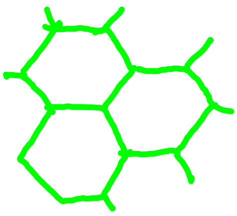
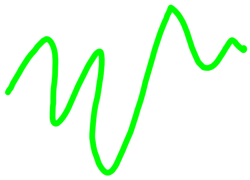
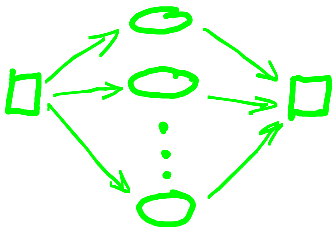
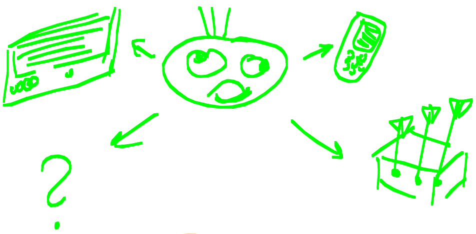
Lattice Coding for Signals & Networks:







Application & Design



Part A: Rami Zamir

Part B: Meir Feder

Tutorial @ ISIT 2012, M.I.T.

Lattice: Definition

Lattice = discrete subgroup of Euclidean space

$$\Lambda = \left\{ \underline{\underline{G}} \cdot \underline{i} \; : \; \underline{i} = \text{vector of integers } (0, \pm 1, \pm 2, \dots) \right\}$$

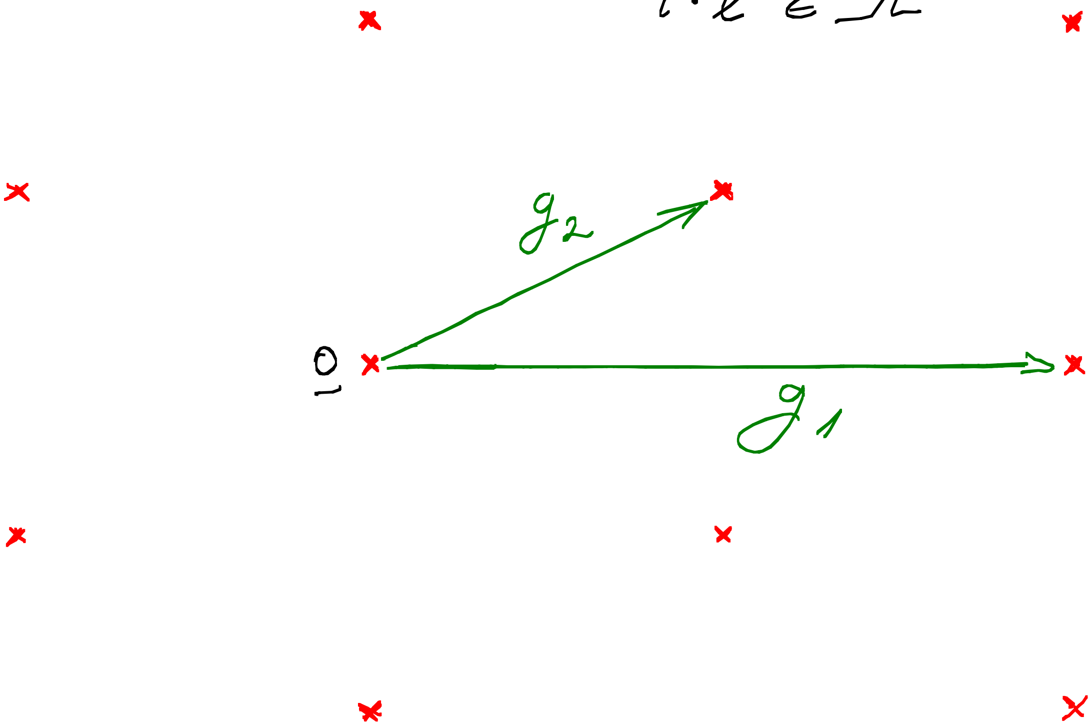
Lattice in $\mathbb{R}^n$

Generator Matrix $n \times n$

Closed under reflection & addition:

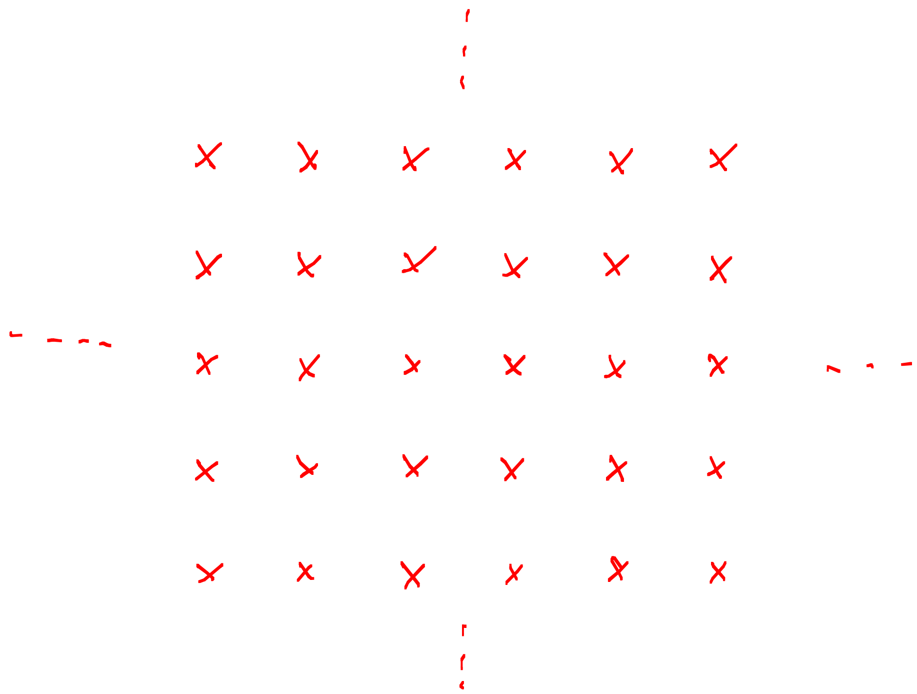
linearity: $l_1, l_2 \in \Lambda \Rightarrow l_1 + l_2 \in \Lambda$

$i \cdot l \in \Lambda$

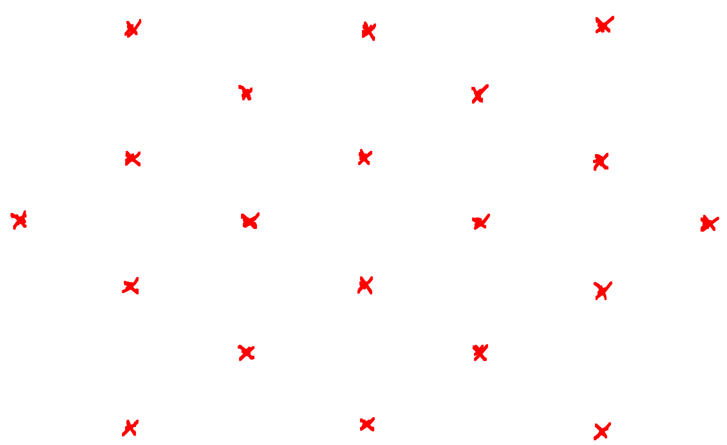


Lattice Codes in Signal Space

square ($\mathbb{Z}$)-lattice $\Rightarrow$ Uncoded constellation



More "interesting" lattice $\Rightarrow$ Coded constellation



What a Lattice Means?...

What a Lattice Means?...

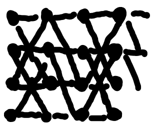
For my 8-year old kid: 

-//- a physicist/crystallographer: 

-//- a mathematician: 

-//- a Computer Scientist: 

-//- a coding theorist: Λ_8, Λ_{24}, ...

 ...

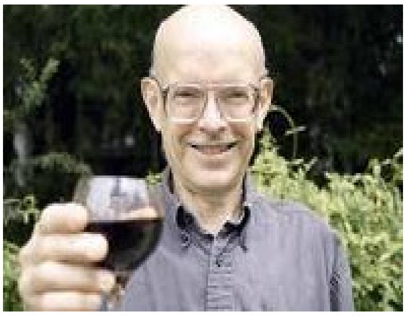
-//- an Information Theorist:

$$n \to \infty$$

People Who Influenced ...

Hermann Minkowski
(1864 – 1909)

Neil Sloane



John Conway

Dave Forney

also, Rudi de Buda, Gregory Poltyrev ...

Why Lattices in Communication?

1

2

3

4

Why Lattices in Communication?

1. a bridge from $n=1$  to $n=\infty$

   = non-asymptotic analysis per dimension

2. Algebraic (low complexity) Binning

   = structured coding schemes for networks

3. 

   bridge from Analog-to-Digital

   = Robust joint source-channel coding

4. Better than Random-Coding!

   in distributed side-information problems

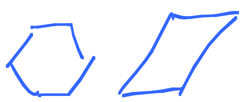
Lattices do things differently:

Randomness $\longrightarrow$

Typicality $\longrightarrow$

Binning $\longrightarrow$

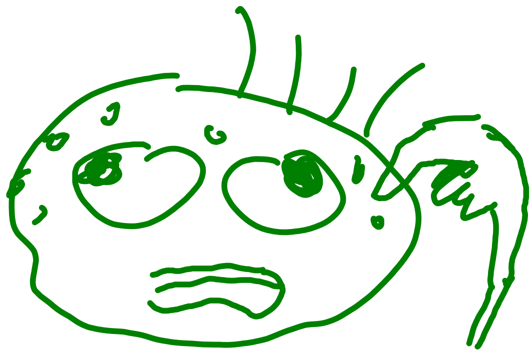
Special features:

* Dither – why? how? is it critical?
* MMSE – estimation or decoding? linear or not?
* Volume & noise → "soft" sphere packing & covering
* Lattices in high dimensions
  - ⟶ white Gaussian noise?
  - ⟶ non-Gaussian noise?
* Nesting, cosets & binning
  * AWGN + dither 
  * Voronoi or Parallelepiped? 
  * Mixed dimensions n_1, n_2
  * noise-matched decoding 

Open Question:

Is the dither really necessary?

?



Book in Writing...

Lattice Coding for Signals and Networks

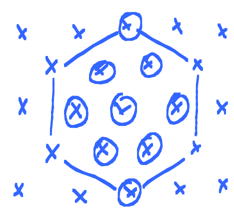
A Structured Coding Approach to Quantization, Modulation and Multiuser Information Theory

Ram Zamir

Draft June 2012

Tutorial - Part A Outline

1. Definitions: Partition, Construction — $Vol(\Lambda)$, Modulo Λ
2. Figures of merit — $G(\Lambda)$
3. Dither & estimation — $noise(\Lambda)$
4. Entropy coding — $H(\Lambda)$
5. Infinite constellation — $Pe(\Lambda + noise)$
6. Asymptotic goodness — $(n \to \infty)$
7. Error exponents
8. Nested lattices — $\Lambda_2 \subset \Lambda_1$
9. Lattice (Voronoi) shaping



10. Side-information problems — $Modulo^2(\Lambda)$
11. Gaussian networks — $Modulo^n(\Lambda)$

Tutorial - Part B :

Design of lattice codes

by Meir Feder

- direct construction in Euclidean space

- efficient encoding & decoding

Our "secret" partner:

Uri Erez: "Coding with Known Interference and Some Results on Lattices ..."

