-Y-1)=-w(\eta,Y)$, while the latter is the condition for a free crack.

The required Fourier transform of the steady-state equation can be obtained from the double transform of the transient equation by the substitution $s=s'+\mathrm{i}kv,\ s'\to+0$ (see Slepyan, 2002, Section 3.3.2). Namely, consider the double convolution, $\Phi(t,m)$, of the linear operator $\mathscr{L}(t,m)$ and a function $f(t,m)$

$$\Phi(t,m)=\sum_{v=-\infty}^{\infty}\int_0^{\infty}\mathscr{L}(t-\theta,m-v)f(\theta,v)\,\mathrm{d}\theta. \tag{22}$$

The double transform, the continuous Laplace transform on t and the discrete Fourier transform on m with the parameters s and k, respectively, leads to the product

$$\Phi^{\mathrm{LD}}(s,k)=\mathscr{L}^{\mathrm{LD}}(s,k)f^{\mathrm{LD}}(s,k). \tag{23}$$

Represent $m=\eta+vt$ $(\eta=m-vt)$ and consider t and η as new independent variables. It is important that η is a continuous variable for any m. We assume that for any η the limits exist as

$$f(\eta)=\lim_{t\to\infty}f(t,\eta+vt),\quad \Phi(\eta)=\lim_{t\to\infty}\Phi(t,\eta+vt). \tag{24}$$

We now consider the continuous Fourier transform of these limiting functions (the transform is marked by the superscript F). The statement is valid that

$$\Phi^{\mathrm{F}}(k)=\int_{-\infty}^{\infty}\Phi(\eta)\mathrm{e}^{\mathrm{i}k\eta}\,\mathrm{d}\eta=\lim_{s'\to0}\mathscr{L}^{\mathrm{LD}}(s'+\mathrm{i}kv,k)f^{\mathrm{F}}(k). \tag{25}$$

So, we have the governing equation (18) with

$$L=L(s'+\mathrm{i}kv,k)=\mathscr{M}(s'+\mathrm{i}kv)Q(s'+\mathrm{i}kv,k),$$

$$\mathscr{M}(s'+\mathrm{i}kv)=\frac{s'+\mathrm{i}kv}{c}\coth[(s'+\mathrm{i}kv)/(2c)],$$

$$Q(s'+\mathrm{i}kv,k)=\sqrt{\frac{h(s'+\mathrm{i}kv,k)}{r(s'+\mathrm{i}kv,k)}} \tag{26}$$

and

$$h(s'+\mathrm{i}kv,k)=M_0[(s'+\mathrm{i}kv)/c]\sinh[(s'+\mathrm{i}kv)/c]+4\cosh[(s'+\mathrm{i}kv)/c]-2\cos k-2,\ r(s'+\mathrm{i}kv,k)=h+4. \tag{27}$$

Eq. (18) can be resolved using the Wiener–Hopf technique. First of all, the function $L(s'+\mathrm{i}kv,k)$ must be factorized, that is, represented as $L=L_+L_-$, where L_+ has no singular points and zeros at $\Im k\geqslant0$, while L_- has no such points at $\Im k\leqslant0$. Note that the function L has no such point on the real k-axis if $s'>0$ and hence the required factorization is not forbidden. Further note that for $v>0$ the function $\mathscr{M}$ in Eq. (26) has such points only above the real axis: $\Im k=\mathrm{i}s'/v$; this function thus belongs to L_-. So, the only problem is the factorization of the

function Q that has such points in the both half-planes (but not on the real axis if $s'>0$).

For any $M_0 \geqslant 0$ and $s'>0$ the statement is valid that

$$Q^2>0 \quad \text{if } \Im Q^2=0. \tag{28}$$

At the same time, for $M_0>0$, $Q^2(\pm\infty)=1$. So, when k runs from $-\infty$ to ∞, the function Q^2 runs on the complex plane along a closed contour that leaves the origin, $Q=0$, in the outer domain. It follows that

$$\operatorname{Ind} Q^2 = \operatorname{Ind} Q = 0 \quad (M_0>0). \tag{29}$$

For $M_0=0$, if Q is a periodic function, that is, if $V=v/c$ is a rational number, the inequality (28) evidences that $\ln Q$ is also a periodic function (with the same period as Q). These statements are essential for the factorization of the function.

