International Journal of Production Research
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tprs20

A new MILP approach for the facility process-layout design problem with rectangular and L/T shape departments

Yossi Bukchin[a] & Michal Tzur[a]
[a] Department of Industrial Engineering, Tel Aviv University, Tel Aviv 69978, Israel
Published online: 11 Jul 2014.

International Journal of Production Research, 2014
http://dx.doi.org/10.1080/00207543.2014.930534

Taylor & Francis
Taylor & Francis Group

A new MILP approach for the facility process-layout design problem with rectangular and L/T shape departments

Yossi Bukchin* and Michal Tzur

Department of Industrial Engineering, Tel Aviv University, Tel Aviv 69978, Israel

(*Received 21 March 2013; accepted 9 May 2014*)

We propose a new approach for the facility process-layout design problem, and introduce new mixed-integer linear programming (MILP) formulations for solving the problem. In this approach, we consider simultaneously, in a process-layout setting, the shape and location of the departments within the facility as well as the internal arrangement of the machines within the departments. Two models are suggested, the first assumes a rectangular shape of the departments and the second allows non-rectangular departments defined by an L/T shape. For the latter model, new constraints are developed to assure a correct design of the L/T shapes and to avoid irregular department shapes, and cuts are added to shorten the solution time. Finally, we conduct an extensive numerical study, in which we show the capabilities of both formulations in solving problems of medium size, and the superiority of the L/T department shape solutions over the rectangular department solutions.

Keywords: facility planning and design; process-layout; mixed integer linear programming; production; optimisation

1. Introduction and literature review

The facility layout problem (FLP) addresses the allocation of space to departments, where each department may contain machines, material handling equipment, aisles, storage areas, etc. The objective function is typically related to transportation costs or non-quantitative closeness performance measures between departments (Francis, McGinnis, and White 1992; Heragu 2006; Tompkins et al. 2010). Traditionally, most approaches for the FLP have followed the systematic layout planning (SLP), which was introduced in Muther (1955). As typical engineering design problems, SLP is based on a hierarchical approach, in which the area is first assigned to departments, and then the same approach is repeated for each department separately, to assign area to each of its components. This approach is also called a top-down approach (Meller, Kirkizoglu, and Chenb 2010), as the block layout problem is solved in the first stage and the detailed layout design is addressed in the second stage.

Most of the research presented in the literature deals with the first stage of the hierarchical approach. Various methods have been suggested to divide a given area among different departments, assuming that the detailed design will be done later on. These methods may be divided into two types. The first assumes a discrete area, where the total area is divided into relatively small squares. Then, various heuristics may be applied to allocate each square to a specific department, see, for example, Bozer, Meller, and Erlebacher (1994) and Meller and Bozer (1996). The second assumes a continuous area, where the borders of each department may be located at any point. The total area is divided among the departments using heuristics, see, for example, Liu and Meller (2007), Bozer and Wang (2012) and the references therein, or mathematical programming (e.g. Mixed Integer Programming) formulations. In the current research, we also assume a continuous area and therefore in the rest of the literature review, we refer mostly to such models.

The first mathematical programming formulation for the FLP was presented by Montreuil (1990). His model addresses the problem of positioning a set of departments in a rectangular facility, where the area size of each department is given, its final shape is rectangular and the dimensions are decision variables. Due to the area constraint, the resulting problem is non-linear. A periodic flow of material between each pair of departments is given, and the objective is to minimise the total transportation costs within the facility, assuming linearity of the costs with the amount of flow and the distance travelled.

Three difficulties are associated with the above formulation, all of which are addressed by our new approach and formulations. The first difficulty is a technical issue, which refers to the non-linearity of the departments' area-related

*Corresponding author. Email: bukchin@tau.ac.il

constraints. Castillo et al. (2005) indicate that replacing the non-linear constraint by a constraint on the perimeter, as suggested in Montreuil (1990), may yield a significant error on the area. Some progress, has been made with respect to this issue by Sherali, Fraticelli, and Melle (2003) by providing piecewise linear approximation methods. An alternative approach to tackle this difficulty was suggested by Castillo and Westerlund (2005), who presented a cutting plane representation of the actual area constraints, which guarantees ε% error of the required area in the optimal solution. Some constraints are added to avoid symmetric solutions and reduce run times, in particular for small values of ε. We overcome this difficulty by presenting a new model, which captures all possible configurations of each department, as described in Sections 3 and 4.

The second difficulty is related to the nature of the hierarchical approach mentioned above. This approach ignores the characteristics of the components to be located later on in each department, after its area has been fixed. In particular, it does not take into consideration the number and dimensions of the machines placed in each department, as we do in this paper. This issue is discussed in Meller, Kirkizoglu, and Chenb (2010) who suggest a formulation, which takes into account a limited number of possible configurations of each department, where in each configuration, the output and input points are pre-specified. Similar to previous research, only rectangular shape departments are considered. Their formulation also involves the sequence-pair concept for improving its tractability, as suggested by Liu and Meller (2007) and Meller, Chen, and Sherali (2007). Some works have considered other types of a more detailed design, but none of them considered the simultaneous department and machine layout planning as we do in this paper. Bock and Hoberg (2007) discuss the importance of a detailed layout. By using a grid-based layout structure, they introduce a detailed layout planning by simultaneously determining department arrangement and transportation paths. Similarly, Ho and Moodie (2000) develop a procedure that simultaneously designs the cell layout and flow path layout for an FCMS in which the flow path configuration is a tree. Kim and Goetschalckx (2005) introduce an integrated approach to obtain a block layout and (heuristically) the location of I/O points. Shebanie (2004) develops a genetic algorithm and a constructive heuristic to the simultaneous determination of the block layout problem and the detailed design problem of locating the input/output stations of departments. Finally, the recent work of Malmborg (2013) and of Leno et al. (2013) considered detailed layout with a bi-objective criterion. The former considers work centres and activity centres within them, where the bi-objective criterion simultaneously minimises material handling cost between work centres and maximises activity relationships within work centres. The latter designs the inter-cell layout and the flow path layout of MHS simultaneously, while minimising the weighted sum of total material handling cost and distance-weighted cost of total closeness.

The third difficulty is related to the rectangular shape constraint imposed on the departments. This restriction is often unnecessary in reality and excludes from consideration high-quality solutions. Bozer and Meller (1997) referred to this issue and envisioned that 'An ideal formulation should allow a set of alternative shapes (rectangular, L-shaped, U-shaped, etc.) for certain departments based on their characteristics'. Bukchin, Meller, and Liu (2006) consider the layout design of multiple given shapes of assembly cells, such as, I, L, T and U. They provide a layout that takes into account both the facility size and the material handling costs. Xie and Sahinidis (2008) address a layout problem of rectangles (facilities) of given size and shape and then extend the analysis to facilities of L/T-shapes, which are also of given size and shape, so that it remains to determine the location of those facilities. In our formulation, we allow the departments to have L/T shapes, where all possible such shapes are considered by the model.

Allowing L/T shape departments, rather than just rectangular shapes, provide significant flexibility to the design, and may better fit real-world environments. In particular, we mention two specific advantages: (1) reducing the material handling cost due to the less constrained problem; and (2) providing feasible solutions where no such solutions are available under the rectangular constraint. To demonstrate the feasibility issue, let us consider a 3-department layout

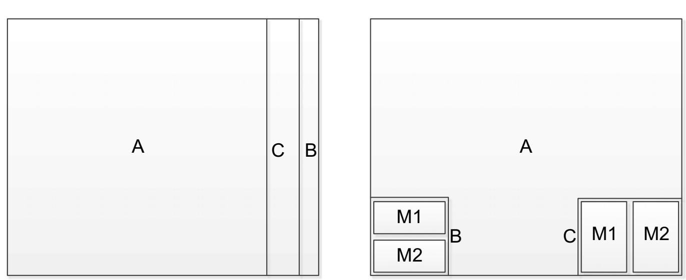

Figure 1. L/T vs. rectangular shape departments.

problem, where one department (A) is much larger than the other two (B, C), and the total area of the building is equal to the sum of areas of the three departments. Two possible layout solutions are shown in Figure 1, where the layout on the left is limited to rectangular departments, while the layout on the right allows L/T shapes (of department A in this figure). One can see that imposing the rectangular constraint results in irregular (long and narrow) shapes of the small departments. In case a regularity measure is used, such as the aspect ratio of the smallest enclosing rectangle (SER) of each department, the rectangular solution will likely be infeasible. The L/T solution, on the other hand, yields regular shapes of all departments, which may satisfy any reasonable value of the aspect ratio. Note that the aspect ratio may be a hard constraint in reality, resulting from the size and shape of the machines. For example, say that each department, B and C, contains two machines, as shown in the L/T layout solution. Clearly, the rectangular solution cannot provide a feasible solution, in which the machines are fully located within the departments.

In our proposed approach, a process layout is considered, where each department is responsible for a certain manufacturing process. To this end, a given number of identical machines need to be placed in each department. This type of layout, which is claimed by Yang, Chang, and Yang (2012) to be 'the most commonly used layout design', is characterised by high flexibility to changes in demands and product line, and is suitable to high variety of products produced in small quantities (Heragu 2006; Tompkins et al. 2010; Yang, Chuang, and Hsu 2011). The material flow in this process layout is mainly interdepartmental, as each product visits a single machine at each department. Consequently, we allocate a continuous area to departments, taking into account the internal setting of the departments (machines arrangement).