PhD. thesis, Tel Aviv Univ. 2002

On-board calculations

1. Crypto lemma + generalized dither
   (the Nyquist-Schuschaman condition: zero-ISI @ dual lattice)
2. Entropy of dithered quantizer (ECDQ)
3. Achieving "Poltyrev's capacity" w random lattice
   (Minkowski-Hlawka-Siegel ensemble)
4. Modulo-Λ channel
   (Shannon meets Wiener)
5. Noise-matched decoding @ low SNR
6. $C_{DMAC} \longrightarrow 0$ @ Costa strategies

Tutorial - Part A Outline

1. **Definitions: Partition, Construction** $\quad Vol(\Lambda)$, Modulo Λ
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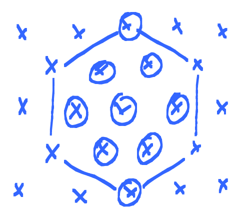
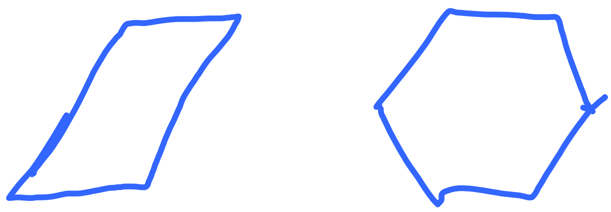


9. Lattice (Voronoi) shaping
10. Side-information problems $\quad$ Modulo$^2(\Lambda)$
11. Gaussian networks $\quad$ Modulo$^n(\Lambda)$

Tutorial - Part A Outline

1. Definitions: Partition, Construction

$\mathrm{Vol}(\Lambda)$



modulo Λ

Lattice: Definition

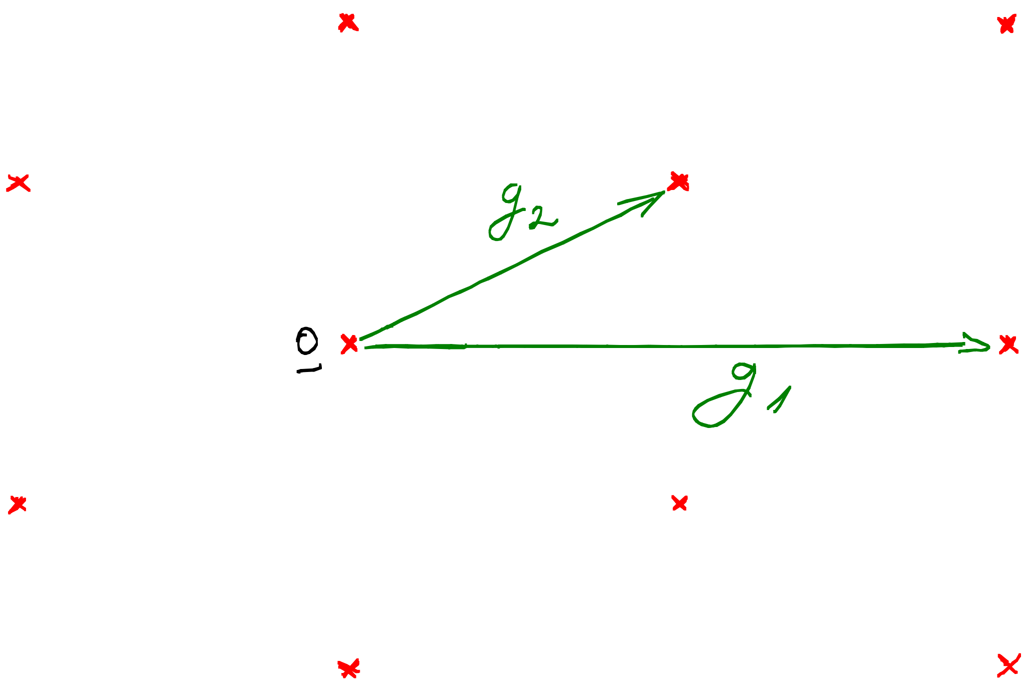
Let $\underline{g}_1, \dots, \underline{g}_n$ – linearly independent vectors in $\mathbb{R}^n$

$$\underline{\underline{G}} = \left(\underline{g}_1 \;\vdots\; \cdots \;\vdots\; \underline{g}_n \right) = \text{generator matrix}$$

$$\Lambda(G) = \{ i_1 \cdot \underline{g}_1 + \dots + i_n \cdot \underline{g}_n : i_1 \dots i_n \in \mathbb{Z} \}$$

$$= \{ \underline{\underline{G}} \cdot \underline{i} : \underline{i} \in \mathbb{Z}^n \}$$

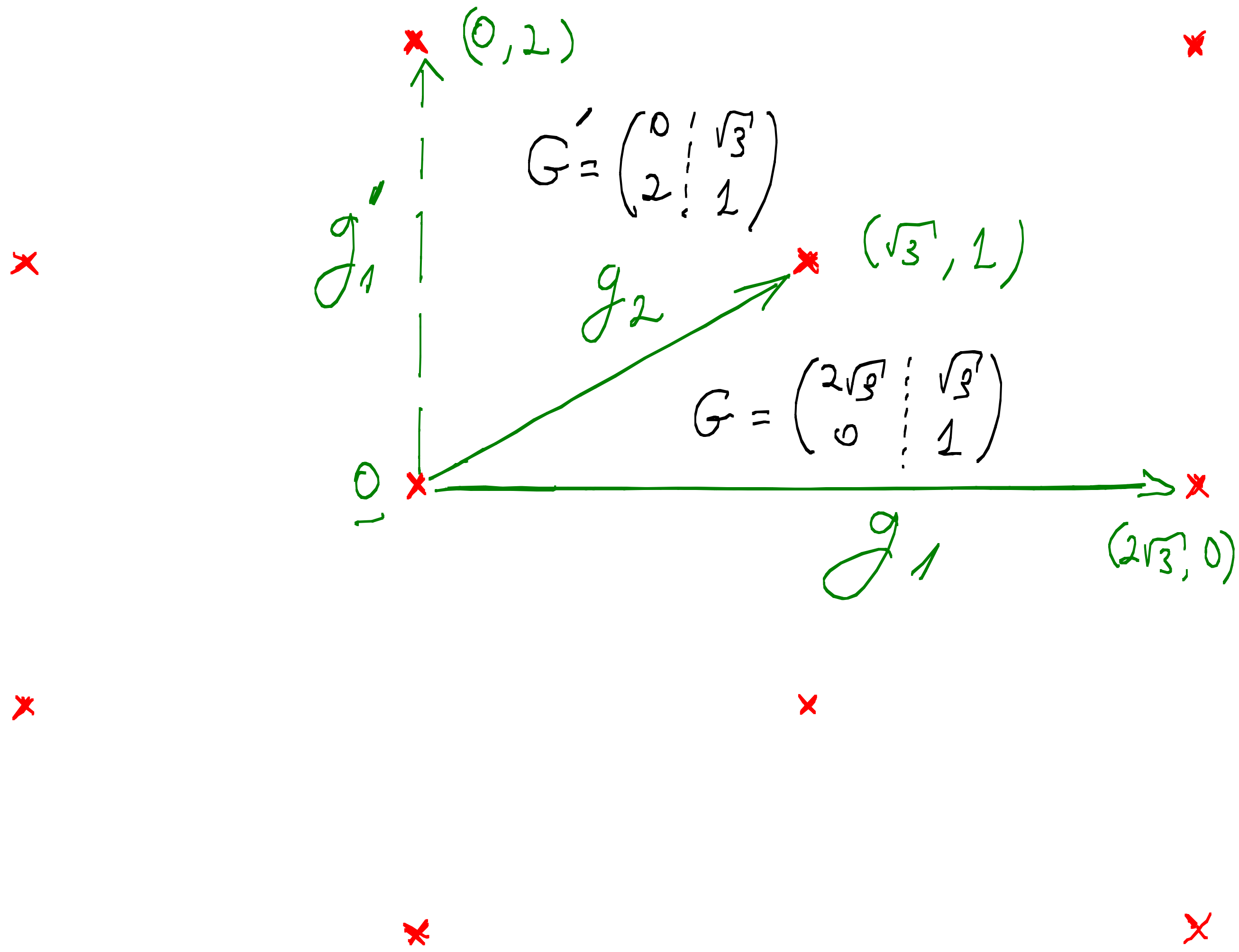
$$= \underline{\underline{G}} \cdot \mathbb{Z}^n$$



Lattice: Equivalent Representations

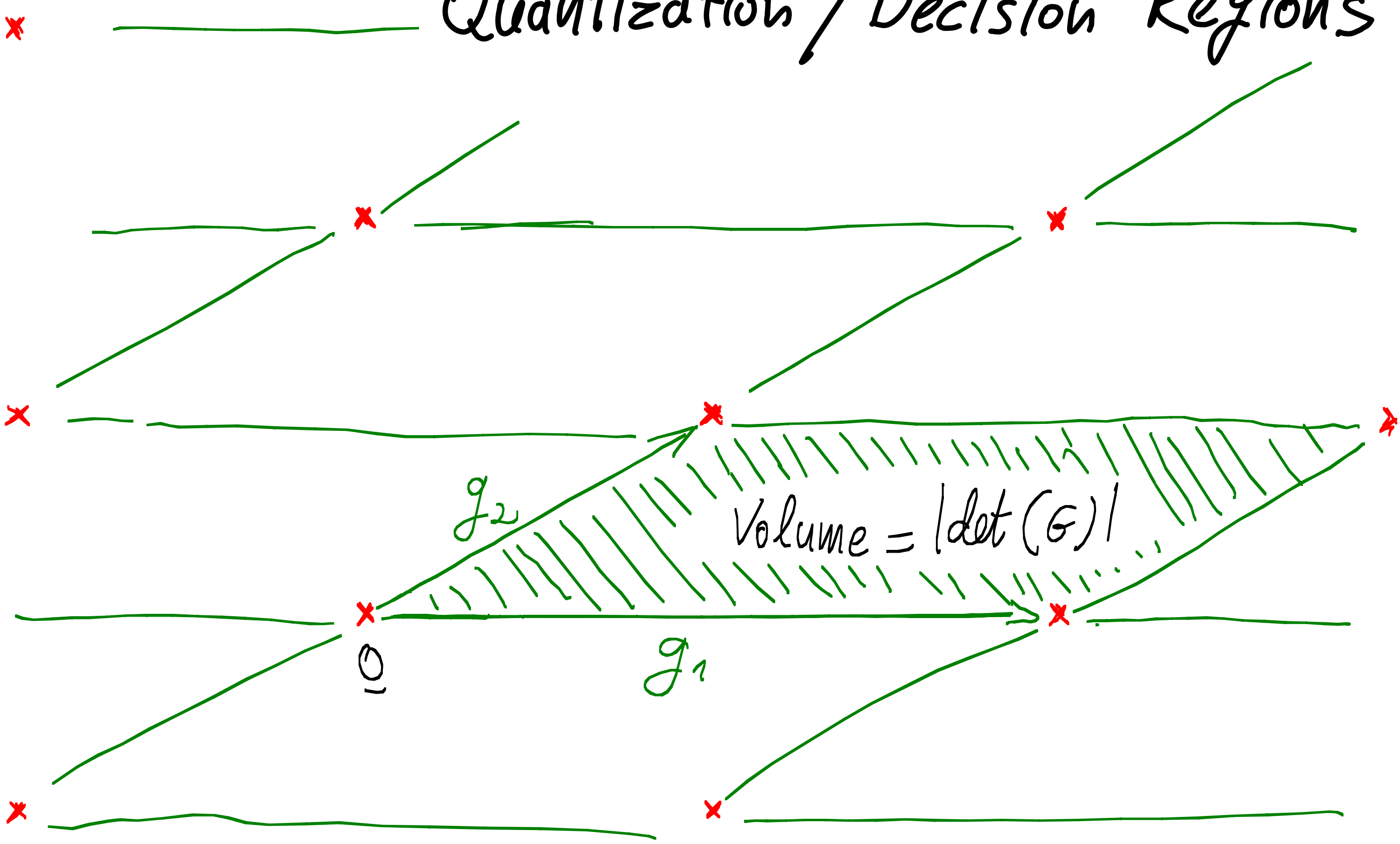
T = unimodular matrix
(integer elements, $\det(T) = \pm 1$)

$$\Rightarrow \quad \Lambda(G \cdot T) = \Lambda(G)$$



Lattice Partition:

Quantization / Decision Regions

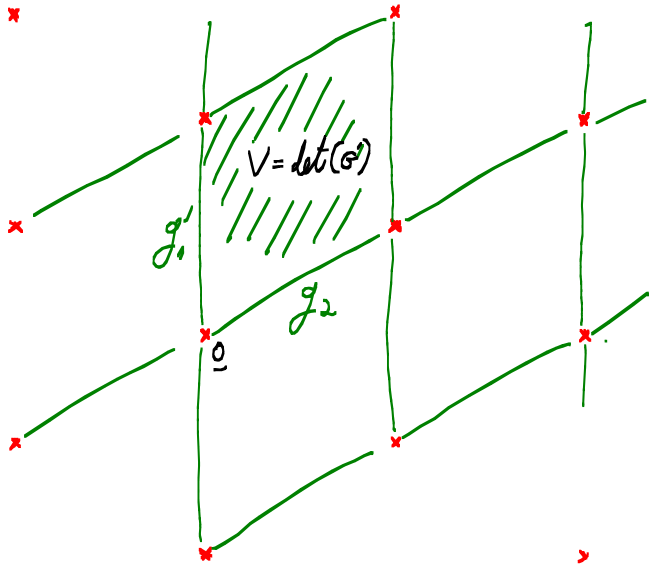


Parallelopipeds

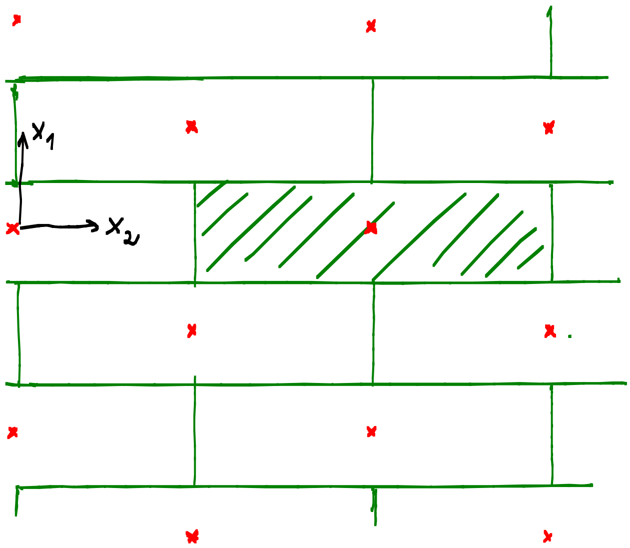
$$P_0 = \{ \alpha_1 g_1 + \alpha_2 g_2 : \quad 0 \le \alpha_1, \alpha_2 \le 1 \}$$

$$\Lambda + P_0 = \mathbb{R}^k$$

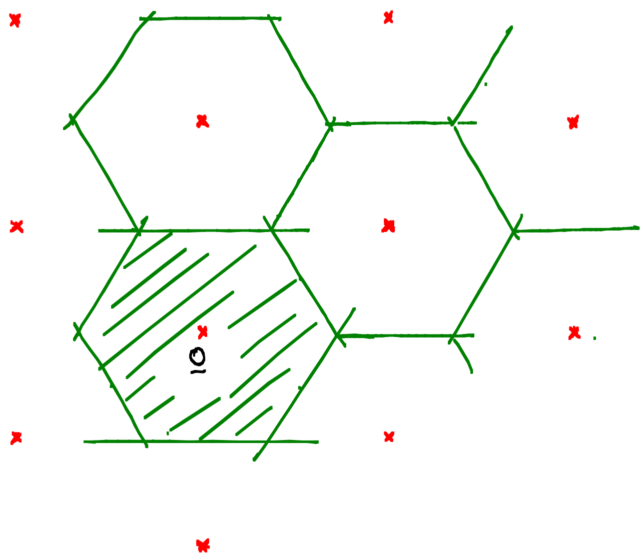
Partitions, Fundamental Cells



Other Basis $\Rightarrow$
other Parallelepiped
$\Rightarrow$ Cell Volume V is
invariant of partition