4.1. The lattice with concentrated masses, $M>0$

In this case,

$$Q(s'+\mathrm{i}kv,k)=\sqrt{\frac{h(s'+\mathrm{i}kv,k)}{r(s'+\mathrm{i}kv,k)}}\to 1 \quad (k\to\infty), \quad \operatorname{Ind} Q(k)=0 \tag{30}$$

and we can represent

$$L_+ = Q_+(s'+\mathrm{i}kv,k), \quad L_- = \frac{s'+\mathrm{i}kv}{c}\coth[(s'+\mathrm{i}kv)/(2c)]Q_-(s'+\mathrm{i}kv,k),$$

$$Q_\pm(s'+\mathrm{i}kv,k)=\exp\left[\pm\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\ln Q(s'+\mathrm{i}\xi v,\xi)}{\xi-k}\,\mathrm{d}\xi\right], \tag{31}$$

where $\Im k>0$ for S_+ and vice versa.

The functions $Q_\pm$ have the following asymptotes:

$$Q_\pm(0+\mathrm{i}kv,k)\sim\left(\frac{1-(M_0+2)v^2/c^2}{4}\right)^{1/4}\mathscr{R}^{\pm1}\sqrt{0\mp\mathrm{i}k} \quad (k\to 0),$$

$$Q_\pm(s'+\mathrm{i}kv,k)=Q_0=1 \quad (k=\pm\mathrm{i}\infty) \tag{32}$$

with

$$\mathscr{R}=\exp\left(\frac{1}{\pi}\int_0^\infty\frac{\operatorname{Arg} Q(0+\mathrm{i}kv,k)}{k}\,\mathrm{d}k\right) \quad [\operatorname{Arg} Q(0,0)=0]. \tag{33}$$

For $s'=+0$ the function $h=h(0+\mathrm{i}h_v v,h_v)$ is real. Consider a point, $k=h_v$, where a plot of h crosses zero: for a small real ε, $h(\mathrm{i}h_v v+\varepsilon,h_v)h(\mathrm{i}h_v v-\varepsilon,h_v)<0$. This point corresponds to a jump of the argument of h by π (with the increase of k). Then the argument does not vary until the next point where h change its sign. Denote $h_v=h_v^+$ if the jump is positive (the argument increases); otherwise, $h_v=h_v^-$. In a regular case,

this means that $\partial h/\partial k \neq 0$ and

$$h_v = h_v^+ \quad \text{if} \quad \frac{\partial h(0+\mathrm{i}h_v v, h_v)}{\partial k}\,\Im h(s'+\mathrm{i}h_v v, h_v) < 0,$$

$$h_v = h_v^- \quad \text{if} \quad \frac{\partial h(0+\mathrm{i}h_v v, h_v)}{\partial k}\,\Im h(s'+\mathrm{i}h_v v, h_v) > 0, \tag{34}$$

where $s' > 0$. Clearly, these considerations are valid for the function r as well.

For the subcritical speed region, $0 < v < c/\sqrt{M_0+2}$, in a vicinity of $k=0$, $\operatorname{Arg} h(0+\mathrm{i}kv, k) = \operatorname{Arg} r(0+\mathrm{i}kv, k) = 0$. It now follows that the relation in Eq. (33) can be represented as

$$\mathscr{R} = \left(\prod_v \frac{h_v^- r_v^+}{h_v^+ r_v^-}\right)^{1/2}, \tag{35}$$

$h_v^\pm$ and $r_v^\pm$ are positive zeros of $h(0+\mathrm{i}kv, k)$ and $r(0+\mathrm{i}kv, k)$, respectively. For $M_0 > 0$ the number of these zeros is finite. Note that the points are meant where the function crosses zero; for example, a zero of an even order is not taken into account since it does not influence the argument.

4.2. *The lattice without concentrated masses, $M = 0$*

In this case, the functions Q and $\ln Q$ are periodic for any rational number $V = v/c$. The period is $T = 2\pi N$, where N is such a minimal integer that NV is integer. For example, if $V = 0.24 = \frac{6}{25}$ then $N = 25$. Here, for the required separation of $\pm$-parts of a periodic function, it is convenient to use the periodic version of the Cauchy type integral in the form (see Eatwell and Willis, 1982; Slepyan, 2002)

$$f_\pm = \pm\frac{1}{4\pi\mathrm{i}N}\int_{-\pi N}^{\pi N} f(\xi)\cot[(\xi-k)/(2N)]\,\mathrm{d}\xi \quad (\pm\Im k > 0). \tag{36}$$