We propose new MILP formulations to solve the resulting combined problem (block/department layout and internal arrangement). The contributions of this paper are the following: First, we suggest a new approach for the facility process-layout design problem, in which the number and dimensions of the machines in each department, rather than the department area, are given as an input data. We formulate the resulting problem as a MILP problem, which finds simultaneously the location of the departments within the facility, as well as the internal arrangement of the machines within the departments. Moreover, we consider non-rectangular department shapes, defined by an L/T shape, and incorporate it in the MILP formulation. Thus, with this approach we are able to address *all* three difficulties mentioned above. Second, we perform a numerical study, which shows that when using our new approach, allowing non-rectangular department shapes, we obtain solutions that outperform design solutions that are limited to rectangular shapes. In particular, when allowing L/T shape departments, the objective function is improved by 18% on average (with a maximal improvement of 32.9%). This is despite the hardness of the resulting formulation and as a consequence the inability of a standard solver to reach an optimal solution for problems beyond a certain size. Overall, this paper provides an efficient and comprehensive new approach to solving the FLP and may form the basis for further developments along its lines.

The above results are achieved through the development of two new formulations, using new types of decision variables that were not previously used in the literature, and which contain information regarding the internal arrangement of the departments. A preliminary version of these formulations was presented in Bukchin and Tzur (2010). The first formulation addresses the rectangular department case, and the second extends it to the non-rectangular case. In the latter, a general L/T department shape is suggested, where each department consists of two rectangles, containing together the given number of machines. To avoid irregular department shapes, these rectangles are forced to be connected in a way that provides a shape similar to the letters L or T. Another way to avoid irregularity is developed for the L/T shaped departments, which can be viewed as an extension of the Aspect Ratio (AR) constraint, commonly used for rectangular shapes. In particular, we bound the AR of the SER as well as the perimeter of the SER.

The rest of the paper is organised as follows. In Section 2, we provide a detailed description of the problem and present pre-processing steps and preliminary results, which are useful to formulating and solving the problem. In Section 3, we provide a formulation of the first variant of the problem, in which the shape of all departments is restricted to be a rectangle, hence referred to as the rectangular shape problem. This section also reports on the results of a numerical study, which evaluated the computation capabilities of this formulation as well as the significant factors that affect it. In Section 4, we extend the formulation and analysis to the non-rectangular, L/T shaped departments, and present cuts that are designed to reduce the solution time and/or the optimality gap. An additional numerical study is presented in Section 5, which examines the computation capabilities of the extended model and compares the qualities of the rectangular and the L/T shape solutions. Finally, Section 6 concludes the paper.

2. Problem description and preliminaries

A set of departments has to be located within a rectangular facility of given length and width. A process layout is considered, in which a given set of identical machines has to be located within the area of each department. The number of machines and their dimensions are department-dependent. Finally, the flow of material between each pair of departments is given. Note that due to the process-layout type of the facility, material flow occurs only between machines of different departments and not between (identical) machines of the same department. Therefore, modelling the latter flow is not required for our problem.

The decisions that need to be made in this problem are where to position each department within the facility, subject to the above mentioned constraints. Note that the area size that each department occupies is not pre-determined, but depends on the internal positioning of its machines, which needs to be determined as well. The objective is to minimise the total transportation costs in the facility, assuming it is obtained by summing up the products of the flow between each pair of departments and the rectilinear distance between the centroids of these departments. Bozer and Meller (1997) suggested an alternative distance measure, called EDIST, which is based on the expected distance between the departments. This measure assumes a uniform distribution of the material flow over the department area. Castillo and Peters (2003) adopted this measure under the name DCTC (distributed centroid to centroid). One may claim that this measure sometimes reflects reality better than the regular centroid to centroid measure. In particular, a significant difference may exist between the two measures when rectangular departments of large aspect ratios are connected to each other along the long edge, or when non-rectangular departments are considered. We have chosen to use the traditional centroid to centroid measure; however, we examined our solutions under the EDIST measure as well.

As discussed in the Introduction, we consider either rectangular shape departments or L/T shape departments. The latter is the case where each department consists of two rectangles, connected in a way that forms a shape similar to the letter L or a T (or a rectangular shape, which may be viewed as a special case of either L or T), as explained in more details in Section 4. Thus, in both cases, possible rectangular shape departments need to be generated. This is achieved through a pre-processing step as described below.

We first present the problem's input parameters, based on which we perform the pre-processing step and generate additional parameters that are used in the problem formulation. In the description below and throughout the paper, we use interchangeably the terms *width* and *length* to refer to the dimension along the x-axis and the y-axis, respectively.

2.1 *Parameters of the problem*

I	number of departments
m_i	number of machines in department i, $i = 1, \ldots, I$
a_i	the width of a machine in department i in its original orientation (defined arbitrarily without loss of generality)
b_i	the length of a machine in department i in its original orientation
A	the width of the facility
B	the length of the facility
f_{ij}	flow between departments i and j

As mentioned above, we consider solutions to the layout problem, where department shapes are based on rectangles. In the basic case, the entire department has a rectangular shape, thus we first compute, for each department i, dimensions of possible rectangles that consist of m_i machines. We refer to each such possibility as a *configuration*. Later, we show how this method is extended to generate L/T configurations, which are made of two rectangles.

2.2 *Rectangles consisting of m_i* machines

We assume that within each rectangle, *all* machines may be placed either in their original orientation, or in a 90° rotated orientation. This leads to the following $2m_i$ possible configurations of rectangles, which consist each of m_i machines for department i.

Configuration r for $r = 1\ldots m_i$ is defined as a configuration, when all machines are positioned in their original orientation, and the row along the x axis with the largest number of machines contains r machines. Configuration r for $r = m_i + 1\ldots 2m_i$ is defined as a configuration, when the machines are in their rotated orientation, and the row along the x axis with the largest number of machines contains $r - m_i$ machines. Note that the dimensions of a rectangle are determined by the length of its longest row in the x-dimension and its longest row in the y-dimension. Thus, the dimensions of the above configurations are computed as follows:

A_{ir} is the width of a rectangle of department i, when choosing configuration r, $r = 1, \ldots, 2m_i$, where

$$A_{ir} = \begin{cases} a_i \cdot r & r = 1, \ldots, m_i \\ b_i \cdot (r - m_i) & r = m_i + 1, \ldots, 2m_i \end{cases}$$

B_{ir} is the length of a rectangle of department i, when choosing configuration r, $r = 1, \ldots, 2m_i$, where

$$B_{ir} = \begin{cases} \lceil \frac{m_i}{r} \cdot b_i \rceil & r = 1, \ldots, m_i \\ \lceil \frac{m_i}{r - m_i} \rceil \cdot a_i & r = m_i + 1, \ldots, 2m_i \end{cases}.$$

A demonstration of all configurations for $m_i = 4$, where $a_i = 2$ and $b_i = 1$, is given in Figure 2.

2.3 *Dominated configurations*

The above specified configurations include all $2m_i$ possibilities, arising from two possible orientations of the machines for each of possibilities of the largest number of machines in the width (x-axis) dimension. Note that the number of machines in one dimension determines the number of machines in the other dimension. However, some of these configurations are dominated by others with respect to the dimensions of the departments, and can be removed from consideration. Specifically, we remove a configuration if there exists another configuration with the same number of machines, whose length and width are smaller than or equal to it. This implies that the area of the former contains the area of the latter, so it is less desirable. Formally, we present the following dominance definitions for two rectangles that have the same number of machines.

Definition 1. Configuration r_g which consists of m_i machines for department i, is weakly dominated by configuration r_h which consists of m_i machines for department i if: $A_{ir_g} \geq A_{ir_h}$ and $B_{ir_g} \geq B_{ir_h}$. Configuration r_g is strongly dominated by configuration r_h if at least one of the above inequalities holds as a strong inequality.

In the example presented in Figure 2, we can see that configuration 3 is strongly dominated by configuration 2, similarly configurations 6 and 7 are strongly dominated by configuration 1, and configuration 8 is weekly dominated by configuration 2 (and vice versa).

Note that when configuration r_g is weakly dominated by configuration r_h, it is also the case that configuration r_h is weakly dominated by configuration r_g. In this case, one of these configurations may be removed from consideration. Dominance can also occur between an original and a rotated configuration. In the rotated configurations of the above example, all but the first configuration (1X8) is dominated by configurations of the original orientation.

2.4 *Irregular configurations*

We assume that department shapes need to satisfy some regularity conditions. Hence, configurations that violate these conditions may be removed. However, these considerations are made separately for the rectangular and the L/T shape departments, and thus, are described in the respective sections.

3. Rectangular shape departments

In this section, we require that each department will have a rectangular shape. We first develop the model and formulation in Sections 3.1 and 3.2 and then present a numerical study in Section 3.3.

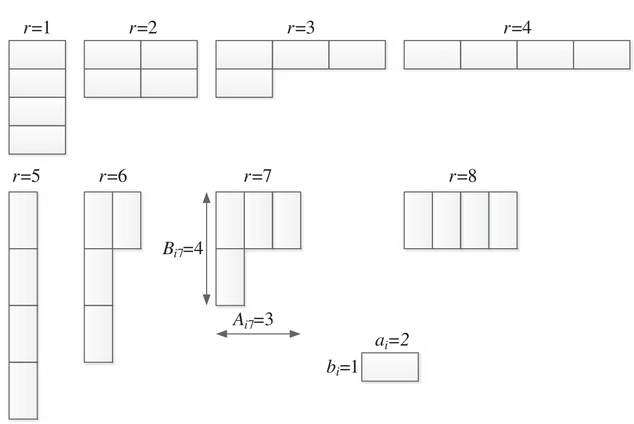

Figure 2. All rectangular configurations for four machines.

3.1 *Model and formulation*

For rectangular shape departments, a commonly used regularity condition in the literature is the aspect ratio (AR), a parameter that bounds the ratio of the two rectangular dimensions (width and length). Thus, adding the AR parameter to the input data of the rectangular shape problem, we use it in the pre-processing step to remove from consideration rectangle shapes, created as explained above, which violate this bound. Alternatively, it is possible to keep these rectangles and remove them later through a constraint in the formulation. However, removing them in the pre-processing step has the advantage that the corresponding decision variables are not needed. Subsequent to this step, removal of dominated configurations according to Definition 1 above is performed on the remaining configurations.