Sequential Quantization



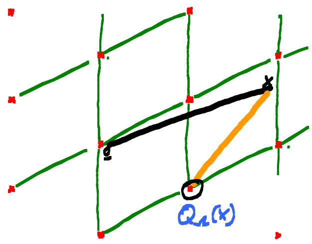
Voronoi Partition

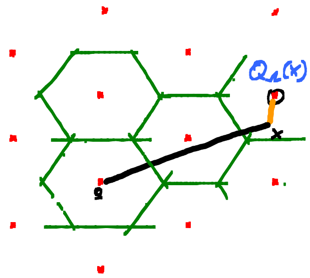
$$P_0 = \left\{ x : \|x\| \le \|x - l_i\| \quad \forall l_i \in \Lambda \right\}$$

Lattice Quantization, Modulo Lattice

$$\boxed{\begin{aligned} Q_{\Lambda,P_0}(x) &= \lambda \quad \text{if} \quad x \in (\lambda + P_0) \\ x \bmod_{P_0} \Lambda &= x - Q_{\Lambda,P_0}(x) \end{aligned}}$$

$\Rightarrow$ $x \in \mathbb{R}^n$ uniquely written as $\underbrace{Q_\Lambda(x)}_{\text{quantization}} + (\underbrace{x \text{ modulo } \Lambda}_{\text{error}})$

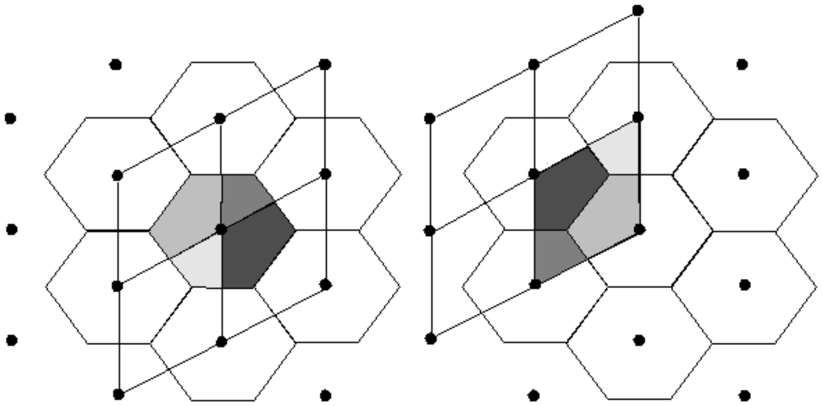




Modulo Laws:

* $a \bmod \Lambda = a + \lambda(a), \quad \lambda(a) \in \Lambda$
* $(a + \lambda) \bmod \Lambda = a \bmod \Lambda, \quad \forall \lambda \in \Lambda$
* $[(a \bmod \Lambda) + b] \bmod \Lambda = (a+b) \bmod \Lambda$
* $(a \bmod_{P_0} \Lambda) \bmod_{Q_0} \Lambda = a \bmod_{Q_0} \Lambda$

Fundamental Cells & Cosets



all fundamental cells are "modulo equivalent"!

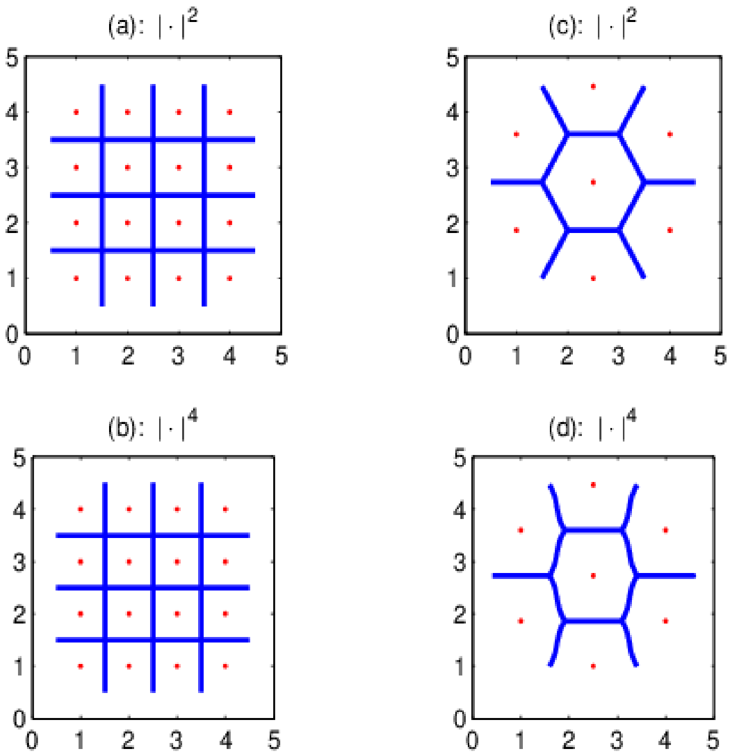
Coset: shift of Λ = points identical modulo Λ

$$\Lambda_v = v + \Lambda$$

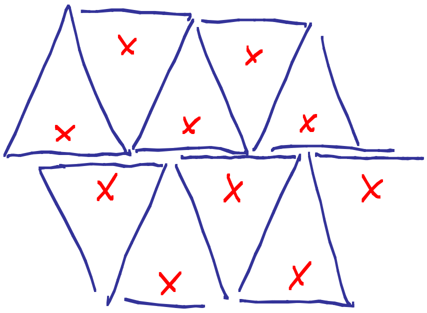
$$= \{ x \in \mathbb{R}^n : x \bmod \mathcal{P}_0 = v \} \quad , \quad v \in \mathcal{P}_0$$

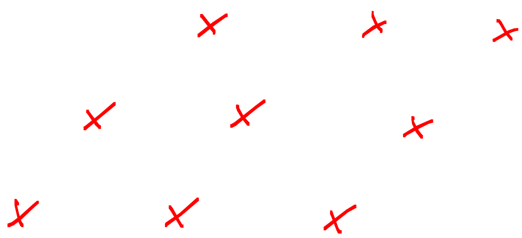
$\therefore$ **Fundamental cell:** a complete set of coset representatives

Non-Euclidean Voronoi partition



Tiling & Tranformation

 $\neq \Lambda$

$A \cdot$  $=$ 

$$\mathrm{Parallelepiped}(A \cdot \Lambda) = A \cdot \mathrm{Parallelepiped}(\Lambda)$$

But ...

$$\mathrm{Voronoi}(A \cdot \Lambda) \neq A \cdot \mathrm{Voronoi}(\Lambda) \quad !$$

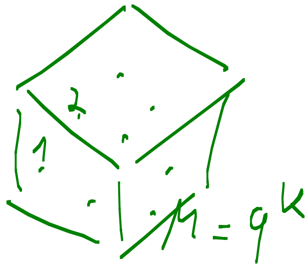
because $\|x\| > \|y\| \not\Rightarrow \|A \cdot x\| > \|A \cdot y\|$

Lattices from Linear Codes ("Construction A")

Let $\mathcal{C}$ = q-ary (n,k) liner code over $\mathbb{Z}_q = \{0,\dots,q-1\}$

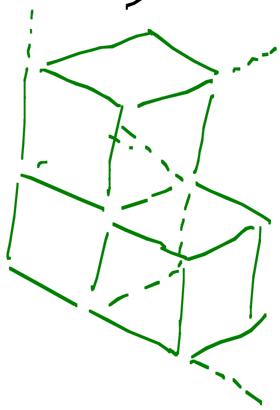
$$= \{\underline{\underline{G}}\cdot \underline{i} \;:\; \underline{i}\in \mathbb{Z}_q^k\}$$

$n \times k$



Let $\Lambda_{\mathcal{C}}$ = modulo-q lattice

$$= \{\lambda\in\mathbb{R}^n \;:\; \lambda \bmod q \in \mathcal{C}\}$$



e.g., E_8 (a lattice in $\mathbb{R}^8$)
is a modulo-2 lattice

$$\Lambda_{\text{Hamming Code }(8,4,4)}$$

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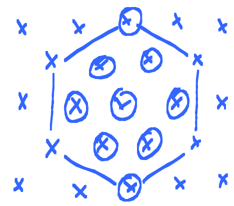


10. Side-information problems $\quad Modulo^2(\Lambda)$
11. Gaussian networks $\quad Modulo^n(\Lambda)$

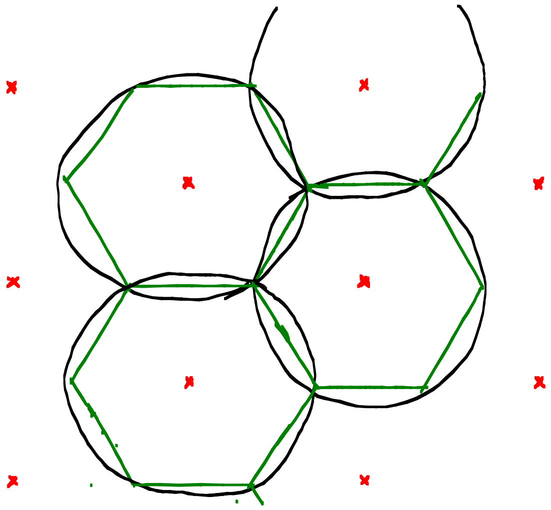
Tutorial - Part A Outline

2. Figures of merit

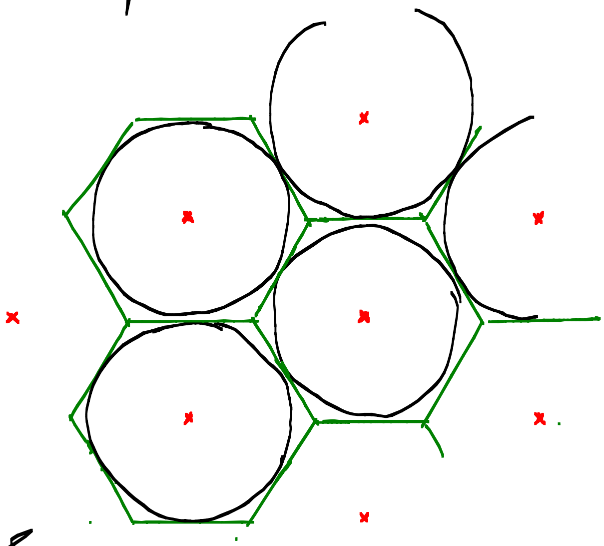
$G(\Lambda)$, $\mu(\Lambda, p_e)$

Covering, Packing, Kissing Number & More ....

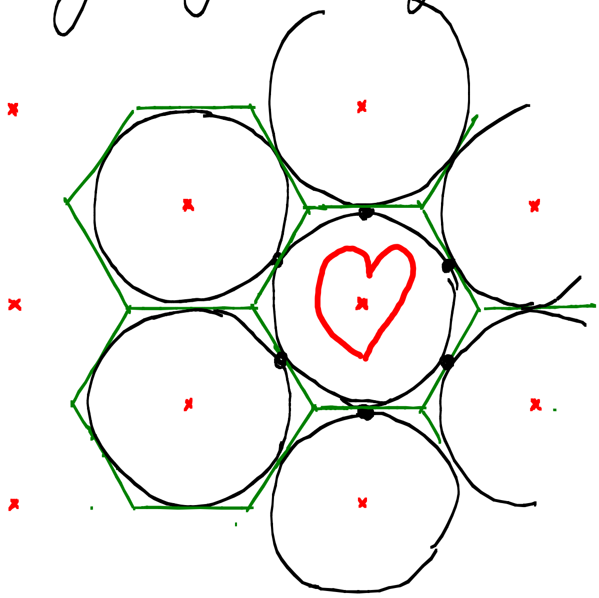
Covering $\mathbb{R}^n$ with (few) spheres



Packing (many) spheres in $\mathbb{R}^n$



Kissing by (many) Spheres

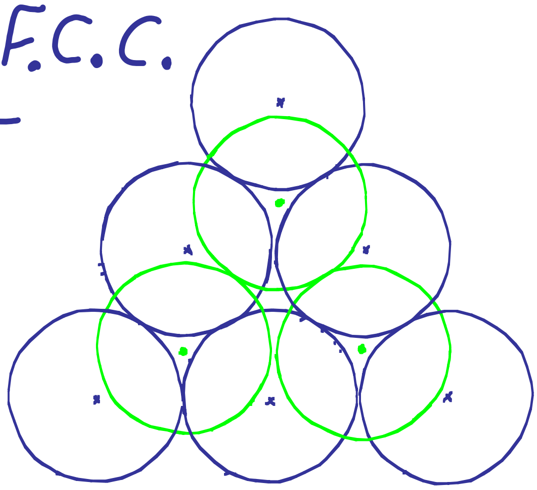


&

good arrangements for quantization and AWGN channel coding

Not an "All-Purpose" Lattice!