The factorization follows as $Q = Q_+ Q_-$ with

$$Q_\pm(s'+\mathrm{i}kv, k) = \exp\left[\pm\frac{1}{4\pi\mathrm{i}N}\int_{-\pi N}^{\pi N} \ln Q(s'+\mathrm{i}\xi v, \xi)\cot\frac{\xi-k}{2N}\,\mathrm{d}\xi\right] \quad (\pm\Im k > 0). \tag{37}$$

$$Q_\pm(0+\mathrm{i}kv, k) \sim \left(\frac{1-2V^2}{4}\right)^{1/4} \mathscr{R}^{\pm 1}\sqrt{0 \mp \mathrm{i}k} \quad (k \to 0),$$

$$Q_\pm = Q_0 = \exp\left[\frac{1}{2\pi N}\int_0^{\pi N} \ln|Q(0+\mathrm{i}\xi v, \xi)|\,\mathrm{d}\xi\right] \quad (\Im k = \pm\mathrm{i}\infty),$$

$$\mathscr{R} = \exp\left[\frac{1}{2\pi N}\int_0^{\pi N} \operatorname{Arg} Q(0+\mathrm{i}\xi v, \xi)\cot\frac{\xi}{2N}\,\mathrm{d}\xi\right]. \tag{38}$$

The same statements concerning $\operatorname{Arg} h$ and $\operatorname{Arg} r$, as in the case $M>0$, are valid here. It now follows that

$$\mathscr{R}=\left(\prod_{v}\frac{\sin[h_v^-/(2N)]\sin[r_v^+/(2N)]}{\sin[h_v^+/(2N)]\sin[r_v^-/(2N)]}\right)^{1/2}. \tag{39}$$

It is of interest that if v/c is a ratio of two odd numbers, then $Q_0=1$. This follows from the fact that, in this case, $|Q(0+\mathrm{i}(k+\pi N)v,k+\pi N)|=1/|Q(0+\mathrm{i}kv,k)|$.

Lastly, the function $\mathscr{M}$ is

$$\mathscr{M}=\mathscr{M}_-=\frac{s'+\mathrm{i}kv}{c}\coth\frac{s'+\mathrm{i}kv}{2c}\to 2\quad(s'+\mathrm{i}kv\to 0),$$

$$\mathscr{M}_-\to\infty\quad(k\to-\mathrm{i}\infty). \tag{40}$$

5. Solution

There exists a macrolevel-associated solution of the homogeneous Eq. (18) ($s=0+\mathrm{i}kv$) corresponding to remote forces driving the crack. It is

$$T_+(k)=\frac{CL_+}{0-\mathrm{i}k},\quad w_-(k)=\frac{C}{(0+\mathrm{i}k)L_-}, \tag{41}$$

where the constant C reflects the existence of these forces; it is defined below in terms of the global energy release rate. From this and the above obtained asymptotes it follows that the tensile force at the moment when the bond breaks, $T(+0)=T_{\mathrm{c}}$, and the macrolevel energy release rate, G, are

$$T(+0)=T_{\mathrm{c}}=\lim_{k\to\mathrm{i}\infty}(-\mathrm{i}k)T_+(k)=CQ_0\quad(\eta=+0),$$

$$G=\lim_{p\to 0}p^2T_+(\mathrm{i}p)w_-(-\mathrm{i}p)=\tfrac{1}{2}C^2\mathscr{R}^2, \tag{42}$$

where the formula used for the energy release rate can be found, for example, in Slepyan (2002, p. 27). Finally,

$$G=R\frac{T_{\mathrm{c}}^2}{2},\quad R=\frac{\mathscr{R}^2}{Q_0^2}. \tag{43}$$

Note that $T_{\mathrm{c}}^2/2$ is the (nondimensional) strain energy of the bond in the case where it is uniformly stressed by the critical force.

In the static case, $v=0$,

$$Q_0=\sqrt{\sqrt{2}-1},\quad \mathscr{R}=1 \tag{44}$$

and

$$G=\frac{\sqrt{2}+1}{2}\,T_{\mathrm{c}}^2=(\sqrt{2}+1)G_0, \tag{45}$$

where G_0 is the critical strain energy of the bond (in statics). This result coincides with that found earlier for the massless-bond lattice (see Slepyan, 1981, 2002) as it must. Indeed, the inertia forces do not appear in statics, and hence the mass distribution is immaterial.