Let R_i be the set of non-dominated rectangle configurations consisting of m_i machines, which satisfy the AR conditions. As defined above, A_{ir} and B_{ir} denote the length and the width of configuration $r \in R_i$. However, since some configurations have been removed, we re-index the configurations, so the values of A_{ir} and B_{ir} do not necessarily follow their values in the pre-processing step. For each department i, one configuration from the set R_i has to be selected. We define appropriate decision variables and present a MILP formulation, referred to as MILP-R.

3.2 *Decision variables for MILP-R*

$w_{ir} = 1$,	if department i is arranged in configuration r, ($w_{ir} = 0$, otherwise), $i = 1, \ldots, I$, $r \in R_i$;
$z^x_{ij} = 1$,	if department i precedes department j along the x direction, i.e., i is to the left of j ($z^x_{ij} = 0$, otherwise), $i = 1, \ldots, I, j = 1, \ldots, I, j \neq i$;
$z^y_{ij} = 1$,	if department i precedes department j along the y direction, i.e., i is lower than j ($z^y_{ij} = 0$, otherwise), $i = 1, \ldots, I, j = 1, \ldots, I, j \neq i$;
x_i	the centroid of department i in the x direction, $i = 1, \ldots, I$;
y_i	the centroid of department i in the y direction, $i = 1, \ldots, I$;
x_{ij}	the horizontal distance (along the x-axis) between the centroids of departments i and j, $i = 1, \ldots, I$, $j = 1, \ldots, I, j \neq i$;
y_{ij}	the vertical distance (along the y-axis) between the centroids of departments i and j, $i = 1, \ldots, I$, $j = 1, \ldots, I, j \neq i$;
x^l_i	the lower end of department i along the x-axis, $i = 1, \ldots, I$;
x^h_i	the higher end of department i along the x-axis, $i = 1, \ldots, I$;
y^l_i	the lower end of department i along the y-axis, $i = 1, \ldots, I$;
y^h_i	the higher end of department i along the y-axis, $i = 1, \ldots, I$;

MILP-R:

$$\min \sum_{i=1}^{I} \sum_{j=i+1}^{I} f_{ij}(x_{ij} + y_{ij}) \tag{1}$$

s.t.

$$x_{ij} \geq x_i - x_j \quad \forall i, \forall j,\ j \neq i \tag{2}$$

$$x_{ij} \geq x_j - x_i \quad \forall i, \forall j \quad j \neq i \tag{3}$$

$$y_{ij} \geq y_i - y_j \quad \forall i, \forall j,\ j \neq i \tag{4}$$

$$y_{ij} \geq y_j - y_i \quad \forall i, \forall j\, j \neq i \tag{5}$$

$$x_i = \frac{x^h_i + x^l_i}{2} \quad \forall i \tag{6}$$

$$y_i = \frac{y^h_i + y^l_i}{2} \quad \forall i \tag{7}$$

$$x_i^h - x_i^l \geqslant \sum_{r \in R_i} A_{ir} w_{ir} \quad \forall i \tag{8}$$

$$y_i^h - y_i^l \geqslant \sum_{r \in R_i} B_{ir} w_{ir} \quad \forall i \tag{9}$$

$$\sum_{r \in R_i} w_{ir} = 1 \quad \forall i \tag{10}$$

$$x_i^h \leq x_j^l + A(1 - z_{ij}^x) \quad \forall i, \forall j, i \neq j \tag{11}$$

$$y_i^h \leq y_j^l + B(1 - z_{ij}^y) \quad \forall i, \forall j, i \neq j \tag{12}$$

$$z_{ij}^x + z_{ji}^x + z_{ij}^y + z_{ji}^y \geq 1 \quad \forall i, \forall j, i \neq j \tag{13}$$

$$x_i^h \leq A \quad \forall i \tag{14}$$

$$y_i^h \leq B \quad \forall i \tag{15}$$

$$w_{ir} \in \{0, 1\} \quad \forall i, \forall r \in R_i \tag{16}$$

$$z_{ij}^x, z_{ij}^y \in \{0, 1\} \quad \forall i, \forall j, i \neq j \tag{17}$$

$$x_i, y_i, x_i^l, x_i^h, y_i^l, y_i^h \geq 0 \quad \forall i \tag{18}$$

$$x_{ij}, y_{ij} \geq 0 \quad \forall i, \ \forall j, i \neq j \tag{19}$$

The objective function (1) minimises the sum of products of the flow parameter and the rectilinear distance between the centroids over all pairs of departments. Constraints (2)–(5) define the distances along the x- and y- axes between pairs of departments as the distance between the departments' centroids. Constraints (6)–(7) define the centroid of each department as the middle of the low and high ends of the department along both axes. Constraints (8) and (9) define the minimal distance between the high and low ends of each department to be no less than the width and length, respectively, of the chosen configuration for that department. Constraint (10) states that exactly one configuration from the set of configurations R_i should be chosen for each department. Constraints (11)–(12) ensure that the precedence relationship between departments i and j along the x and y directions, respectively, is satisfied with respect to department i's high end and department j's low end positions. Constraint (13) ensures that departments i and j do not overlap, by requiring that at least one precedence relationship exists between them. Constraints (14)–(15) require that departments are positioned within the facility dimensions. Finally, Constraints (16)–(17) and Constraints (18)–(19) define the problem variables to be binary and continuous, respectively.

Formulation MILP-R represents the first step of our new approach, and is innovative by itself in applying the concept of simultaneously selecting the location of the departments within the facility and the internal arrangement of the machines within each department. To test its practicality and gain insights on the main factors affecting its solution time, we conducted a numerical study with varying parameter values, as described in the next section.

3.3 *Numerical study*

A full factorial experiment was designed and executed, for identifying the size of the problem that can be solved in a reasonable amount of time and the effect of the problem parameters on the run time. The following factors and parameter values were considered:

A – number of departments: 8, 9 and 10 departments;

B – average number of machines in each department: 4 and 6 machines;

C – tightness of the facility, expressed by the ratio of the total available area and the net area consumed by the machines: 1.4 and 1.8;

D – density of the flow matrix: low (positive values only in the diagonal) and high (all matrix elements are positive). The former is denoted as *diagonal flow* while the latter as *full flow*;

E – maximal aspect ratio: 1.5 and 2.

Each dimension of each of the machines was randomly generated from a discrete uniform distribution between 1 and 5.

Thus, a total of 48 problems were solved, where the largest instance included 10 departments and 60 machines. The run time for each problem was limited to six hours. The results indicate that the run times were characterised by a large variability; 22 out of 48 problems were solved within one minute, 15 problems were solved within one minute to one hour, eight problems were solved in more than one hour and three problems (of 10 departments) could not be solved within the time limit. The detailed run times of all problems are presented in Appendix A.

The large variability of the run times can be partially explained by a statistical analysis of the results. The results show that the run time significantly increases (with significance level of 0.05) with the number of departments and the flow density. The average run time of 8-, 9- and 10-department problems, shown in Figure 3(a), is 22.2, 228.7 and 760.7 s, respectively. The effect of the flow density on the average run time is presented in Figure 3(b), where it is indicated that the average run time for problems with a diagonal flow is equal to 18.8 s, while for problems with a full flow this value goes up to 1569.8 s. Also note that the confidence intervals of the average run times (shown in the figure by the vertical lines) increase drastically with the values of these two factors. The analysis shows that the run time also increases with the maximal value of the aspect ratio, however, with a higher p-value of 0.07. For problems with low and high aspect ratio, the average run time is equal to 113.0 and 261.8 s, respectively.

The effect of the number of departments on the average run time is quite intuitive since the number of decision variables increases with the number of departments. As for the density effect, we believe the explanation is related to the number of decision variables, which have non-zero coefficients in the objective function. This number is very small for the diagonal flow problems, and much larger for the full flow problems. The less significant effect of the aspect ratio can be explained by noting that a large value of this factor leads to a larger feasible region.

Finally, it was interesting to find out that the results did not show any effect of the average number of machines on the run time. This is due to the fact that the average number of machines did not have a significant effect on the number of configurations created in the pre-processing step, which in turn determined the number of decision variables.

To summarise this part of the numerical study, we note that the results provide important insights regarding the expected run time of a problem as a function of its parameters. One can see that when a flow shop setting (diagonal flow) is considered along with a relatively strict aspect ratio, larger problems can be solved.

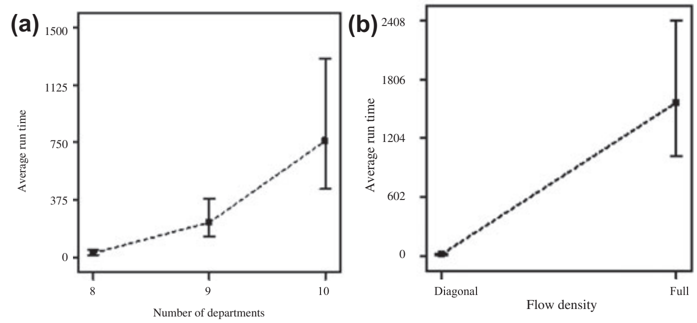

Figure 3. The effect of the number of departments and the flow density on the run time.

4. L/T shape departments

In this section, we relax the requirement to construct rectangular shape departments and consider departments of an L/T shape. We first develop the model and formulation in Sections 4.1–4.3 and describe cut developments and evaluation in Section 4.4. The performance of the L/T shape formulation is analysed in Section 5, by comparing to the rectangular solutions.

4.1. *Model and formulation*

An L/T shape department is created by connecting two rectangles in a way that forms a shape similar to the letter L or a T shape (or a rectangular shape, as a special case of each of the above two shapes). In particular, we have the following definition.

Definition 2. An L/T shape is a shape created by two rectangles such that one edge of one of the rectangles is attached along its entire length to one of the edges of the other rectangle.