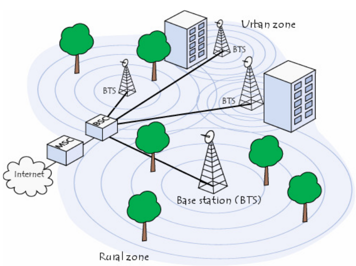
* **Best 3-dim Packing: F.C.C.**

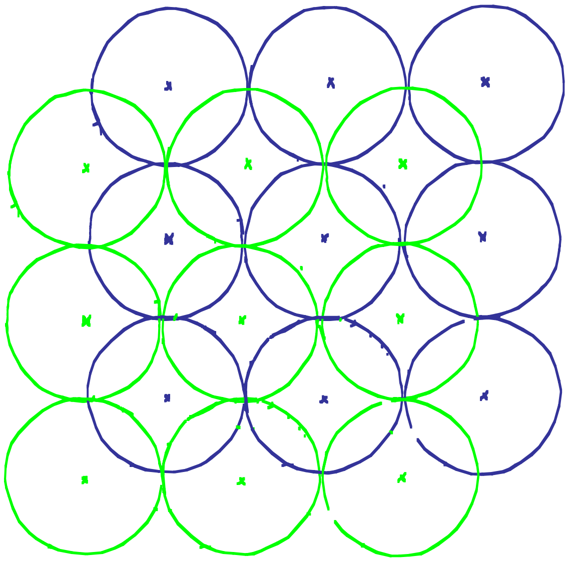


each layer = hexagonal Λ
layers are staggered

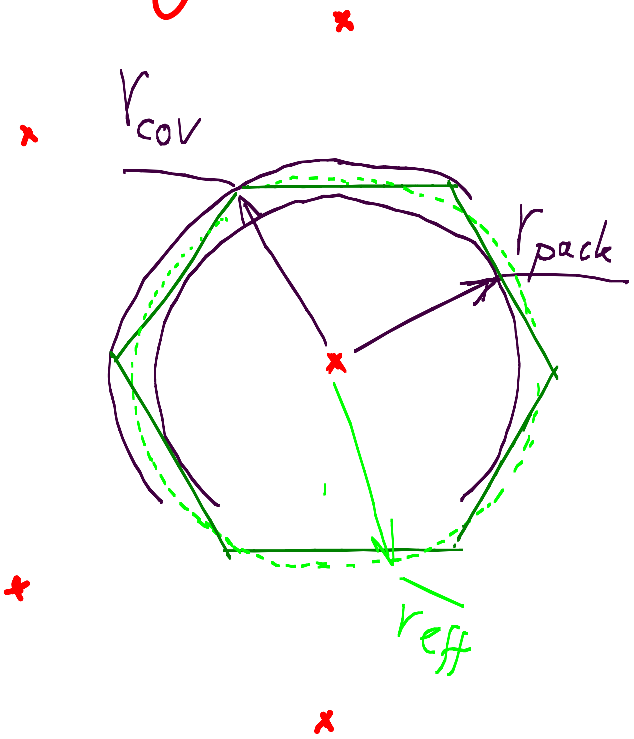
* **Best 3-dim Covering: B.C.C.**

each layer = cubic Λ
layers are staggered





Figures of Merit



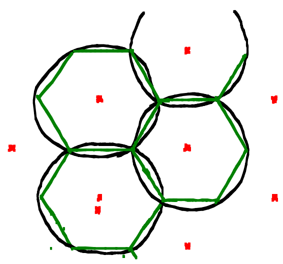
Radiuses:

r_{cov} = min sphere containing V_0

r_{pack} = max sphere contained in V_0
$= d_{min} / 2$

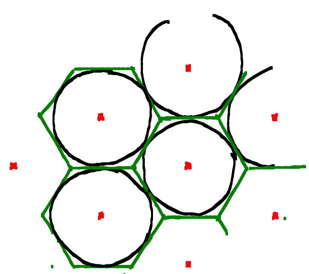
r_{eff} = Sphere with same volume

- **Covering efficiency:**



$$\rho_{cov}(\Lambda) = \frac{r_{cov}}{r_{eff}} > 1$$

- **Packing efficiency:**



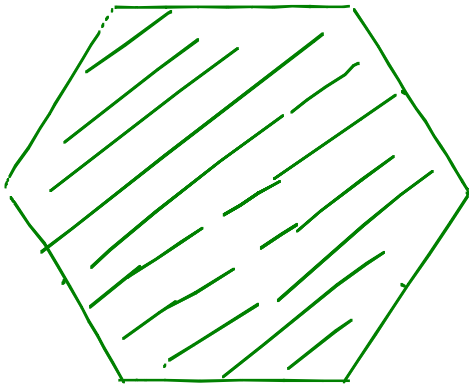
$$\rho_{pack}(\Lambda) = \frac{r_{pack}}{r_{eff}} < 1$$

Figures of Merit (Continued)

- Quantization efficiency:

$\underline{X} \sim \text{Uniform}(V_0)$

$\sigma^2(\Lambda) \triangleq \frac{1}{n} E\|X\|^2$



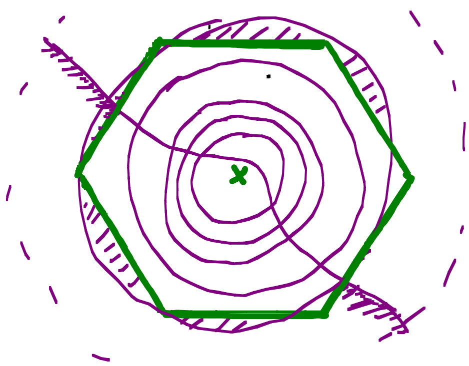
$$G(\Lambda) \triangleq \frac{\sigma^2(\Lambda)}{V^{2/n}}$$ = normalized second moment

Figures of Merit (Continued)

- AWGN coding efficiency: $\underline{Z} \sim$ AWGN $\mathcal{N}(0, \sigma^2)$

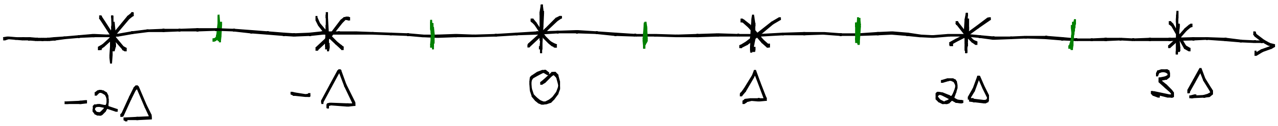
$$\mu(\Lambda, \sigma^2) \triangleq \frac{V^{2/n}}{\sigma^2} = \text{Volume-to-Noise Ratio}$$

$$P_e \triangleq \Pr\{\underline{Z} \notin \mathcal{V}_0\}$$



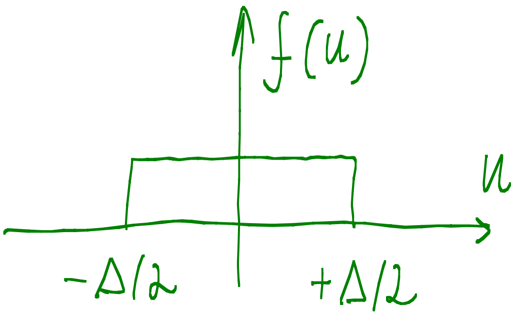
$$\mu(\Lambda, P_e) \triangleq \left.\frac{V^{2/n}}{\sigma^2}\right|_{@ P_e}$$

Example: One dimensional lattice
(Voronoi cell = interval)



1. NSM

u = dither uniform on Voronoi cell $= (-\Delta/2, +\Delta/2)$



$$V(\Lambda) = \Delta$$

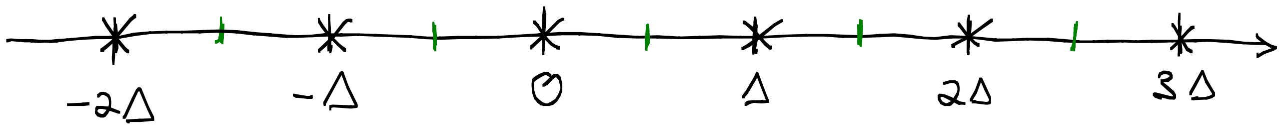
$$Eu^2 = \frac{\Delta^2}{12}$$

$$\Rightarrow \quad G(\square) = \frac{Eu^2}{V^2(\Lambda)} = \frac{\Delta^2/12}{\Delta^2} = \frac{1}{12}$$

invariant of Δ

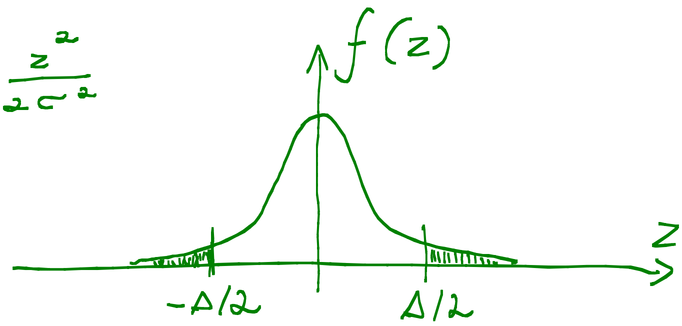
Example: One dimensional lattice

(Voronoi cell = interval)



2. NVNR

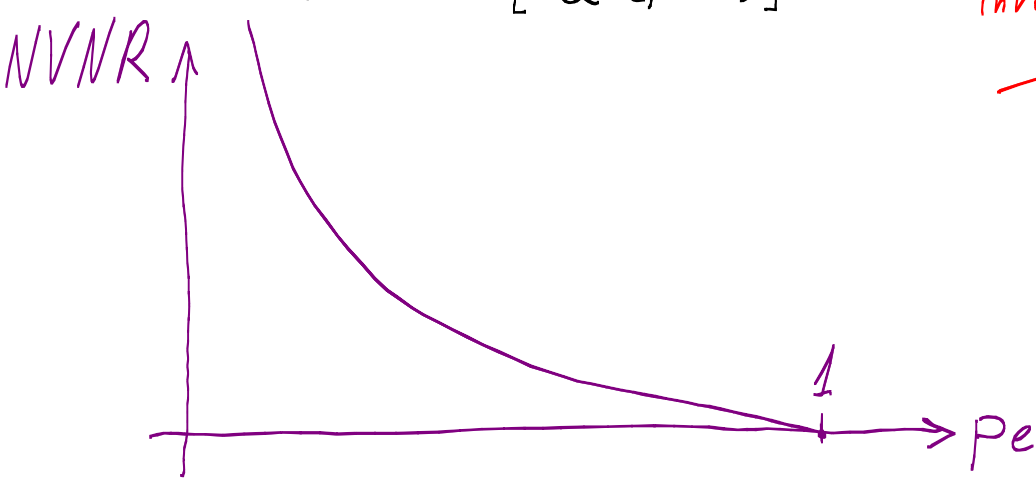
$$Z \sim \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}}$$