Finally, it should be emphasized that result (43) is obtained using the limit from the right, $\eta = +0$. This fact becomes significant for any point of discontinuity of R as a function of the crack speed. For a given V this means that the result corresponds to $t = m/V - 0 = m/(V + 0)$, that is, it corresponds to the right limit of R: $R(V + 0)$ if it exists (this also follows from the consideration of waves in the bonds, see Section 8). In particular, this conclusion allows us to give a correct interpretation of the below results for special points, $V = \frac{1}{2}, \frac{1}{4}$. As to the left limit of $R(V)$, it cannot be obtained so simply since $t = m/V + 0$ belongs to the after-the-break time when the considered tensile force is at zero. Practically, it is defined by the analytically determined values of R for some discrete points of V approaching the considered one from the left (see Section 8).

6. Two special points: $V = \frac{1}{2}, \frac{1}{4}$

At least at these points, for $M_0 = 0$ each of the functions, h and r, can be represented as a finite product 'on its zeros'. This allows the required factorization of Q to be obtained without invoking the Cauchy type integral. These points are of special interest since the resulting dependence in Eq. (43) has a jump discontinuity at these points (see Section 8 and Figs. 3 and 4). The straightforward solution for these points also serves to justify the results obtained by a more sophisticated general method.

6.1. $V = \frac{1}{2}$

In this case

$$h(\mathrm{i}kv, k) = 4\cos k/2 - 2(1 + \cos(k)) = 8\cos k/2 \sin^2(k/4) \tag{46}$$

and [see Eq. (27); $M_0 = 0$]

$$h(0 + \mathrm{i}kv, k) = 16 \sinh\frac{0 - \mathrm{i}k}{4} \sinh\frac{0 + \mathrm{i}k}{4} \sinh\frac{0 - \mathrm{i}(\pi - k)}{4} \sinh\frac{0 + \mathrm{i}(\pi + k)}{4}, \tag{47}$$

while for the function r we have

$$\begin{aligned} r(\mathrm{i}kv, k) &= 4\cos k/2 + 4(1 - \cos^2 k/2) \\ &= -4(\cos k/2 - \cos k_1/2)(\cos k/2 - \cos k_2/2) \end{aligned} \tag{48}$$

and

$$\begin{aligned} r(0 + \mathrm{i}kv, k) = 16 &\sinh\frac{0 - \mathrm{i}(k_1 - k)}{4} \sinh\frac{0 + \mathrm{i}(k_1 + k)}{4} \\ &\times \sinh\frac{-\mathrm{i}(k_2 + k)}{4} \sinh\frac{-\mathrm{i}(k_2 - k)}{4}, \end{aligned} \tag{49}$$

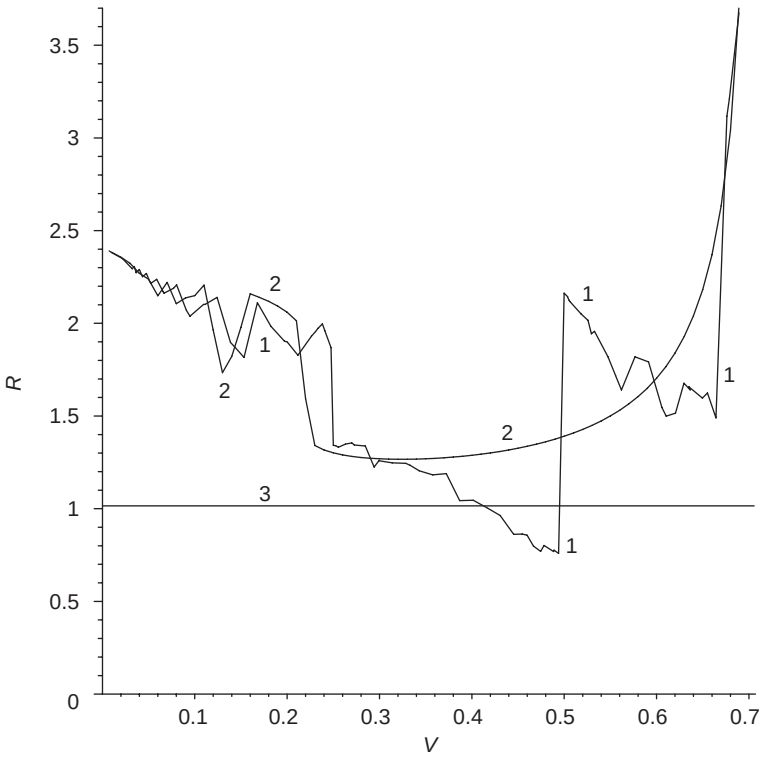

Fig. 3. Normalized macrolevel energy release rate, $R = 2G/T_c^2$, as a function of the crack speed, $V = v/c = v/(\sqrt{2}c_0)$. 1. Material-bond lattice without concentrated masses. 2. Massless-bond lattice of the same averaged density. 3. Continuous elastic material (here $R = 1$ is the global-to-local energy release ratio).