Figure 4 presents a layout that consists of several L/T departments. The rounded rectangles within each department denote the machines, and the different shades distinguish between the two internal rectangles of each department. One can see that department 2 is rectangular (still consisting of two smaller rectangles), department 1 is of T shape and departments 3, 4 and 5 are of L shape.

We create L/T shape departments by defining, for each department i, all rectangles that consist of up to $(m_i - 1)$ machines. Then two rectangles are selected for each department, such that the total number of machines in both of them is m_i. We further make sure through constraints in the MILP formulation that the two rectangles are connected as described in Definition 2. To achieve this goal, we extend some of the definitions and procedures described in the previous section, as detailed next.

The generation of all possible rectangles for each department i is performed in the pre-processing step. When a rectangle consists of, say, l machines, all configurations with l machines are generated in exactly the same way as explained in Section 2 for m_i machines. Thus, $2l$ configurations are generated for all $l = 1, \ldots, m_i - 1$, each consisting of l machines, for a total of $2\sum_{l=1}^{m_i-1} l = (m_i - 1)(m_i - 2)$ configurations. L/T shapes with m_i machines are then obtained by selecting two configurations, each with $1 \le l \le m_i - 1$ machines, such that their sum is equal to m_i. In this section, we use the term configuration (and the index r) to denote a rectangle with $1 \le l \le m_i - 1$ machines, rather than the final department's shape.

The dimensions A_{ir} and B_{ir} have now the following meaning:
A_{ir} is the width of configuration r of department i,
B_{ir} is the length of configuration r of department i.

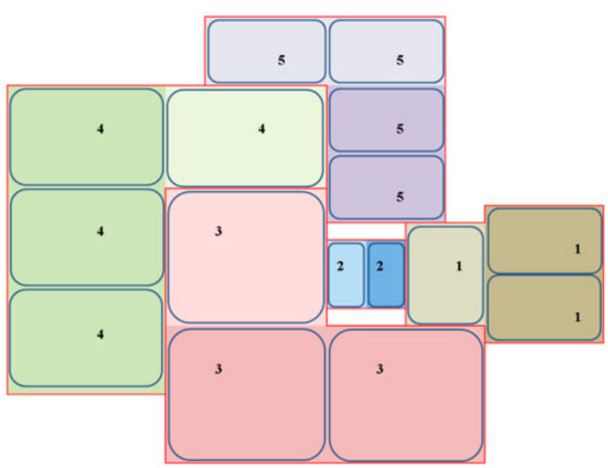

Figure 4. An example of an L/T layout solution.

4.2. *Irregular configurations*

No regularity condition specific to L/T-shapes is known in the literature; although several regularity conditions are defined with respect to general shapes (see Liggett and Mitchell 1981 and Bozer, Meller, and Erlebacher 1994). Here we define, in addition to the standard AR condition, a new regularity condition that is designed to handle specifically the L/T shapes considered here. One of the challenges in defining regularity conditions is our intent to include them as linear constraints in the MILP model.

The proposed regularity conditions refer to the entire department's width and length rather than the separate rectangles. For that purpose, we define the width and length of department i, LA_i and LB_i, respectively, as the dimensions of the SER of the department. This is illustrated in Figure 5 for three different departments, where the final shape of each department is shown along with its respective dimensions.

The first regularity condition we introduce is the standard aspect ratio (AR) condition, with respect to the entire department's dimensions. This condition precludes department shapes whose dimensions, even though referring to the SER (which typically has a more regular shape), differ from each other considerably.

A second condition is required since with non-rectangular departments, irregular shapes can be obtained even when satisfying the aspect ratio constraint (see, for example, department k in Figure 5). To handle that, we add a constraint, which refers to the sum of the width and length of the department. Specifically, we require that this sum will be bounded by a constant multiplied by twice the square root of the total area occupied by the machines. Note that the latter, i.e. $(m_i a_i b_i)^{0.5}$, represents the length of a minimal square that can contain the entire area of all the machines, and thus twice that value represents the minimal sum of length and width of any rectangle that contains the entire area of all the machines. The constant by which it is multiplied, which we denote by *PF* (perimeter factor), enables creating shapes other than squares, and in particular L/T shapes, which typically have 'extra' area within the smallest enclosing rectangle of the department, compared to the actual machine area. Requiring the above two conditions ensures that shapes of departments will not be unusually irregular, and this is achieved through the MILP formulation, where these requirements are added as constraints.

Due to these regularity constraints, we do not remove in the L/T case rectangles that are dominated according to Definition 1. This is since the final department shape consists of a combination of two rectangles, which needs to satisfy the above regularity constraints. Then, a dominated rectangle could have been a part of a feasible combination, whereas the rectangle that dominates it may violate some regularity condition when considered instead. Thus, removing the dominated rectangles may completely exclude some combinations from consideration, and therefore, this is not performed.

A final note refers to the objective function of the MILP formulation, which has to reflect the fact that each department is combined now of two rectangles (configurations), formed in an L/T shape. The centroid point along each axis of such a shape is obtained by an average of the two centroid points of the two configurations, weighted by the relative area of each configuration out of the entire area. In the MILP formulation, this raises a difficulty, since the respective area of each configuration is a decision variable by itself, so that multiplying it by the distance to another department (which is also a decision variable) would create a non-linear expression. We overcome this difficulty by keeping track (in the decision variables) of the number of machines included in each configuration, and using the respective number of machines as a measure for the respective area. This measure is accurate, up to the existence of an empty space that might be included in each configuration (for example, when a configuration has two machines in one row and one machine in the other, as may also occur in the rectangle case). Note that in either case, and thus also in the formulation below, the centroid point of a department may not reside within the area of the department. A further discussion of this and other issues regarding the objective function is provided in the numerical study section.

To achieve the above goal, we define for all $i = 1, \ldots, I, k = 1, \ldots, m_i - 1$:

R_{ik} = set of rectangle configurations for department i, which contain k machines.

We observe that when choosing for some department i two configurations, together consisting of m_i machines, they will be chosen from different R_{ik} sets, unless m_i is an even number, in which case two configurations may be selected

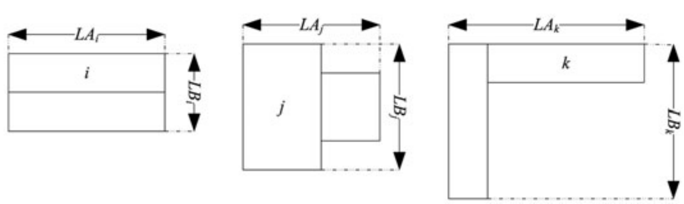

Figure 5. Examples of the department dimensions.

from the set $R_{im_i/2}$. The latter case requires slightly more work, but can be handled through variable duplication; here, we avoid this technicality and assume w.l.o.g. that m_i is an odd number.

We are now ready to present our MILP formulation for the L/T shape department case, referred to as MILP-LT.

4.3 *Model formulation for L/T-shape departments*

As discussed above, the MILP-LT formulation uses the following parameters in addition to those used in the MILP-R formulation:

AR aspect ratio of the SER of the L/T department (this parameter was used in MILP-R with respect to the rectangular shape departments, and now it is defined in a more general way)

PF perimeter factor of the SER of the L/T department

The decision variables are identical, generalised or expanded relative to those used in the MILP-R formulation:

x_i, y_i, x_{ij}, y_{ij} are Identical to those used in MILP-R;

The variables w_{ir} have the same meaning as previously, with respect to a rectangle/configuration, which is a part of a department:

w_{ir} = 1, if configuration r of department $i$is chosen (=0, otherwise), $i = 1, \ldots, I$, $r \in \bigcup_k \{R_{ik}\}$.

Analogous to the variables $x_i^h, x_i^l, y_i^h, y_i^l$ in the rectangle case, each variable is now expanded, to indicate the number of machines included in the configuration:

x_{ik}^h, x_{ik}^l are the higher and lower ends along the x-axis of a configuration of department i, which contains k machines, $i = 1, \ldots, I$, $k = 1, \ldots, m_i - 1$, respectively;

y_{ik}^h, y_{ik}^l, are defined similarly, along the y-axis;

For each department i, these variables would obtain (in the MILP formulation) the actual higher and lower ends for the two chosen values of k, and would be zero for all other values of k. Similarly, the variables described next would obtain the value zero if the respective configuration is not chosen.

The analogous to the precedence (non-overlapping) variables z_{ij}^x, z_{ij}^y are:

$z_{ik_1jk_2}^x = 1$, if a configuration of department i, which includes k_1 machines, precedes a configuration of department j $(i \neq j)$, which includes k_2 machines in the x direction (i.e., i with k_1 is to the left of j with k_2), $k_1 = 1, \ldots, m_i - 1, k_2 = 1, \ldots, m_j - 1$;

$z_{ik_1ik_2}^x$=1, if a configuration of department i, which includes k_1 machines, precedes a configuration of the *same* department i, which includes k_2 machines in the x direction (i.e., i with k_1 is to the left of i with k_2), $\forall i, k_1, k_2 = 1, \ldots, m_i - 1$;

$z_{ik_1jk_2}^y$ and $z_{ik_1ik_2}^y$ are defined similarly, along the y-axis; (i.e., i with k_1 is lower than j or i with k_2);

$\bar{x}_{ik}$ = the centroid of a chosen configuration of department i in the x direction, when this configuration contains k machines, $i = 1, \ldots, I$, $k = 1, \ldots, m_i - 1$;

$\bar{y}_{ik}$ is defined similarly, along the y-axis.