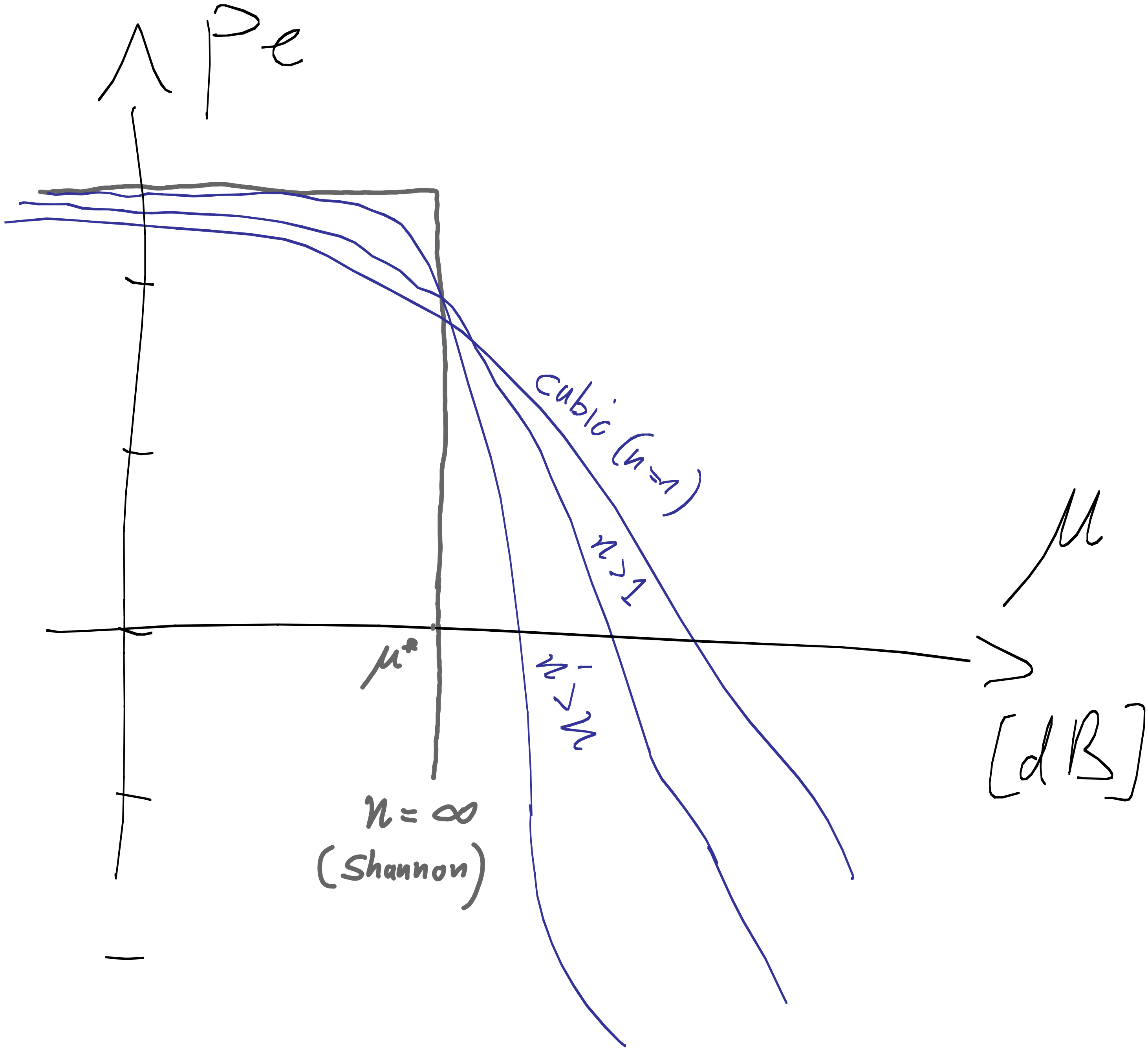
$$P_e = \Pr\left\{|Z| > \frac{\Delta}{2}\right\} = 2 \cdot Q\left(\frac{\Delta/2}{\sigma}\right)$$

$$\Rightarrow \mu(\Lambda, P_e) = \frac{V^2(\Lambda)}{\sigma^2_{P_e}} = \left[\frac{\Delta}{\frac{\Delta/2}{Q^{-1}(P_e/2)}}\right]^2 = \left[2 \cdot Q^{-1}\left(\frac{P_e}{2}\right)\right]^2$$

invariant of Δ



Pe versus V.N.R.



$G(\Lambda_n)$ and $\mu(\Lambda_n, P_e)$ as a function of n

[Conway & Sloane Book 1988]

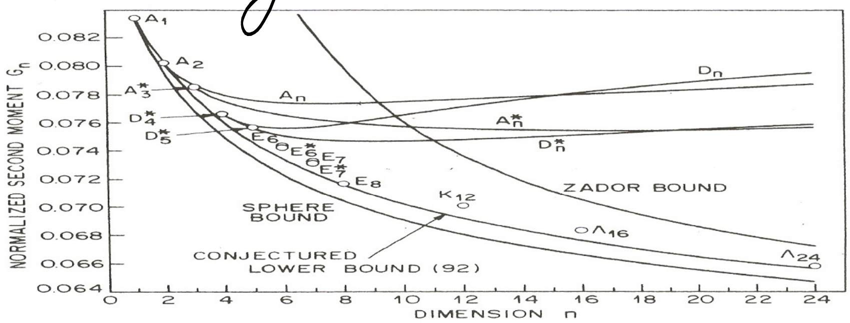
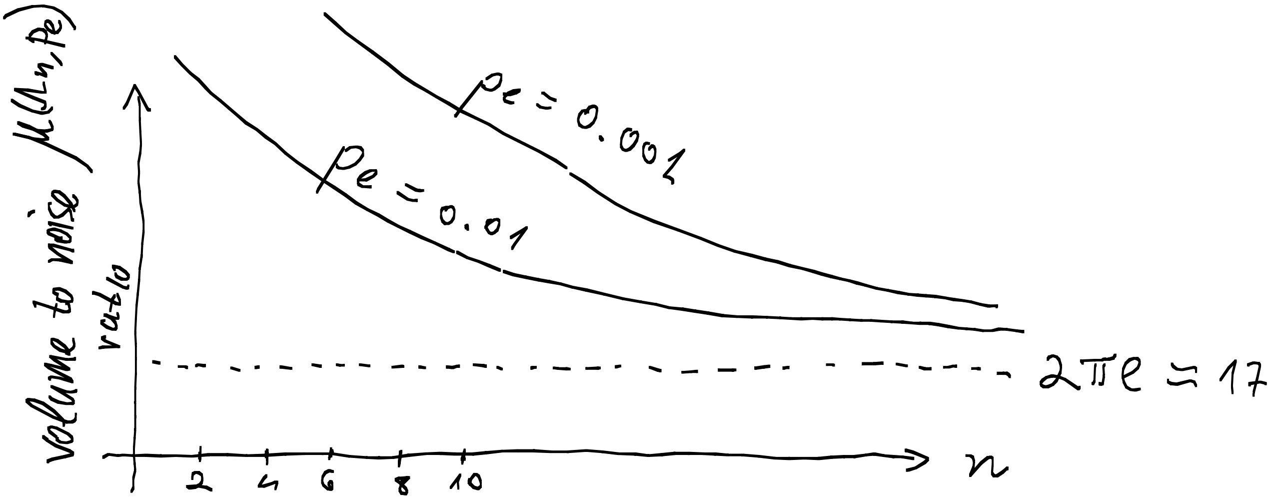


Figure 2.9. The best quantizers known in dimensions $n \leqslant 24$. $\frac{1}{2\pi e} \simeq 0.058$



$\Lambda_k^{opt} \longrightarrow$

$G_k \longrightarrow$?

$\mu_k \longrightarrow$

Tutorial - Part A Outline

1. Definitions: Partition, Construction — $Vol(\Lambda)$, Modulo Λ
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3. **Dither & estimation** — $noise(\Lambda)$
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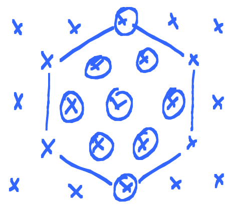


10. Side-information problems — $Modulo^2(\Lambda)$
11. Gaussian networks — $Modulo^n(\Lambda)$

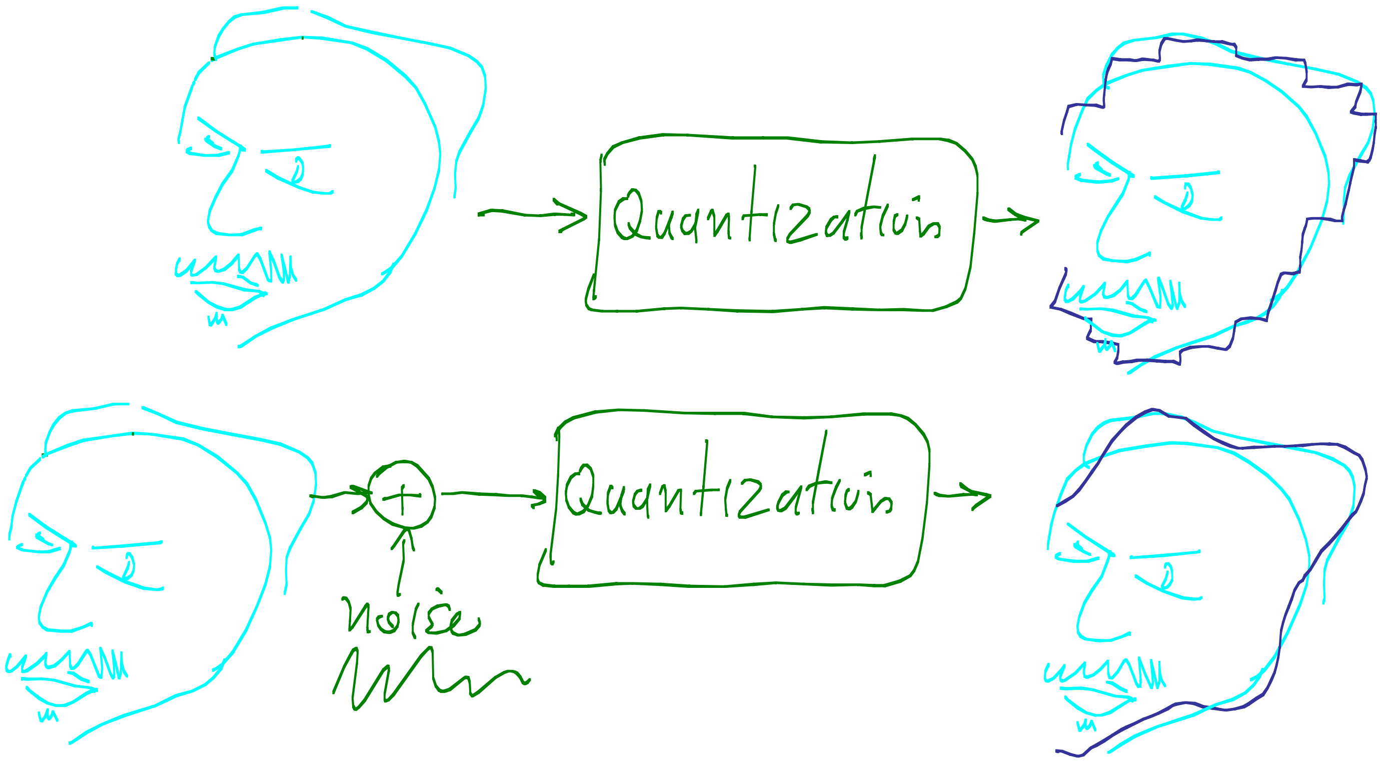
Tutorial-Part A Outline

3. Dither & estimation

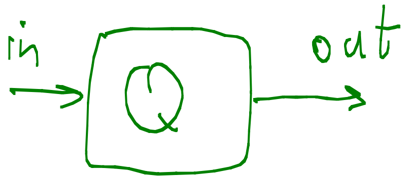
noise (1)

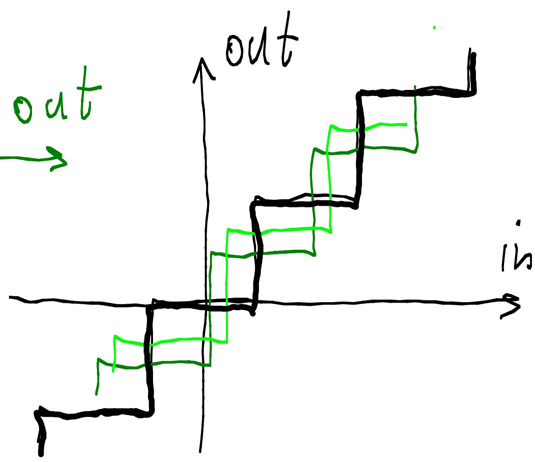
Dithered Quantization

- dither for perceptual reasons:

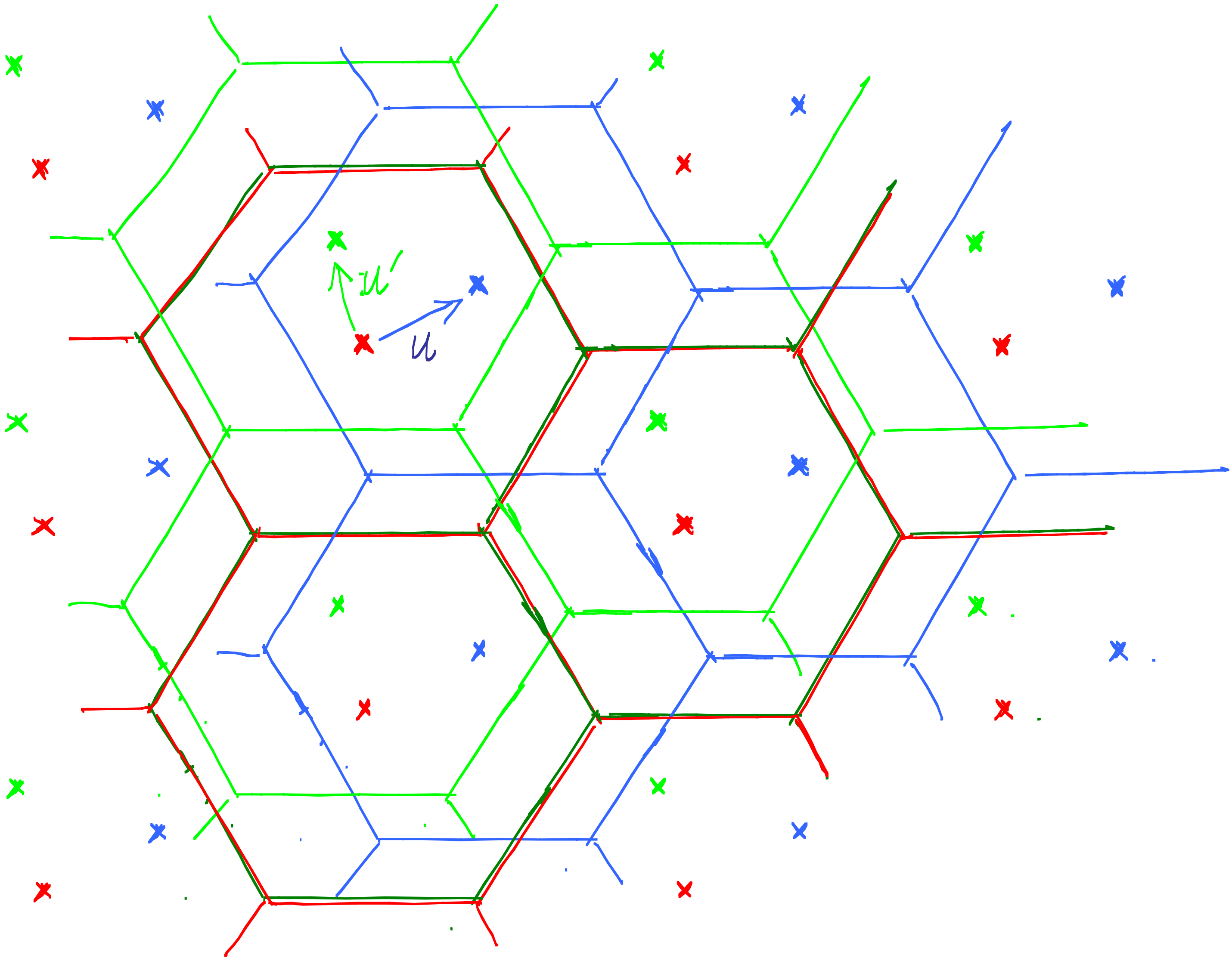


- dither for analytical reasons:





$$Q_\Lambda(x+u)-u$$



$\Rightarrow$ Random shift of the lattice quantizer

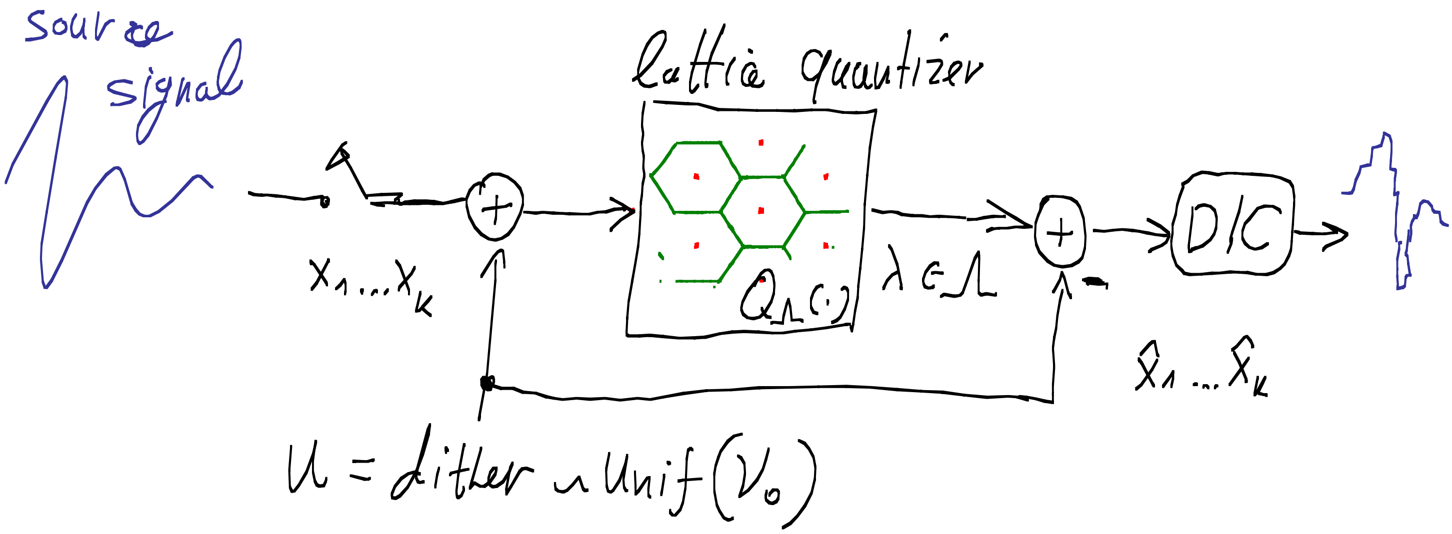
The Crypto-Lemma

Let $\boxed{x \bmod \Lambda \triangleq x - Q_\Lambda(x)}$

$$\boxed{\begin{array}{l}\text{If } \quad U \sim \mathrm{unif}(\mathcal{P}_0), \quad \text{then} \\ (x+U) \bmod \Lambda \sim \mathrm{unif}(\mathcal{P}_0), \quad \forall x\end{array}}$$

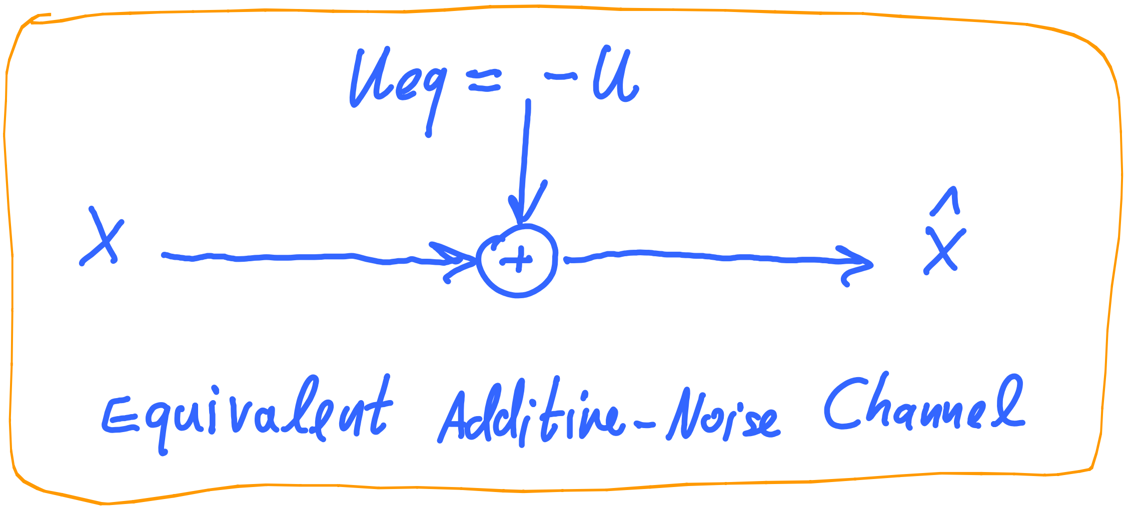
Proof: View as a modulo-additive noise channel, with a uniform noise.

Dithered Quantization Error



Crypto Lemma $\Rightarrow$

Thm. 1: quantization error $Q(x+U)-x-U$ is independent of input x, and uniform over (reflection of) lattice cell:



Generalized Dither

Def. $\mathcal{U}$ is G.D. if $\boxed{(s+\mathcal{U}) \bmod \Lambda \sim \mathrm{Unif}(\mathcal{P}_0)}$ $\forall s$

Necessary condition on $f_{\mathcal{U}}(\cdot)$ for G.D. ?

[Schuschman, Stockham - Gray, others]

[Tamas Linder ]

Generalized Dither

Def. $\mathcal{U}$ is G.D. if $\boxed{(s+\mathcal{U}) \bmod \Lambda \sim \mathrm{Unif}(\mathcal{P}_0)}$ $\forall s$

Necessary condition for G.D.?

1. $\mathcal{U}$ is G.D. iff $\mathcal{U} \bmod \Lambda \sim \mathrm{Unif}(\mathcal{P}_0)$

2. $\mathcal{U}$ is G.D. iff $f_{\mathcal{U}_{rep}}(x) = \text{constant}$

   where,

   $f_{rep}(x) \triangleq$ periodic replication $f(x) \triangleq \sum_{\lambda \in \Lambda} f(x-\lambda)$

3. $\mathcal{U}$ is G.D. iff its characteristic function is zero on the dual lattice:

$$\mathcal{F}\{f_{\mathcal{U}}(\cdot)\} = 0 \quad \text{on } \Lambda^* \setminus \underline{0}$$

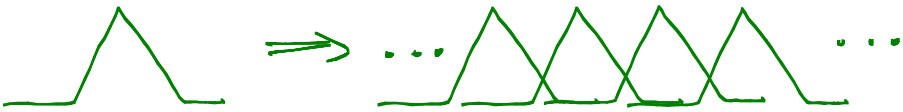
where $\Lambda^* = \text{dual lattice} = \Lambda(G^{-t})$

Generalized Dither

Def. $\mathcal{U}$ is G.D. if $\boxed{(s+\mathcal{U}) \bmod \Lambda \sim \mathrm{Unif}(\mathcal{P}_0)}$ $\forall s$

Necessary condition for G.D. ?

$f_{rep}(x) \triangleq$ periodic replication $f(x) \triangleq \sum_{\lambda\in\Lambda} f(x-\lambda)$



Claims

1. $f_{rep}(x)$ is periodic-Λ in space
2. If $X\sim f(x)$, and $\mathcal{P}_0$ = fundmental cell of Λ, then

$$f_{X \bmod \Lambda}(x) = \begin{cases} f_{rep}(x), & x\in\mathcal{P}_0 \\ 0, & \text{o.w.}\end{cases}$$

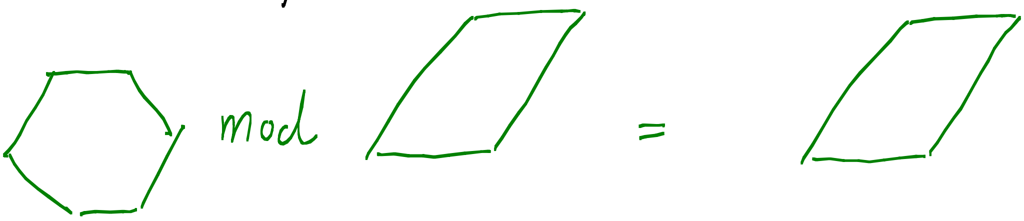
3. $X \bmod \Lambda \sim \mathrm{Unif}(\mathcal{P}_0)$ iff $f_{rep}(x) = \text{constant}$
4. $\mathcal{U}$ is generalized dither iff $f_{\mathcal{U}rep}(x) = \text{constant}$

Generalized Dither: Examples

1. Uniform over <u>any</u> fundmental cell

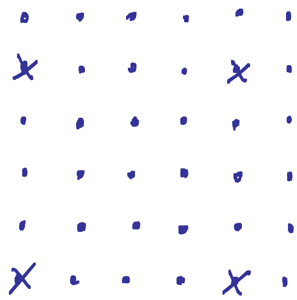
$$\text{Unif}(Q_0) \bmod_{P_0} \Lambda \sim \text{Unif}(P_0)$$

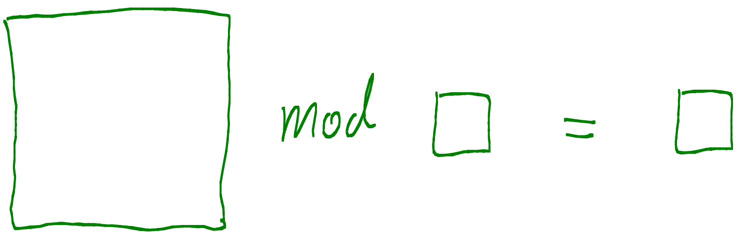
where Q_0, P_0 = fundmental cells of Λ.