where

$$k_1 = \arccos\frac{1-\sqrt{5}}{2} \approx 4.474071519,$$

$$k_2 = 2\arccos\frac{1+\sqrt{5}}{2} \approx 2.122550124\,\mathrm{i}. \tag{50}$$

It follows that

$$Q_+^2(k) = \frac{\sinh[(0-\mathrm{i}k)/4]}{\sinh[-\mathrm{i}(k_2+k)/4]},$$

$$Q_-^2(k) = \frac{\sinh[(0+\mathrm{i}k)/4]\sinh[(0-\mathrm{i}(\pi-k))/4]\sinh[(0+\mathrm{i}(\pi+k))/4]}{\sinh[(0-\mathrm{i}(k_1-k))/4]\sinh[(0+\mathrm{i}(k_1+k))/4]\sinh[(-\mathrm{i}(k_2-k))/4]} \tag{51}$$

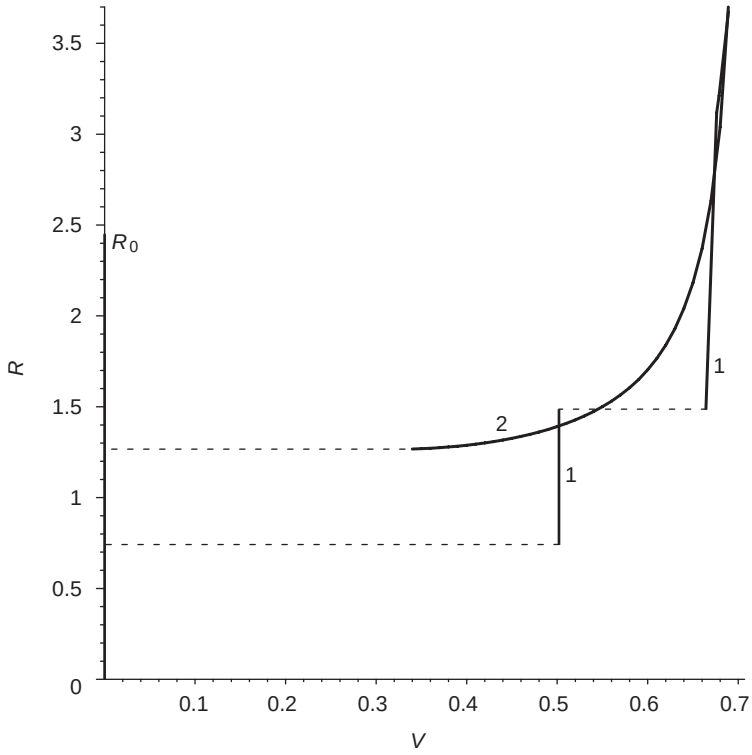

Fig. 4. Admissible branches of the macrolevel energy-release-rate dependencies (solid lines). 1. Material-bond lattice without concentrated masses. 2. Massless-bond lattice of the same averaged density. $R_0 = \sqrt{2}+1$ is the crack initiation value for both models.

and [see Eq. (38)]

$$Q_0^2 = Q_+^2(\mathrm{i}\infty) = \mathrm{e}^{\mathrm{i}k_2/4} \approx 0.5882298354,$$

$$\mathscr{R}^2 = \lim_{p\to 0} \frac{Q_+(\mathrm{i}p)}{Q_-(-\mathrm{i}p)} = \sqrt{2}\sin k_1/4 \approx 1.272019649. \tag{52}$$

Finally

$$R = [\mathscr{R}/Q_0]^2 = 2.162453470. \tag{53}$$

6.2. $V = \frac{1}{4}$

In this case

$$h(\mathrm{i}kv, k) = 4\cos k/4 - 2(1+\cos k) = 4(z - 4z^4 + 4z^2 - 1)$$
$$= 4(1-z)(4z^3 + 4z^2 - 1), \quad z = \cos k/4. \tag{54}$$