The rest of the decision variables refer to the structure of the L/T shape; they make sure that these shapes are created correctly:

$\tilde{A}_i$ = the width of the configuration, which is the longest (along the x-axis) of the two chosen configurations of department i;

$\tilde{B}_i$ = the length of the configuration, which is the longest (along the y-axis) of the two chosen configurations of department i;

$s_{ik}^x = 1$ when $\tilde{A}_i$ is determined by the width of a configuration, which contains k machines, $i = 1, \ldots, I$, $k = 1, \ldots, m_i - 1$;

$s_{ik}^y = 1$ when $\tilde{B}_i$ is determined by the length of a configuration, which contains k machines, $i = 1, \ldots, I$, $k = 1, \ldots, m_i - 1$;

s_i = 1 when the width of the department is equal to the largest width of the two chosen configurations, i.e., the two configurations of department i are one on top of the other (s_i = 0 when the length of the department is equal to the largest length of the chosen configurations, i.e., the two configurations of department i are besides each other), $i = 1, \ldots, I$;

LA_i and LB_i are the width and length, respectively, of the SER of the department. Thus, either LA_i obtains the value of $\tilde{A}_i$ (when the configurations are on top of each other) or LB_i obtains the value of $\tilde{B}_i$ (when the configurations are besides each other).

Given the above definitions, the MILP for the L/T department shapes is formulated as follows.

MILP-LT:

$$\min \sum_{i=1}^{I} \sum_{j=i+1}^{I} f_{ij}(x_{ij} + y_{ij}) \tag{20}$$

s.t.

$$x_{ij} \geq x_i - x_j, \quad \forall i, \forall j, j \neq i \tag{21}$$

$$x_{ij} \geq x_j - x_i \quad \forall i, \forall j, j \neq i \tag{22}$$

$$y_{ij} \geq y_i - y_j \quad \forall i, \forall j, j \neq i \tag{23}$$

$$y_{ij} \geq y_j - y_i \quad \forall i, \forall j\, j \neq i \tag{24}$$

$$x_i = \sum_{k=1}^{m_i-1} \frac{k\bar{x}_{ik}}{m_i} \quad \forall i \tag{25}$$

$$y_i = \sum_{k=1}^{m_i-1} \frac{k\bar{y}_{ik}}{m_i} \quad \forall i \tag{26}$$

$$\bar{x}_{ik} = \frac{x^h_{ik} + x^l_{ik}}{2} \quad \forall i, k = 1, \ldots, m_i - 1 \tag{27}$$

$$\bar{y}_{ik} = \frac{y^h_{ik} + y^l_{ik}}{2} \quad \forall i, k = 1, \ldots, m_i - 1 \tag{28}$$

$$\bar{x}_{ik} \leq A \sum_{r \in R_{ik}} w_{ir} \quad \forall i, k = 1, \ldots, m_i - 1 \tag{29}$$

$$\bar{y}_{ik} \leq B \sum_{r \in R_{ik}} w_{ir} \quad \forall i, k = 1, \ldots, m_i - 1 \tag{30}$$

$$x^h_{ik} - x^l_{ik} \geqslant \sum_{r \in R_{ik}} A_{ir} w_{ir} \quad \forall i,\ k = 1, \ldots, m_i - 1 \tag{31}$$

$$y^h_{ik} - y^l_{ik} \geqslant \sum_{r \in R_{ik}} B_{ir} w_{ir} \quad \forall i,\ k = 1, \ldots, m_i - 1 \tag{32}$$

$$\sum_{k=1}^{m_i-1} \sum_{r \in R_{ik}} k w_{ir} = m_i \quad \forall i \tag{33}$$

$$\sum_{k=1}^{(m_i-1)/2} \sum_{r \in R_{ik}} w_{ir} = 1 \quad \forall i \tag{34}$$

$$\sum_{k=(m_i-1)/2+1}^{m_i-1} \sum_{r \in R_{ik}} w_{ir} = 1 \quad \forall i \tag{35}$$

$$x^h_{ik_1} \leq x^l_{jk_2} + A(1 - z^x_{ik_1jk_2}) \quad \forall i, \forall j \neq i, k_1 = 1, \ldots, m_i - 1,\ k_2 = 1, \ldots, m_j - 1 \tag{36}$$

$$x^h_{ik_1} \leq x^l_{i(m_i-k_1)} + A(1 - z^x_{ik_1i(m_i-k_1)}) \quad \forall i,\ k_1 = 1, \ldots, m_i - 1 \tag{37}$$

$$y^h_{ik_1} \leq y^l_{jk_2} + B(1 - z^y_{ik_1jk_2}) \quad \forall i, \forall j \neq i, k_1 = 1, \ldots, m_i - 1,\ k_2 = 1, \ldots, m_j - 1 \tag{38}$$

$$y^h_{ik_1} \le y^l_{i(m_i-k_1)} + B(1 - z^y_{ik_1 i(m_i-k_1)}) \quad \forall i, k_1 = 1, \ldots, m_i - 1 \tag{39}$$

$$z^x_{ik_1jk_2} + z^x_{jk_2ik_1} + z^y_{ik_1jk_2} + z^y_{jk_2ik_1} \ge \sum_{r \in R_{ik_1}} w_{ir} + \sum_{r \in R_{jk_2}} w_{jr} - 1 \quad \forall i, \forall j \ne i, k_1 = 1, \ldots, m_i - 1, k_2 = 1, \ldots, m_j - 1 \tag{40}$$

$$z^x_{ik_1 i(m_i-k_1)} + z^x_{i(m_i-k_1)ik_1} + z^y_{ik_1 i(m_i-k_1)} + z^y_{i(m_i-k_1)ik_1} \ge \sum_{r \in R_{ik_1}} w_{ir} + \sum_{r \in R_{i(m_i-k_1)}} w_{ir} - 1 \quad \forall i, k_1 = 1, \ldots, (m_i - 1)/2 \tag{41}$$

$$x^h_{ik_1} - x^l_{i(m_i-k_1)} \le \sum_{k=1}^{m_i-1} \sum_{r \in R_{ik}} A_{ir} w_{ir} \quad \forall i,\ k_1 = 1, \ldots, m_i - 1 \tag{42}$$

$$y^h_{ik_1} - y^l_{i(m_i-k_1)} \le \sum_{k=1}^{m_i-1} \sum_{r \in R_{ik}} B_{ir} w_{ir} \quad \forall i, k_1 = 1, \ldots, m_i - 1 \tag{43}$$

$$\tilde{A}_i \ge \sum_{r \in R_{ik}} A_{ir} w_{ir} \quad \forall i, k = 1, \ldots, m_i - 1 \tag{44}$$

$$\tilde{B}_i \ge \sum_{r \in R_{ik}} B_{ir} w_{ir} \quad \forall i, k = 1, \ldots, m_i - 1 \tag{45}$$

$$\tilde{A}_i \le \sum_{r \in R_{ik}} A_{ir} w_{ir} + A(1 - s^x_{ik}) \quad \forall i,\ k = 1, \ldots, m_i - 1 \tag{46}$$

$$\tilde{B}_i \le \sum_{r \in R_{ik}} B_{ir} w_{ir} + B(1 - s^y_{ik}) \quad \forall i, k = 1, \ldots, m_i - 1 \tag{47}$$

$$\sum_{k=1}^{m_i-1} s^x_{ik} \ge 1 \quad \forall i \tag{48}$$

$$\sum_{k=1}^{m_i-1} s^y_{ik} \ge 1 \quad \forall i \tag{49}$$

$$LA_i \le \tilde{A}_i + A(1 - s_i) \quad \forall i \tag{50}$$

$$LB_i \le \tilde{B}_i + Bs_i \quad \forall i \tag{51}$$

$$LA_i \ge x^h_{ik} - x^l_{im_i-k} \quad \forall i,\ k = 1, \ldots, m_i - 1 \tag{52}$$

$$LA_i \ge x^h_{ik} - x^l_{ik} \quad \forall i,\ k = 1, \ldots, m_i - 1 \tag{53}$$

$$LB_i \ge y^h_{ik} - y^l_{im_i-k} \quad \forall i,\ k = 1, \ldots, m_i - 1 \tag{54}$$

$$LB_i \ge y^h_{ik} - y^l_{ik} \quad \forall i,\ k = 1, \ldots, m_i - 1 \tag{55}$$

$$LA_i \le \sum_{k=1}^{m_i-1} \sum_{r \in R_{ik}} A_{ir} w_{ir} \quad \forall i \tag{56}$$

$$LB_i \leq \sum_{k=1}^{m_i-1} \sum_{r \in R_{ik}} B_{ir} w_{ir} \quad \forall i \tag{57}$$

$$LA_i \leq AR \cdot LB_i \quad \forall i \tag{58}$$

$$LB_i \leq AR \cdot LA_i \quad \forall i \tag{59}$$

$$LA_i + LB_i \leq PF \cdot 2(m_i a_i b_i)^{0.5} \quad \forall i \tag{60}$$

$$w_{ir} = 0, 1 \quad \forall i, \ \forall r \in R_{ik}, \quad \forall k \tag{61}$$

$$z^x_{ik_1jk_2}, z^y_{ik_1jk_2} = 0, 1 \quad \forall i, \ \forall j \neq i, \ k_1 = 1, \ldots, m_i - 1, k_2 = 1, \ldots, m_j - 1 \tag{62}$$

$$s_i = 0, 1 \quad \forall i \tag{63}$$

$$s^x_{ik}, s^y_{ik} = 0, 1 \quad \forall i, k = 1, \ldots, m_i - 1 \tag{64}$$

$$x_i, y_i \geq 0 \quad \forall i \tag{65}$$

$$x_{ij}, y_{ij} \geq 0 \quad \forall i, \forall j, i \neq j \tag{66}$$

$$\bar{x}_{ik}, \bar{y}_{ik}, x^h_{ik}, x^l_{ik}, y^h_{ik}, x^l_{ik} \geqslant 0 \quad \forall i, k = 1, \ldots, m_i - 1 \tag{67}$$

The objective function (20) minimises the sum of products of the flow parameter and the rectilinear distance between the centroids over all pairs of departments. In the rest of the explanation of MILP-LT, we refer to the constraints by their equation number and classify them into groups.