2. Uniform over a <u>nested</u> coarse lattice cell

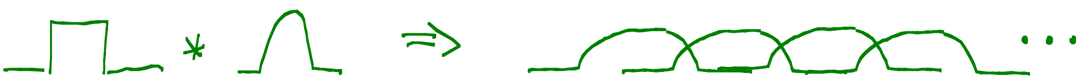
Q_0 = fundmental cell of $\Lambda_c \subset \Lambda$





3. Spreading

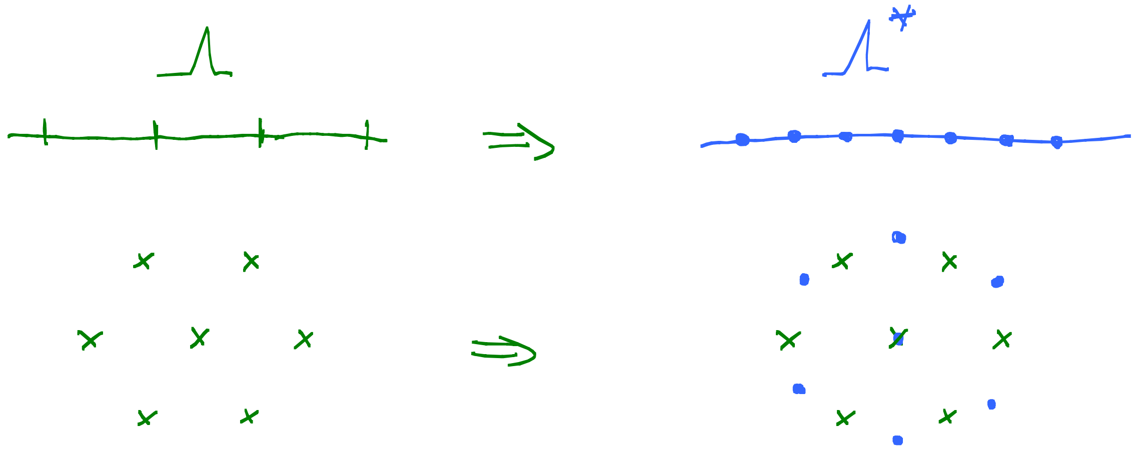
$$\{f_U(\cdot)\}_{rep} = \text{constant} \Rightarrow \{f_U(\cdot) * \tilde{f}(\cdot)\}_{rep} = \text{constant}$$



Generalized Dither $\Rightarrow$ Zeroes on Dual Lattice

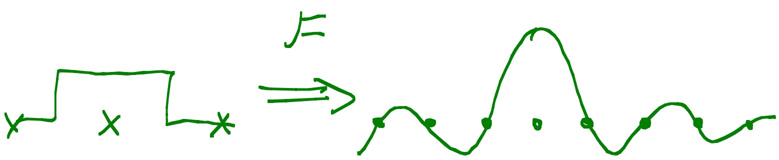
Def. $\Lambda^* =$ dual lattice of $\Lambda(G)$

$$= \Lambda(G^{-t})$$



Claim: U is G.D. iff its characteristic function is zero on the dual lattice:

$$F\{f_U(\cdot)\} = 0 \quad \text{on } \Lambda^* \setminus \underline{0}$$



Good lattice $\Rightarrow$ white dither

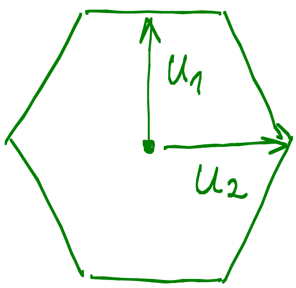
$\underline{\underline{R}}_Q \triangleq$ dither auto-correlation matrix $= E\{\underline{u} \cdot \underline{u}^t\}$

$M_u \triangleq \frac{1}{n} \text{trace}\{\underline{R}_Q\} \geq \sigma^2(\Lambda)$

equality if Voronoi cells

Thm.: If Λ is an optimal lattice quantizer in $\mathbb{R}^n$ (minimizes N.S.M. $G(\Lambda)$), then $\underline{u}$ is white:

$$\underline{R}_Q = \sigma^2(\Lambda) \cdot \underline{\underline{I}}_n$$



u_1 and u_2 are dependent

but $Var(u_1) = Var(u_2)$

$E\{u_1 \cdot u_2\} = 0$

Proof :

1. $\Lambda, \mathcal{V}_0 \longrightarrow$ whitenning (orthonormal) transformation $\longrightarrow \Lambda', P_0'$

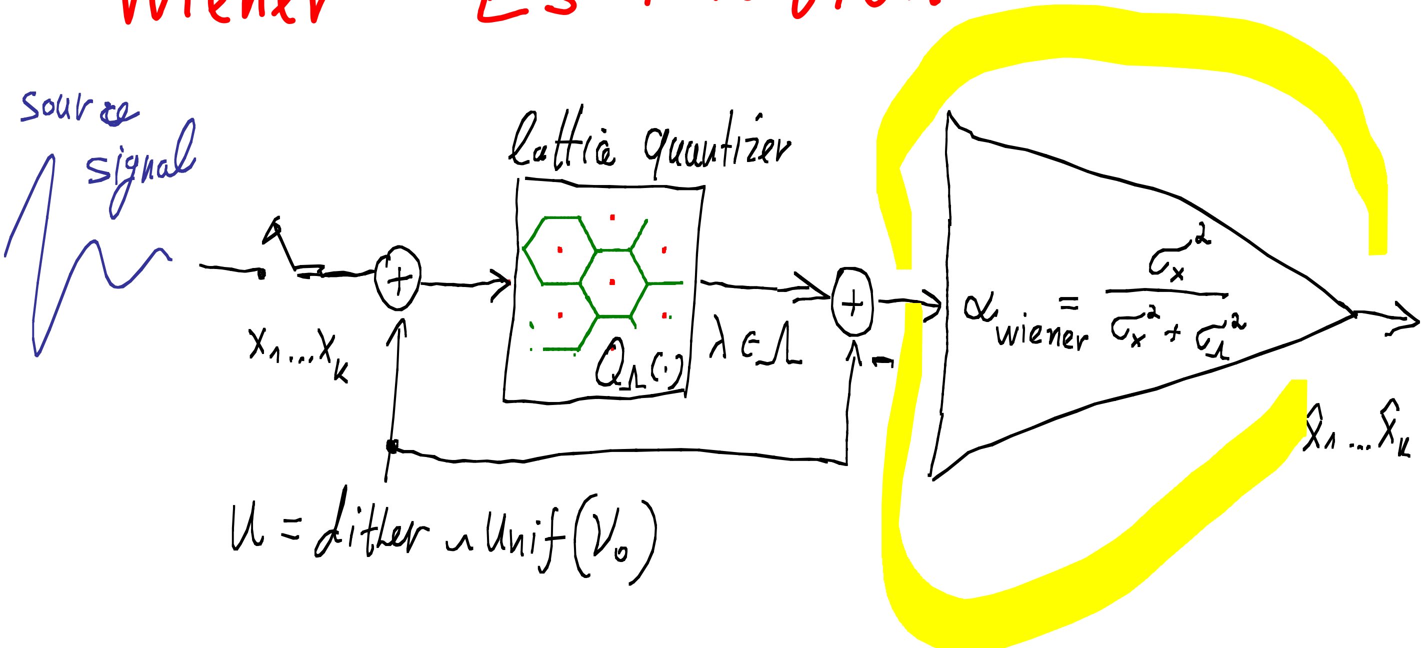
2. $\Lambda', P_0' \longrightarrow$ Voronoi Partition $\longrightarrow \Lambda', \mathcal{V}_0'$

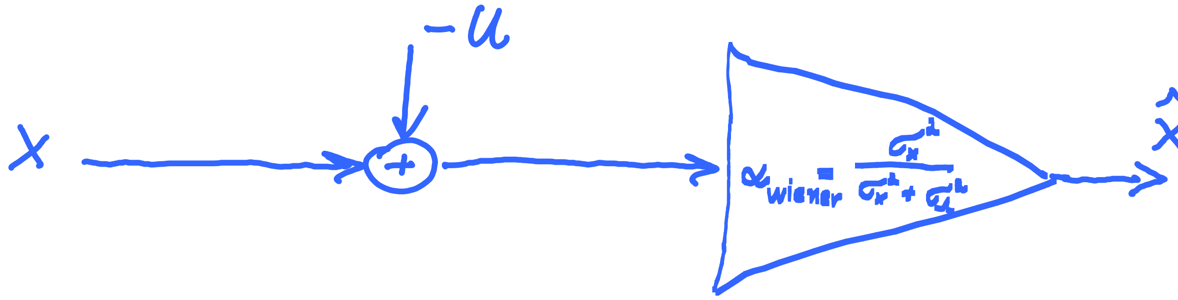
and repeat ...

$$\Longrightarrow \quad G(\Lambda) \geq G(\Lambda') \geq G(\Lambda'') \geq \dots$$

w. equality iff Λ is white !

Wiener Estimation





Equivalent Additive-Noise Wiener-Estimated Channel

$\Rightarrow$ distortion: $\sigma_\Lambda^2 \longrightarrow \frac{\sigma_x^2 \sigma_\Lambda^2}{\sigma_x^2 + \sigma_\Lambda^2}$

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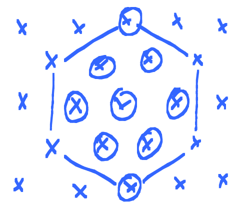