The zeros of this polynomial are defined as

$$z = z_v \quad v = 2, \ldots, 5; \quad z_2 = 1, \quad z_3 \approx 0.4196433776,$$

$$z_{4,5} = -0.7098216888 \pm 0.3031453648\,\mathrm{i}. \tag{55}$$

If one defines

$$\cos k_v/4 = z_v, \tag{56}$$

then the zeros of $h(\mathrm{i}kv, k)$ are

$$k_3 \approx 4.550975732, \quad k_{4,5} \approx 9.158065424 \mp 1.570428240\,\mathrm{i} \tag{57}$$

for $\Re k > 0$, and the symmetrical points for $\Re k < 0$. The function $h(0 + \mathrm{i}kv, k)$ can be represented as

$$h(0 + \mathrm{i}kv, k) = 32 \prod_{v=2}^{5} H_v,$$

$$H_2 = \sinh(0 - \mathrm{i}k)/8 \sinh(0 + \mathrm{i}k)/8,$$

$$H_3 = \cos k/4 - \cos k_3/4 = 2\sinh[(0 - \mathrm{i}(k_3 - k))/8]\sinh[(0 + \mathrm{i}(k_3 + k))/8],$$

$$H_4 = \cos k/4 - \cos k_4/4 = 2\sin[(k_4 - k)/8]\sin[(k_4 + k)/8],$$

$$H_5 = \cos k/4 - \cos k_5/4 = 2\sin[(k_5 - k)/8]\sin[(k_5 + k)/8]. \tag{58}$$

For the function $r(\mathrm{i}kv, k)$ we get

$$r(\mathrm{i}kv, k) = 4(\cos k/4 + \sin^2 k/2) = 4z(1 + 4z - 4z^3). \tag{59}$$

All zeros of the latter polynomial, $z = z_v$, $v = 6, \ldots, 9$, are real:

$$z_6 \approx= -0.8375654353, \quad z_7 \approx -0.2695944364, \quad z_8 \approx 1.107159872, \quad z_9 = 0. \tag{60}$$

As in the previous representation we define

$$\cos k_v/4 = z_v, \quad v = 6, \ldots, 9. \tag{61}$$

It follows that for $\Re k \geqslant 0$

$$k_6 = 10.25443220, \quad k_7 = 7.375072704, \quad k_8 = 1.835637216\,\mathrm{i}, \quad k_9 = 2\pi. \tag{62}$$

The function $r(0 + \mathrm{i}kv, k)$ can thus be represented as

$$r(0 + \mathrm{i}kv, k) = 16 \prod_{v=6}^{9} H_v,$$

$$H_6 = \cos k/4 - \cos k_6/4 = 2\sinh[(0 - \mathrm{i}(k_6 - k))/8]\sinh[(0 + \mathrm{i}(k_6 + k))/8],$$

$$H_7 = \cos k/4 - \cos k_7/4 = 2\sinh[(0 + \mathrm{i}(k_7 - k))/8]\sinh[(0 - \mathrm{i}(k_7 + k))/8],$$

$$H_8 = \cos k/4 - \cos k_8/4 = 2\sinh[(-\mathrm{i}k_8 - \mathrm{i}k)/8]\sinh[(-\mathrm{i}k_8 + \mathrm{i}k)/8],$$

$$H_9 = \cos k/4 - \cos k_9/4 = 2\sinh[(0 + \mathrm{i}(k - 2\pi))/8]\sinh[(0 + \mathrm{i}(k + 2\pi))/8]. \tag{63}$$

This yields

$$Q_+^2(k) = \frac{\sinh[(0 - \mathrm{i}k)/8]\sin[(k_4 - k)/8]\sin[(k_5 + k)/8]}{\sinh[(0 - \mathrm{i}(k - k_7))/8]\sinh[(0 - \mathrm{i}(k + k_7))/8]\sinh[(-\mathrm{i}k_8 - \mathrm{i}k)/8]},$$

$$Q_-^2(k) = \frac{\sinh[(0 + \mathrm{i}k)/8]\sinh[(0 - \mathrm{i}(k_3 - k))/8]\sinh[(0 + \mathrm{i}(k_3 + k))/8]}{\sinh[(0 + \mathrm{i}(k - k_6))/8]\sinh[(0 + \mathrm{i}(k + k_6))/8]\sinh[(-\mathrm{i}k_8 + \mathrm{i}k)/8]} \times \frac{\sin[(k_4 + k)/8]\sin[(k_5 - k)/8]}{\sinh[(0 + \mathrm{i}(k - 2\pi))/8]\sinh[(0 + \mathrm{i}(k + 2\pi))/8]}. \tag{64}$$