Constraints (21)–(30) – distance definition: the distances between the centroids are obtained by (21)–(24) while the centroids of the departments are calculated in (25)–(26), based on the centroids of the chosen configurations, defined in (27)–(28). (Note the dependency on the number of machines in (25)–(26)). The centroids of the non-chosen configurations are set to zero in (29)-(30), so they don't affect the centroid point calculations.

Constraints (31)–(32) – configuration dimensions: the minimal dimension values (along both axes) of each of the two chosen configurations are set. (Exact dimensions and positions of the entire department are determined through other constraints, detailed later.)

Constraints (33)–(35) – number of machines: assure that for each department i the chosen configurations contain in total exactly m_i machines, where one configuration includes up to $(m_i - 1)/2$ machines, and the other contains between $(m_i - 1)/2 + 1$ and $m_i - 1$ machines. Note that Constraints (34)–(35) are implied by (33), but are included in the formulation to represent the clear choice of one configuration from each of the two sets.

Constraints (36)–(41) – no overlapping: required between all configurations, within and among departments. Constraints (36)–(39) address the precedence of the chosen configurations along the x and y axes. Note that in (37) and (39), as well as other constraints presented in the sequel that refer to the configurations of the same department, we make use of the fact that when one chosen configuration has k_1 machines, the other must have $m_i - k_1$ machines. Constraints (40)–(41) prevent overlapping and define the relative position of all chosen configurations. Note that the right hand side of each of these constraints could have been simply one, however the constraint is effective only when the respective configurations are chosen (and redundant otherwise). In (41), $\sum_{r \in R_{ik_1}}$ implies that $\sum_{r \in R_{i(m-k_1)}}$ as well, thus it suffices to require this constraint for $k_1 = 1, \ldots, (m_i - 1)/2$ only.

Constraints (42)–(57) – *department position and dimensions*: Constraints (42)–(43) assure that chosen configurations belonging to the same department are attached to each other. Constraints (44)–(49) set for each department the value of the longest dimension (obtained from one of the two chosen configurations) along each axis. These values are used in determining the length along both dimensions of the SER of each department in (50)–(51), which is used in turn in setting the exact position of the configurations in (52)–(55). Note that (50)–(51) enforce the L/T shape of the departments according to Definition 2, as they assure that at least one edge of a configuration belonging to a certain department will be attached along its entire length to the other configuration of that department. In (56) and (57), we make sure that none of the dimensions of the SER exceeds the sum of the two configurations along that dimension.

Constraints (58)–(60) – *regularity constraints*: Constraints (58)–(59) correspond to the standard AR constraint, however, with respect to the SER of each department, while (60) represents the newly defined regularity constraint, limiting the sum of both dimensions of the SER.

Finally, (61)–(67) – *decision variables definitions*: these constraints classify the decision variables as either continuous or binary.

4.4 *Developments and evaluation of valid inequalities*

Since the above formulation involves a considerable number of binary variables, its solution time when using a general MILP solver (Cplex 12.2) is quite long and in many cases the optimal solution is not achievable in a reasonable amount of time. Thus, we developed a set of valid inequalities (that may form cuts), whose purpose is to shorten the problem solution time and/or reduce the optimality gap obtained at the end of a time-constrained run.

However, adding valid inequalities (VI) does not always have a positive effect on the solution time and/or optimality gap, since handling the additional constraints requires additional efforts (computer resources), which may or may not offset the advantages of having a reduced solution space to the problem. Therefore, we conducted an ad hoc experiment whose purpose is to determine which, if at all, valid inequality sets, are useful when added to the above problem formulation. Here, we present only those inequality sets that we found helpful, and which we included in the formulation when conducting the numerical experiment discussed in the next section.

Valid Inequality – Set 1 (VI-1):

$$z^x_{ik_1i(m_i-k_1)} + z^x_{i(m_i-k_1)ik_1} \geq \sum_{r \in R_{ik_1}} w_{ir} - s_i \quad \forall i, k_1 = 1, \ldots, (m_i - 1)/2 \tag{68}$$

$$z^y_{ik_1i(m_i-k_1)} + z^y_{i(m_i-k_1)ik_1} \geq \sum_{r \in R_{ik_1}} w_{ir} - 1 + s_i \quad \forall i, k_1 = 1, \ldots, (m_i - 1)/2 \tag{69}$$

VI-1 is a constraint that strengthens constraint (41). It refers to the relative position of the two configurations constructing department i. In particular, it is based on the observation that given the value of s_i, it becomes known whether at least one of the two z^x (or the two z^y) variables of department i must be equal to one. For example, when $s_i = 0$, so that the two configurations of department i are besides each other, one of the two z^x variables must be one. Moreover, this applies only to those z^x variables whose configurations were chosen, as represented by the w-variables on the RHS of (68). Similarly, (69) becomes active with respect to the z^y variables when $s_i = 1$.

Valid Inequality – Set 2 (VI-2):

$$\sum_{k_1=1}^{m_i-1} z^x_{ik_1i(m_i-k_1)} = 1 - s_i \quad \forall i \tag{70}$$

$$\sum_{k_1=1}^{m_i-1} z^y_{ik_1i(m_i-k_1)} = s_i \quad \forall i \tag{71}$$

VI-2 is similar to VI-1 in making use of the information on s_i and its implications on the associated z-variables. However, in VI-2 the z-variables are restricted in the opposite direction to that of VI-1, so that when the value of s_i imposes no restriction on one of the axes, the respective z-variables are constrained to be zero. In addition, to set to zero as many z-variables as possible, no dependency on the w-variables are introduced in the RHS of (70) and (71). In particular, when $s_i = 1$, (70) implies that all z^x-variables must be zero and when $s_i = 0$, (71) implies that all z^y-variables must be zero.

Valid Inequality – Set 3 (VI-3):

$$z^x_{ik_1jk_2} + \left(z^x_{jk_2i(m_i-k_1)} + z^x_{j(m_j-k_2)i(m_i-k_1)} + z^x_{j(m_j-k_2)ik_1} \right)/3 \leq 1 \quad \forall i, \forall j \neq i, k_1 = 1, \ldots, m_i - 1, k_2 = 1, \ldots, m_j - 1 \tag{72}$$

$$z^y_{ik_1jk_2} + \left(z^y_{jk_2i(m_i-k_1)} + z^y_{j(m_j-k_2)i(m_i-k_1)} + z^y_{j(m_j-k_2)ik_1} \right)/3 \leq 1 \quad \forall i, \forall j \neq i, k_1 = 1, \ldots, m_i - 1, k_2 = 1, \ldots, m_j - 1 \tag{73}$$

While VI-1 and VI-2 are concerned with two configurations of the same department, VI-3 is concerned with all four configurations associated with two different departments. The inequality builds on the definition of an L/T shape and prohibits positioning that violates it. In particular, if a configuration with k_1 machines of department i is located to the left of a configuration with k_2 machines of department j $(i \neq j)$ then: (1) configuration j with k_2 machines cannot be located to the left of the other configuration of department i, with $(m_i - k_1)$ machines; (2) the other configuration of department j, with $(m_j - k_2)$ machines, cannot be located to the left of configuration i with $k_1$machines; and (3) configuration j with $(m_j - k_2)$ machines cannot be located to the left of configuration i with $(m_i - k_1)$ machines. This is captured by Constraint (72). Note that dividing by three in this constraint assures that the constraint remains valid when the configuration with k_1 machines of department i is not located to the left of the configuration with k_2 machines of department j. The constraint is defined similarly with respect to the y-axes (Constraint (73)).

5. Numerical study

The L/T shape department model is a relaxation of the rectangular department model. As such, an improvement in the material flow objective is expected. In this section, a comparison between the objective values obtained by MILP-R and MILP-LT is conducted. Since the L/T problem is more difficult to solve than the rectangular problem, we created instances of 5 and 8 departments with an average number of machines in each department of 4 and 5, so that the largest problem included 40 machines.

The rest of the factors remained the same as in the rectangular model experiment (Section 3.2). The perimeter factor (PF), which is used only in the L/T formulation, was set to 1.25 and 1.5 for AR values of 1.5 and 2, respectively.

A total of 32 problems were run by each of the two formulations and the run time of each problem was limited to four hours. All 32 problems were solved to optimality when applying the rectangular formulation, however, no solution was confirmed as optimal when applying the L/T formulation. Hence, for the L/T model, the objective values presented below represent the values of the best solution obtained after running the problem for four hours. This implies that any improvement observed in our experiments by using the L/T model over the rectangular model is a lower bound on the actual difference between the optimal solutions of both models.

The results are presented in Table 1. Columns 1–6 contain the instance number and the problem parameters. The rectangular (denoted by R from now on) and L/T objective values are presented in columns 7 and 8, respectively, while the difference (in %) between the objective values appears in column 9. One can see that in all problems but one (problem #25) the L/T objective value outperforms that of the R objective value. The average improvement of the L/T over the R objective value is 17.97% with a maximal improvement of 32.9%.

The main significant factors affecting the above difference are the number of departments and the aspect ratio. The average improvement of the L/T solution as a function of the number of departments is shown in Figure 6(a). We can see that this value is equal to 20.6% for five departments and it drops down to 9.6% for eight departments. The advantage of the L/T over R is more significant when the number of departments is smaller, since relaxing the department shape restriction is more meaningful when the area size of each department out of the total area is relatively large. To continue with this intuitive argument, one can claim that when the number of departments is very large (and as a result, the relative area of each department is small) the problem becomes similar to a location problem rather than a layout problem. Hence, the effect of the area shape of each department on the objective becomes negligible. The second significant factor indicates that the difference between the L/T and the R objectives increases with the aspect ratio. This value, shown in Figure 6(b), is equal to 11.4% for the low AR value and 18.9% for the high AR value. This effect can be explained by the fact that increasing the aspect ratio increases the number of layout options for the L/T shape model, more than it does to the R model. The average difference for the combination of large aspect ratio and small number of departments is as large as 27.2%.

Figure 7 illustrates an 8-department solution example, solved by MILP-R and MILP-LT. This example refers to problem #27 in Table 1. The 'x' in each department denotes its centroid point. We can see that the L/T solution utilises all possible shapes as it contains three rectangular departments (1, 2, 4), four L-shaped departments (3, 5, 6, 8) and one T-shaped department (7). This flexibility of the L/T shape clearly enables achieving a better solution value than that obtained by the R-shape solution (9.34% in this example) and in some cases smaller departments. One can also observe that in spite of the different designs, the relative position of most departments remains similar. One can also see that since a full flow density is applied in this example, the smaller departments are typically located in the centre of the layout, while the larger departments are pushed to the sides.

The above analysis was based on the common quantitative objective in the facility layout literature, namely the material flows between departments' centroids. The justification for using this objective is twofold. First, the distance between centroids may be used as a good estimate for the average distance between departments. The other reason is

Table 1. L/T vs. R – comparison results.

No.	No. of dept.	No. of mach.	Tightness	Flow density	Aspect ratio	R-obj.	L/T-obj.	Diff-(%)	R-unif.	L/T-unif.	Diff-unif.(%)
1	5	4	1.4	D	1.5	51.0	47.0	7.92	82.4	71.0	13.85
2	5	4	1.4	D	2	151.5	121.9	19.54	144.4	115.1	20.30
3	5	4	1.4	F	1.5	192.0	182.7	4.85	257.9	238.3	7.61
4	5	4	1.4	F	2	232.5	178.0	23.43	283.1	248.9	12.08
5	5	4	1.8	D	1.5	65.0	59.3	8.77	141.7	126.3	10.84
6	5	4	1.8	D	2	101.0	69.4	31.30	182.6	138.3	24.27
7	5	4	1.8	F	1.5	431.5	408.9	5.24	372.9	347.6	6.78
8	5	4	1.8	F	2	328.0	251.0	23.49	454.6	400.7	11.87
9	5	5	1.4	D	1.5	188.5	155.8	17.32	244.8	218.6	10.70
10	5	5	1.4	D	2	139.5	93.6	32.90	234.0	216.2	7.60
11	5	5	1.4	F	1.5	372.0	294.2	20.92	478.8	389.2	18.71
12	5	5	1.4	F	2	438.5	332.9	24.08	558.5	523.4	6.29
13	5	5	1.8	D	1.5	110.0	74.1	32.60	146.8	111.0	24.40
14	5	5	1.8	D	2	128.0	87.8	31.43	180.7	157.4	12.92
15	5	5	1.8	F	1.5	487.0	416.2	14.54	550.0	483.0	12.18
16	5	5	1.8	F	2	323.5	222.0	31.39	496.4	420.6	15.27
17	8	4	1.4	D	1.5	200.5	164.6	17.91	247.1	219.9	11.00
18	8	4	1.4	D	2	183.5	164.8	10.20	154.5	144.1	6.77
19	8	4	1.4	F	1.5	1238.9	1073.4	13.36	1640.2	1552.0	5.38
20	8	4	1.4	F	2	1431.4	1274.5	10.96	1565.7	1437.7	8.18
21	8	4	1.8	D	1.5	211.0	185.7	11.99	315.2	279.7	11.28
22	8	4	1.8	D	2	189.5	169.7	10.46	234.5	222.3	5.23
23	8	4	1.8	F	1.5	1516.8	1412.9	6.85	1673.8	1492.3	10.84
24	8	4	1.8	F	2	1245.4	1119.1	10.14	1602.4	1449.7	9.53
25	8	5	1.4	D	1.5	239.5	239.6	−0.04	211.4	197.5	6.58
26	8	5	1.4	D	2	165.5	159.1	3.87	316.6	320.1	−1.10
27	8	5	1.4	F	1.5	1263.0	1145.0	9.34	1372.1	1249.1	8.96
28	8	5	1.4	F	2	1286.9	1218.5	5.31	1584.4	1498.9	5.39
29	8	5	1.8	D	1.5	196.0	184.3	5.97	260.1	234.0	10.01
30	8	5	1.8	D	2	171.0	137.0	19.90	320.0	302.3	5.54
31	8	5	1.8	F	1.5	1430.4	1368.5	4.32	2052.9	1913.1	6.81
32	8	5	1.8	F	2	1519.8	1311.0	13.74	1791.8	1665.3	7.06
Average								17.97			0.41

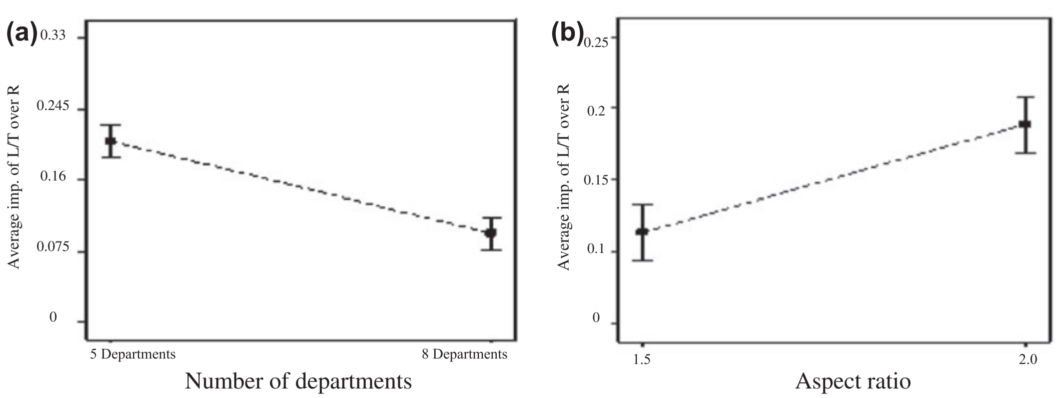

Figure 6. Significant main effects.

based on the claim that the layout problem should not consider the internal flows within each department, but only the flow between departments. Hence, when the output/input points of the departments are not known (as in our case), it is reasonable to consider the distance between the departments centroids. Still, when comparing the R and L/T shaped departments, one may claim that since the R shape model is a special case of the L/T model, the latter is more flexible in achieving a good centroid based objective, although this solution may not represent well the real flows. For example, when an L shaped department is chosen, the average distance to all locations of the department may be larger than the distance to its centroid.

To study the effect of the above considerations, we examine the results obtained above with the EDIST objective, suggested by Bozer and Meller (1997), which is denoted here as the *uniform* objective. As noted above, this objective assumes that the flow between departments i and j occurs uniformly from any location in department i to any location in department j. When the R model is considered, expression (74) represents the uniform objective.

$$\sum_{i=1}^{I-1}\sum_{j=i+1}^{I}\frac{\left[\int_{x_i^l}^{x_i^h}\int_{y_i^l}^{y_i^h}\int_{x_j^l}^{x_j^h}\int_{y_j^l}^{y_j^h} f_{ij}(|x_i-x_j||y_i-y_j|)dy_j dx_j dy_i dx_i\right]}{(x_i^h-x_i^l)(y_i^h-y_i^l)(x_j^h-x_j^l)(y_j^h-y_j^l)} \tag{74}$$

In (74), the distance between any point in department i to any point in department j is multiplied by the flow parameter between the departments, and this term is integrated over the ranges in which they reside, thus accounting for all (uniformly) possible combinations of this distance. This expression is then normalised (in the denominator) by the product of the areas of both departments.

To express the uniform L/T objective, we define x'^h_{ip} (y'^h_{ip}) as the right (upper) end of the p^{th} chosen rectangle ($p = 1, 2$) of department i ($i = 1..I$), and x'^l_{ip} (y'^l_{ip}) as the left (lower) end of the p^{th} chosen rectangle ($p = 1, 2$) of department i ($i = 1..I$). Then, (75) denotes the uniform objective of the L/T shaped department model.

$$\sum_{i=1}^{I-1}\sum_{j=i+1}^{I}\sum_{p=1,2}\sum_{q=1,2}\frac{\left[\int_{x'^l_{ip}}^{x'^h_{ip}}\int_{y'^l_{ip}}^{y'^h_{ip}}\int_{x'^l_{jq}}^{x'^h_{jq}}\int_{y'^l_{jq}}^{y'^h_{jq}} f_{ij}(|x_i-x_j||y_i-y_j|)dy_j dx_j dy_i dx_i\right]}{(x'^h_{ip}-x'^l_{ip})(y'^h_{ip}-y'^l_{ip})(x'^h_{jq}-x'^l_{jq})(y'^h_{jq}-y'^l_{jq})} \tag{75}$$

Expression (75) generalises (74) by summing the uniform flows between each of the two rectangles of department i to each of the two rectangles of department j.

The uniform objective values were calculated for the layout solutions in the experiment discussed above. The solution values are presented in columns 10–12 of Table 1. We can see that in general, the objective value is indeed larger in most cases, both for the R and L/T solutions, compared to the original objective. But moreover, we can see that even with the new objective, the L/T outperforms the R solutions in 35 out of the 36 instances, with an average gap of 10.4%. The number of department was found to be the only significant factor; the average gap was equal to 13.5% for the 5-department case and 7.3% for the 8-department case. Thus, even under this conservative measure, L/T solutions are better than R solutions.