Now

$$Q_0^2 = Q_+^2(\mathrm{i}\infty) = \exp[\Im(k_5 - k_4 - k_8)/8] \approx 1.177216093,$$

$$\mathscr{R}^2 = \lim_{k\to 0} \frac{Q_+(\mathrm{i}p)}{Q_-(-\mathrm{i}p)} = \frac{\sin k_6/8}{\sqrt{2}\sin k_3/8 \sin k_7/8} \approx 1.579215613. \tag{65}$$

Lastly

$$R = [\mathscr{R}/Q_0]^2 \approx 1.341483201. \tag{66}$$

7. The corresponding massless-bond lattice

For the dynamic case, $v > 0$, it is of interest to compare the results for the lattice-network (43) and corresponding massless-bond lattice of the same averaged density, that is, the lattice with $c = \infty$ ($\varrho = 0$) and $M = 2$. From the corresponding relations in Sections 4.1 and 5 it follows that

$$R = \prod_{v=1} \frac{h_{2v} r_{2v-1}}{h_{2v-1} r_{2v}}, \tag{67}$$

where $h_n u$ are the positive zeros of the function $h(\mathrm{i}kv, k)$, $0 < h_v \leqslant h_{v+1}$, while r_v are the positive zeros of the function $r(\mathrm{i}kv, k)$, $0 < r_v \leqslant r_{v+1}$ with

$$h(\mathrm{i}kv, k) = 2(1 - \cos k) - 2k^2V^2, \quad r(\mathrm{i}kv, k) = 2(1 - \cos k) - 2k^2V^2 + 4. \tag{68}$$

In these relations, V is the nondimensional crack speed: $V = v/(\sqrt{2}c_0)$, where c_0 is the speed of a long wave in the lattice. So, as in the case of the material-bond lattice, the critical crack speed is $v = c_0$ ($V = 1/\sqrt{2}$).

8. Discussion

The function $R(V)$ for the lattice-network, $M=0$, is presented in Fig. 3, curve 1. It is plotted based on the values of $R(V)$ calculated for a discrete set of points where V is a ratio of two odd integers. Recall that in this case $Q_0=1$ and hence $R=\mathscr{R}^2$. In addition, to justify the results, the two values, $R(\frac{1}{2})$ and $R(\frac{1}{4})$, calculated in a different way in Section 6 are taken into account. A good agreement is found:

$$R(1/2+0)\approx 2.162453470, \quad R(1/4+0)\approx 1.341483201,$$

$$R(69/137)\approx 2.14393716, \quad R(25/99)\approx 1.340436849. \tag{69}$$

Also, in this figure, the function $R(V)$ is presented for the corresponding massless-bond lattice, curve 2. The calculations are performed using *Maple* (with the help of the Programming Guide by Monagan et al., 2001).

Clearly, in the main, these dependencies are similar. They show an essential difference between the criteria for the crack initiation, $R(0)$, and the crack propagation, $R(V_*)=R_{\min}$; however the corresponding values of the crack speed, V_*, are different: $V_*\approx 0.33$ ($v/c_0\approx 0.46$) for the massless-bond lattice, and $V_*=0.5$ ($v/c_0=1/\sqrt{2}$) for the other.

The main difference is that dependence 1, in contrast to dependence 2, is discontinuous. This is a result of the fact that, after each break, step waves propagate along the material bonds, and the reflection and refraction at the lattice knots changing the wave amplitude do not change the wave shape. In particular, if $V=\frac{1}{2}-0$ the step waves caused by the bond break reach the middle of the next bond at the moment of its break (the distance along the bonds is equal to two, while the crack advance is equal to one). These waves additionally stress the bond, and this leads to a dramatic decrease in the required macrolevel energy release rate in comparison with that required for $V=\frac{1}{2}+0$. Note that for $V=\frac{1}{2}-0$ not only the break of the nearest bond, but all the previous breaks create the discontinuity of the considered function; however, the number of knots where the wave refracts increases with the distance, and the wave amplitude decreases. The jump discontinuity at $V=\frac{1}{2}$ is found using the results obtained in Section 6 for some discrete point of V approaching $V=\frac{1}{2}$ from the left. It is found that $R(\frac{1}{2}+0)-R(\frac{1}{2}-0)\approx 1.40$ [$R(\frac{1}{2}-0)\approx R(0.4939759036)=0.7591886165$].