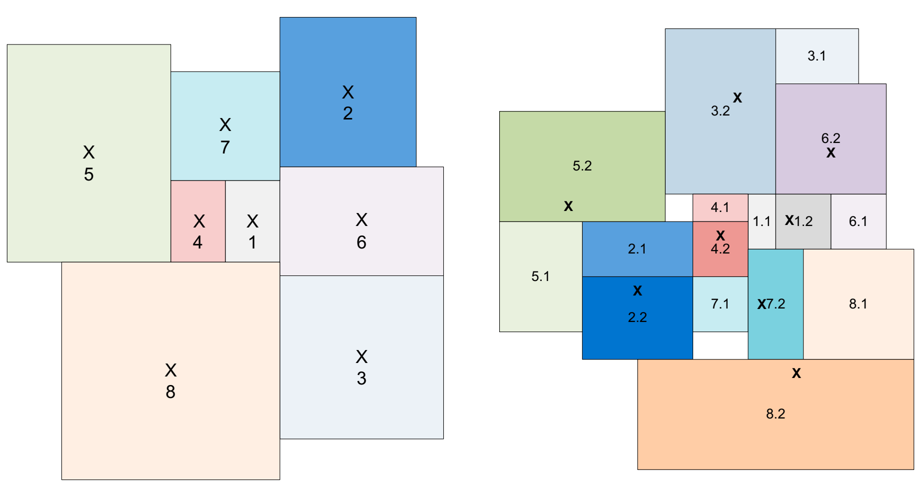

Figure 7. Rectangular vs. L/T 8-department example.

6. Conclusions

In this paper, we proposed a new simultaneous approach for the FLP, assuming a process layout, where each department contains a given number of machines of the same type. We suggested two new MILP formulations; the first assumes traditional rectangular shape departments, and the second is an extension that allows L/T shaped departments. The contribution of the new approach is threefold. First, it considers simultaneously the location of the departments within the facility and the internal arrangement of the machines. Second, the non-linear area constraint of previous formulations is avoided so that our new formulation has a linear structure. Third, the proposed approach allows non-rectangular department shapes, in particular L/T department shapes. There are examples where the latter shape exists in practice, and there are a number of benefits associated with it (Bozer and Meller 1997). In particular, allowing L/T shape departments decreases the material handling cost and provides feasible solutions in cases where such solutions cannot be obtained under the rectangular shape constraint, due to the machine size and shape and aspect ratio.

A numerical study was conducted to examine the layout design and performance of the proposed formulations. The results of the R-shape model show that we can solve to optimality problems with about the same number of departments that were solved in previous studies, even though in our case the simultaneous approach results in a more complex formulation, leading to a more detailed solution. The L/T formulation was much harder to solve, and most problems were not solved to optimality. Still, allowing L/T department shapes resulted in an average improvement of 18% over the traditional (optimal) R-shape solutions, with a maximal difference of 32.9%. Another objective function was tested, assuming that the flow to/from each department is distributed uniformly over the department area. Computing the value of this objective for the solutions obtained for the previous objective resulted in an average difference of 10.4% in favour of the L/T shape solutions.

References

Bock, S., and K. Hoberg. 2007. “Detailed Layout Planning for Irregularly-shaped Machines with Transportation Path Design.” *European Journal of Operational Research* 177 (2): 693–718.

Bozer, Y. A., and R. D. Meller. 1997. “A Reexamination of the Distance-based Facility Layout Problem.” *IIE Transactions* 29 (7): 549–560.

Bozer, Y. A., and C. T. Wang. 2012. “A Graph-pair Representation and MIP-model-based Heuristic for the Unequal-area Facility Layout Problem.” *European Journal of Operational Research* 218: 382–391.

Bozer, Y. A., R. D. Meller, and S. J. Erlebacher. 1994. “An Improvement-type Layout Algorithm for Single and Multiple-floor Facilities.” *Management Science* 40 (7): 918–932.

Bukchin, Y., and M. Tzur. 2010. “A New MILP Approach for the Facility Layout Design Problem with Rectangular and L/T Shaped Departments.” *11th International Material Handling Research Colloquium, 2010, Milwaukee, WI, USA, June 21–24*.

Bukchin, Y., R. D. Meller, and Q. Liu. 2006. “Assembly System Facility Design.” *IIE Transactions* 38: 53–65.

Castillo, I., and B. A. Peters. 2003. “An Extended Distance-based Facility Layout Problem.” *International Journal of Production Research* 41 (11): 2451–2479.

Castillo, I., and T. Westerlund. 2005. “An ε-accurate Model for Optimal Unequal-area Block Layout Design.” *Computers & Operations Research* 32: 429–447.

Castillo, I., J. Westerlund, S. Emet, and T. Westerlund. 2005. “Optimization of Block Layout Design Problems with Unequal Areas: A Comparison of MILP and MINLP Optimization Methods.” *Computers & Chemical Engineering* 30: 54–69.

Francis, R. L., L. F. McGinnis, and J. A. White. 1992. *Facility Layout and Location: An Analytical Approach*. Englewood Cliffs, NJ: Prentice-Hall.

Heragu, S. S. 2006. *Facility Design*. 2nd ed. New York: Universe Inc.

Ho, Y.-C., and C. L. Moodie. 2000. “A Hybrid Approach for Concurrent Layout Design of Cells and Their Flow Paths in a Tree Configuration.” *International Journal of Production Research* 38 (4): 895–928.

Kim, J. G., and M. Goetschalckx. 2005. “An Integrated Approach for the Concurrent Determination of the Block Layout and the Input and Output Point Locations Based on the Contour Distance.” *International Journal of Production Research* 43 (10): 2027–2047.

Leno, I. J., S. S. Sankar, M. V. Raj, and S. G. Ponnambalam. 2013. “An Elitist Strategy Genetic Algorithm for Integrated Layout Design.” *The International Journal of Advanced Manufacturing Technology* 66: 1573–1589.

Liggett, R. S., and W. J. Mitchell. 1981. “Optimal Space Planning in Practice.” *Computer-Aided Design* 13: 277–288.

Liu, Q., and R. D. Meller. 2007. “A Sequence-pair Representation and MIP-model-based Heuristic for the Facility Layout Problem with Rectangular Departments.” *IIE Transactions* 39 (4): 377–394.

Malmborg, C. J. 2013. “The Concurrent Block Layout Problem.” *International Journal of Production Research* 51 (6): 1745–1761.

Meller, R. D., and Y. A. Bozer. 1996. “A New Simulated Annealing Algorithm for the Facility Layout Problem.” *International Journal of Production Research* 34 (6): 1675–1692.

Meller, R. D., W. Chen, and H. D. Sherali. 2007. “Applying the Sequence-pair Representation to Optimal Facility Layout Designs.” *Operations Research Letters* 35 (5): 651–659.

Meller, R. D., Z. Kirkizoglu, and W. Chenb. 2010. “A New Optimization Model to Support a Bottom-up Approach to Facility Design.” *Computers & Operations Research* 37: 42–49.

Montreuil, B. 1990. “A Modeling Framework for Integrating Layout Design and Flow Network Design.” *In: Proceedings from the Material Handling Research Colloquium, Hebron, Kentucky* : 43–58.

Muther, R. 1955. *Practical Plant Layout*. New York: McGraw Hill.

Shebanie, C. R. 2004. “An Integrated, Evolutionary Approach to Facility Layout and Detailed Design.” M.S. Thesis, University of Pittsburgh.

Sherali, H. D., B. M. P. Fraticelli, and R. D. Meller. 2003. “Enhanced Model Formulations for Optimal Facility Layout.” *Operations Research* 51 (4): 629–644.

Tompkins, J. A., J. A. White, Y. A. Bozer, and J. M. A. Tanchoco. 2010. *Facilities Planning*. 4th ed. Hoboken, NJ: John Wiley and Sons.

Xie, W., and N. V. Sahinidis. 2008. “A Branch-and-Bound Algorithm for the Continuous Facility Layout Problem.” *Computers & Chemical Engineering* 32: 1016–1028.

Yang, C. L., S. P. Chuang, and T. S. Hsu. 2011. “A Genetic Algorithm for Dynamic Facility Planning in Job Shop Manufacturing.” *The International Journal of Advanced Manufacturing Technology* 52: 303–309.

Yang, T., Y. C. Chang, and Y. H. Yang. 2012. “Fuzzy Multiple Attribute Decision-Making Method for a Large 300-mm Fab Layout Design.” *International Journal of Production Research* 50 (1): 119–132.

Appendix A. Run time analysis of rectangular shaped departments

No.	No. of dept.	No. of mach.	Tightness	Flow density	Aspect ratio	Run time (sec)
1	8	4	1	1	1	2.79
2	8	4	1	1	2	6.44
3	8	4	1	2	1	20.67
4	8	4	1	2	2	4901.89
5	8	4	2	1	1	4.12
6	8	4	2	1	2	2.64
7	8	4	2	2	1	60.53
8	8	4	2	2	2	637.74
9	8	6	1	1	1	3.23
10	8	6	1	1	2	6.85
11	8	6	1	2	1	106.11
12	8	6	1	2	2	12.62
13	8	6	2	1	1	23.31
14	8	6	2	1	2	11.62
15	8	6	2	2	1	86.92
16	8	6	2	2	2	538.17
17	9	4	1	1	1	13.43
18	9	4	1	1	2	59.17
19	9	4	1	2	1	303.09
20	9	4	1	2	2	859.86
21	9	4	2	1	1	13.32
22	9	4	2	1	2	10.84
23	9	4	2	2	1	11857.36
24	9	4	2	2	2	8057.77
25	9	6	1	1	1	20.90
26	9	6	1	1	2	11.92
27	9	6	1	2	1	14372.65
28	9	6	1	2	2	2546.77
29	9	6	2	1	1	5.79
30	9	6	2	1	2	35.10
31	9	6	2	2	1	2148.70
32	9	6	2	2	2	4929.75
33	10	4	1	1	1	90.53
34	10	4	1	1	2	48.94
35	10	4	1	2	1	1064.06
36	10	4	1	2	2	NS[1]
37	10	4	2	1	1	116.16
38	10	4	2	1	2	148.11
39	10	4	2	2	1	NS
40	10	4	2	2	2	NS
41	10	6	1	1	1	46.25
42	10	6	1	1	2	21.40
43	10	6	1	2	1	1412.77
44	10	6	1	2	2	19147.01
45	10	6	2	1	1	16.16
46	10	6	2	1	2	486.53
47	10	6	2	2	1	7782.18
48	10	6	2	2	2	4945.43

[1]NS means that no optimal solution was verified within six hours.