The above obtained analytical solution must be admissible in the sense that the stress in the crack-line bonds must be below the critical value at $\eta>0$. This condition but for any η also concerns all the bonds outside the crack line; otherwise, the considered single-line crack propagation does not exist (in this connection, see Marder and Gross, 1995; Slepyan, 2000). It follows that the branches where R generally decreases as V increases are inadmissible (see Slepyan and Ayzenberg-Stepanenko, 2004, pp. 1465–1466). This concerns the left range of crack speed, $0<V<V_*$ [$R(V_*)=R_{\min}$]. For the dependence 1 the region $\frac{1}{2}<V<0.664$ is also forbidden. In both cases, $R\to\infty$ when V tends to the critical value. [Recall that the critical crack speed is $v=c_0$ ($V=1/\sqrt{2}$).]

Assume that the horizontal bonds are strong enough, and the crack really propagates between the two parallel layers of the lattice knots (at $Y=-\frac{1}{2}$ in the case of the material-bond lattice without concentrated masses). In this case, the admissible regions of the crack speed for the considered two lattice models differ greatly, Fig. 4. Indeed, for the massless-bond lattice such a region is $0.33<V<1/\sqrt{2}$ $(0.46<v/c_0<1)$, while for the material-bond lattice without concentrated masses it consists of a point, $V=\frac{1}{2}$ $(v/c_0=1/\sqrt{2})$, and a very narrow range, $0.664<V<0.707$ $(0.94<v/c_0<1)$.

The fact is of special interest that due to this discontinuity there exists a range of R corresponding to the fixed crack speed, $V=\frac{1}{2}$ $(v/c_0=1/\sqrt{2})$. As the (normalized) global energy release rate increases from the minimal value, from $R\approx 0.76$, to $R\approx 1.49$ $[R(V_1)\approx 1.489860661, V_1\approx 0.6642335766)]$, the crack speed cannot change its value, $V=\frac{1}{2}$ (for $R>1.49$ it can jump to the right branch of the curve 1, $V>V_1$). So, at least in this range, $0.76<R<1.49$, the crack speed is insensitive to variation of the global energy release rate.

Thus, the function $R(V)$ was obtained for a series of $V\neq\frac{1}{2}$ and for $V=\frac{1}{2}+0$. For $V=\frac{1}{2}$ it is not defined uniquely; however, the inverse function, $V(R)$, in the admissible range, $0.76<R<1.49$, is uniquely defined: $V(R)=\frac{1}{2}$ for any R in this range.

Lastly, we recall that the critical strain energy of a uniformly stressed massless-bond is equal to the fracture energy of the 'corresponding' continuous material. So, for the massless-bond lattice the inequality $R(V)\geqslant 1$ is true since the energy dissipation cannot be negative. However, for the material-bond lattice this inequality does not hold. Indeed, in this case, the bond is not stressed uniformly at the moment when the stress at its center reaches the critical value. This is reflected in Fig. 3: in a range of V, it appears that $R(V)<1$, that is, the crack propagation requires less energy than in the case of the continuous material model. Note that in the continuous model there is no radiation in the subcritical steady-state regime; hence the local and the global energy release rates are equal, $R\equiv 1$, Fig. 3(3).

Acknowledgements

This research was supported by The Israel Science Foundation, Grant no. 1155/04, and ARO Grant no. 45584-MA.

References

Eatwell, G.P., Willis, J.R., 1982. The excitation of a fluid-loaded plate stiffened by a semi-infinite array of beams. IMA J. Appl. Mech. 29 (3), 247–270.

Marder, M., Gross, S., 1995. Origin of crack tip instabilities. J. Mech. Phys. Solids 43, 1–48.

Monagan, M.B., Geddes, K.O., Heal, K.M., Labahn, G., Vorkoetter, S.M., McCarron, J., DeMarko, P., 2001. Maple 7, Programming guide. Waterloo Maple Inc. [the book is represented at http://www.math.mcgill.ca/sysdocs/maple7/prguide.pdf].

Slepyan, L.I., 1981. Dynamics of a crack in a lattice. Sov. Phys. Dokl. 26, 538–540.

Slepyan, L.I., 2000. Dynamic factor in impact, phase transition and fracture. J. Mech. Phys. Solids 48, 927–960.

Slepyan, L.I., 2002. Models and Phenomena in Fracture Mechanics. Springer, Berlin.

Slepyan, L.I., Ayzenberg-Stepanenko, M.V., 2004. Localized transition waves in bistable-bond lattices. J. Mech. Phys. Solids 52, 1447–1479.