10. Side-information problems $\quad Modulo^2(\Lambda)$
11. Gaussian networks $\quad Modulo^n(\Lambda)$

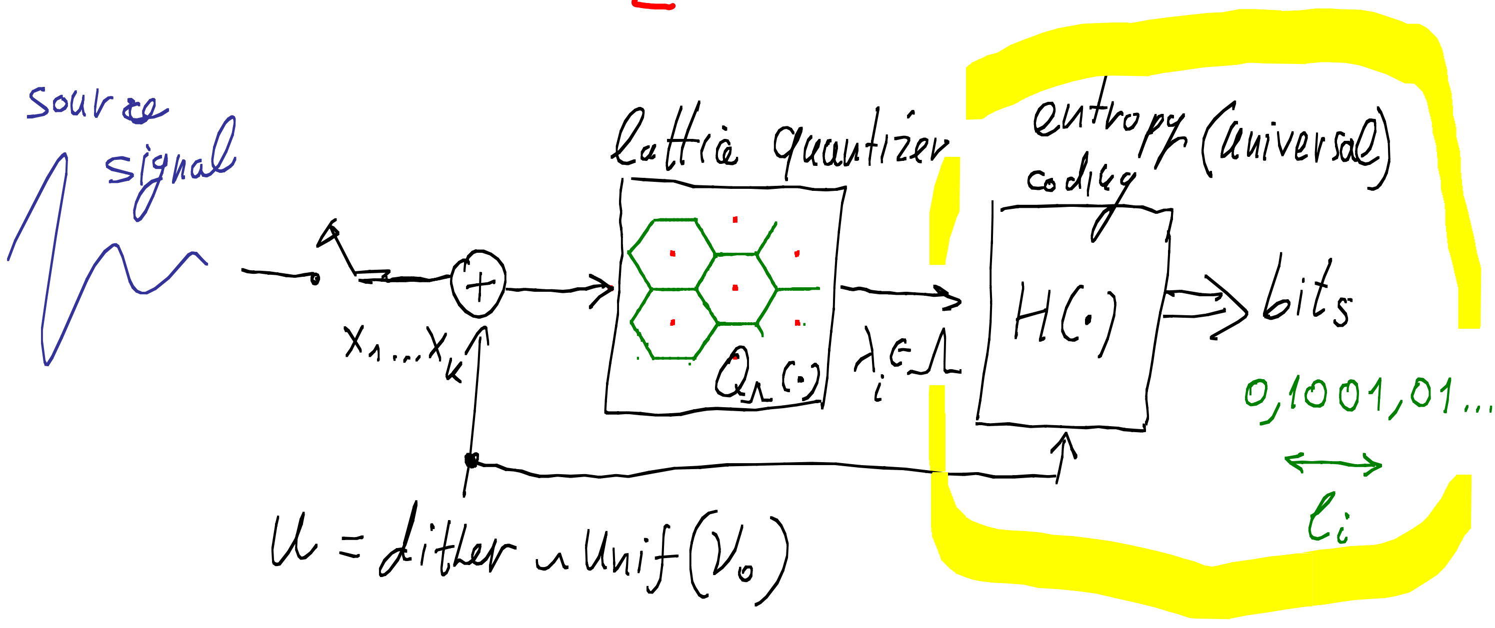
Tutorial - Part A Outline

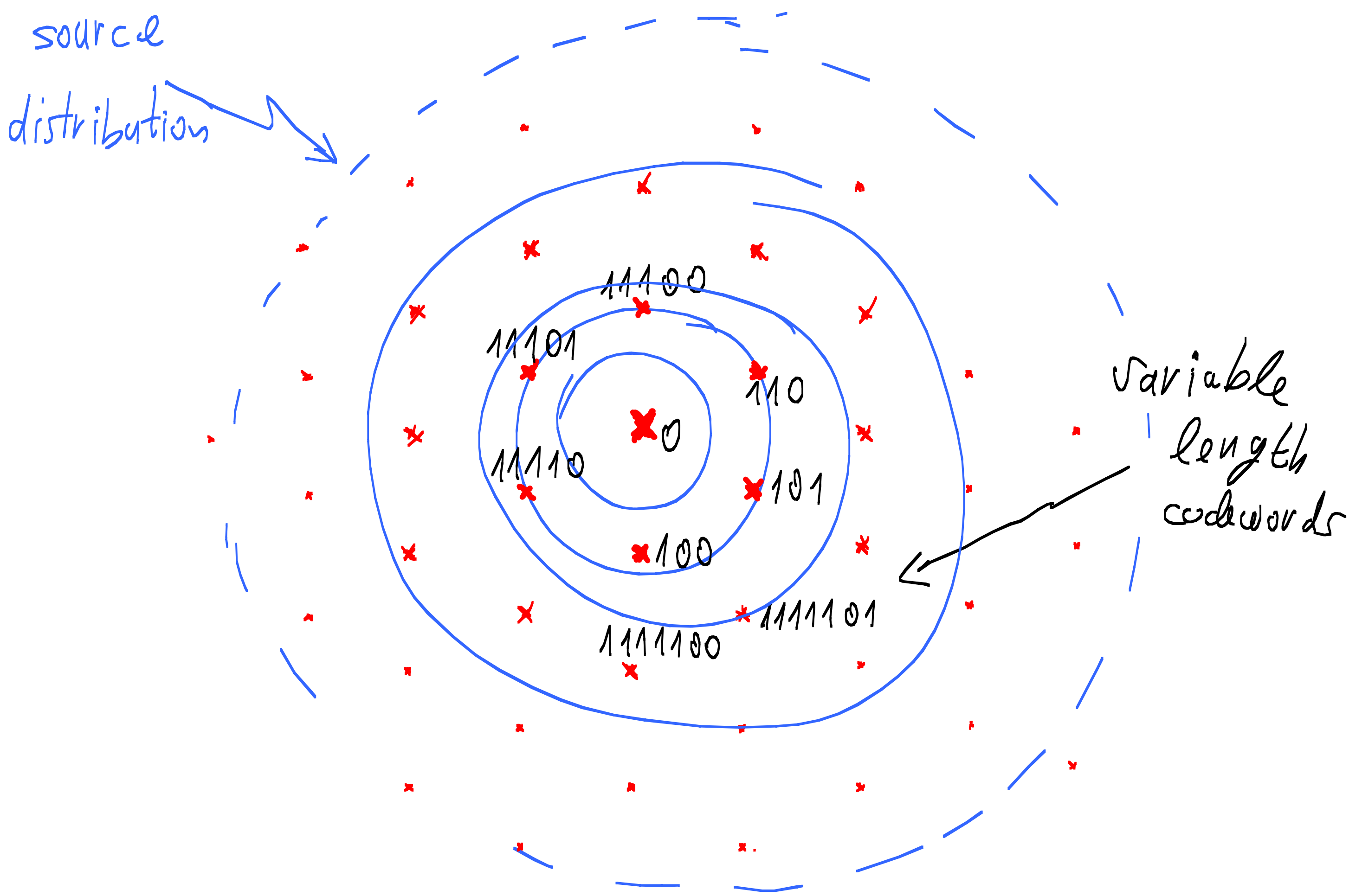
4. Entropy coding

$H(\Lambda)$

Entropy Coded Dithered Quantization

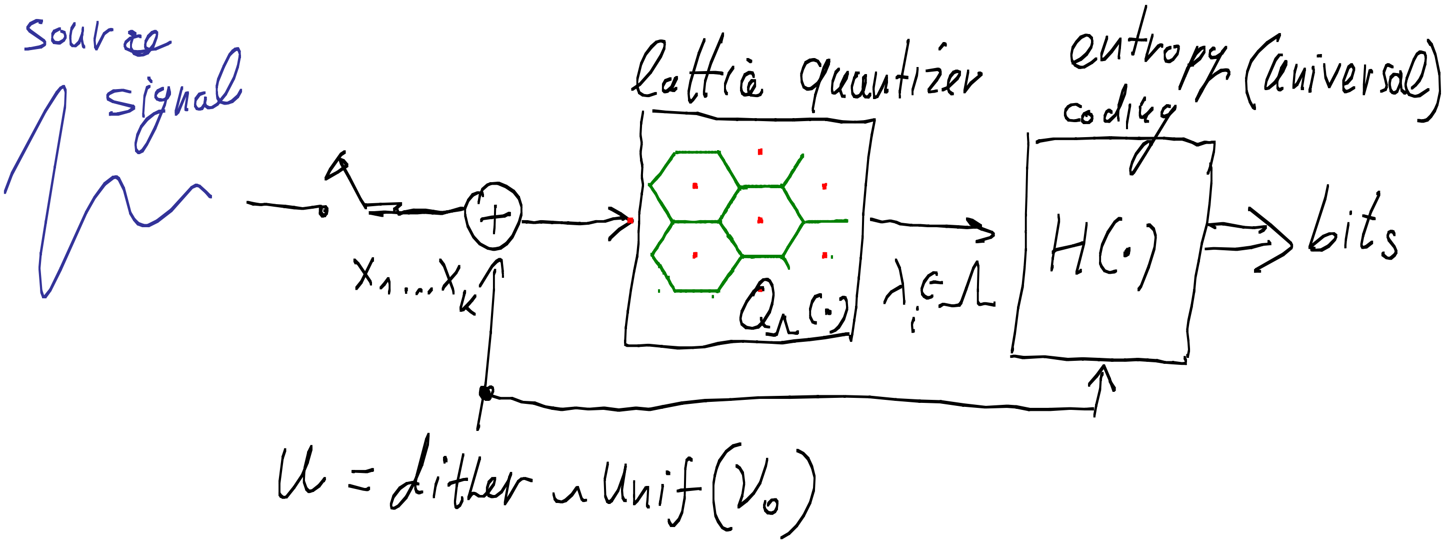
[ziv85]





Entropy Coded Dithered Quantization

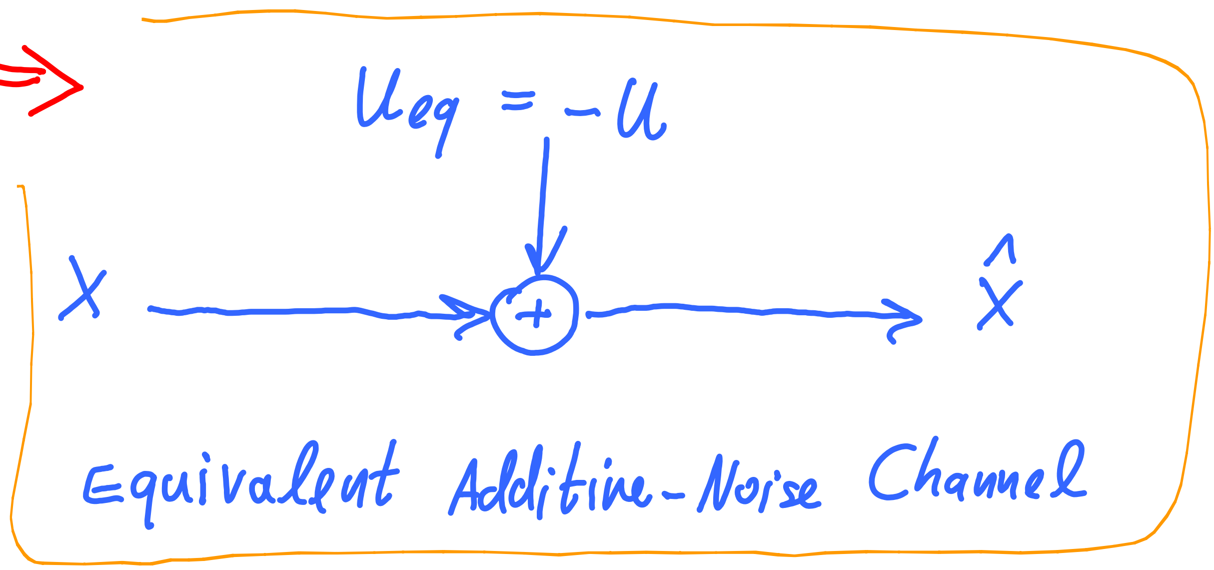
[ziv85]



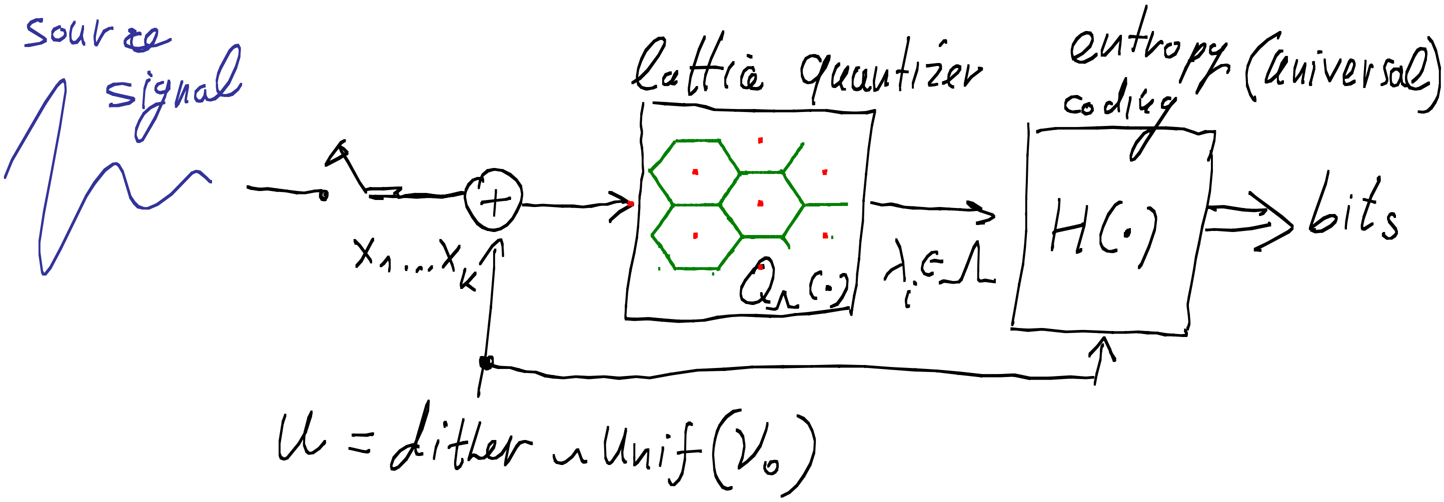
• For any source X and lattice Λ:

$$H(Q(X+U) \mid U) = I(X; X-U)$$

$\Rightarrow$



Entropy Coded Dithered Quantization [Ziv 85]



- For any source X and lattice Λ:

$$H(Q(X+U)\,|\,U) = I(X;\, X-U)$$

- and if X is Gaussian under MSE:

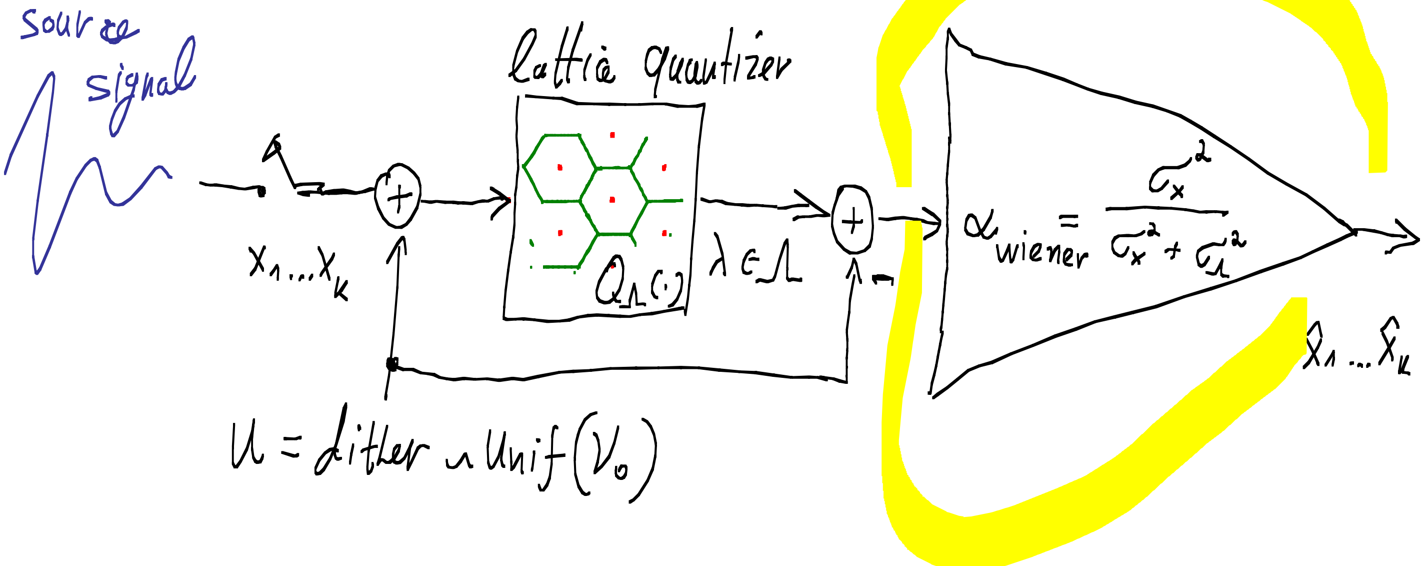
$$\lesssim \underbrace{\frac{1}{2}\log\left(1+\frac{\sigma_x^2}{D}\right)}_{R(D) \text{ upto loss of "1"}} + \underbrace{\frac{1}{2}\log\left(2\pi e\, G(\Lambda)\right)}_{\text{divergence of dither from AWGN}}$$

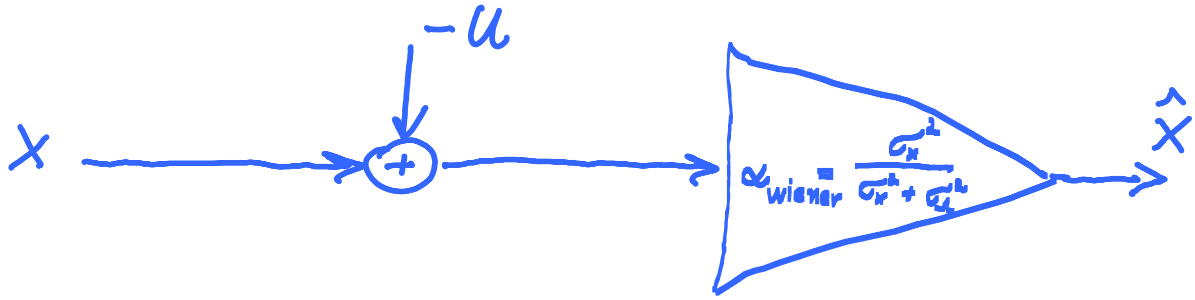
R(D) upto loss of "1"

divergence of dither from AWGN

$\frac{1}{2}\log(2\pi e/12) \approx 0.254 \frac{\text{bit}}{\text{sample}}$ for $\Lambda = \mathbb{Z}^n$

Wiener Estimation



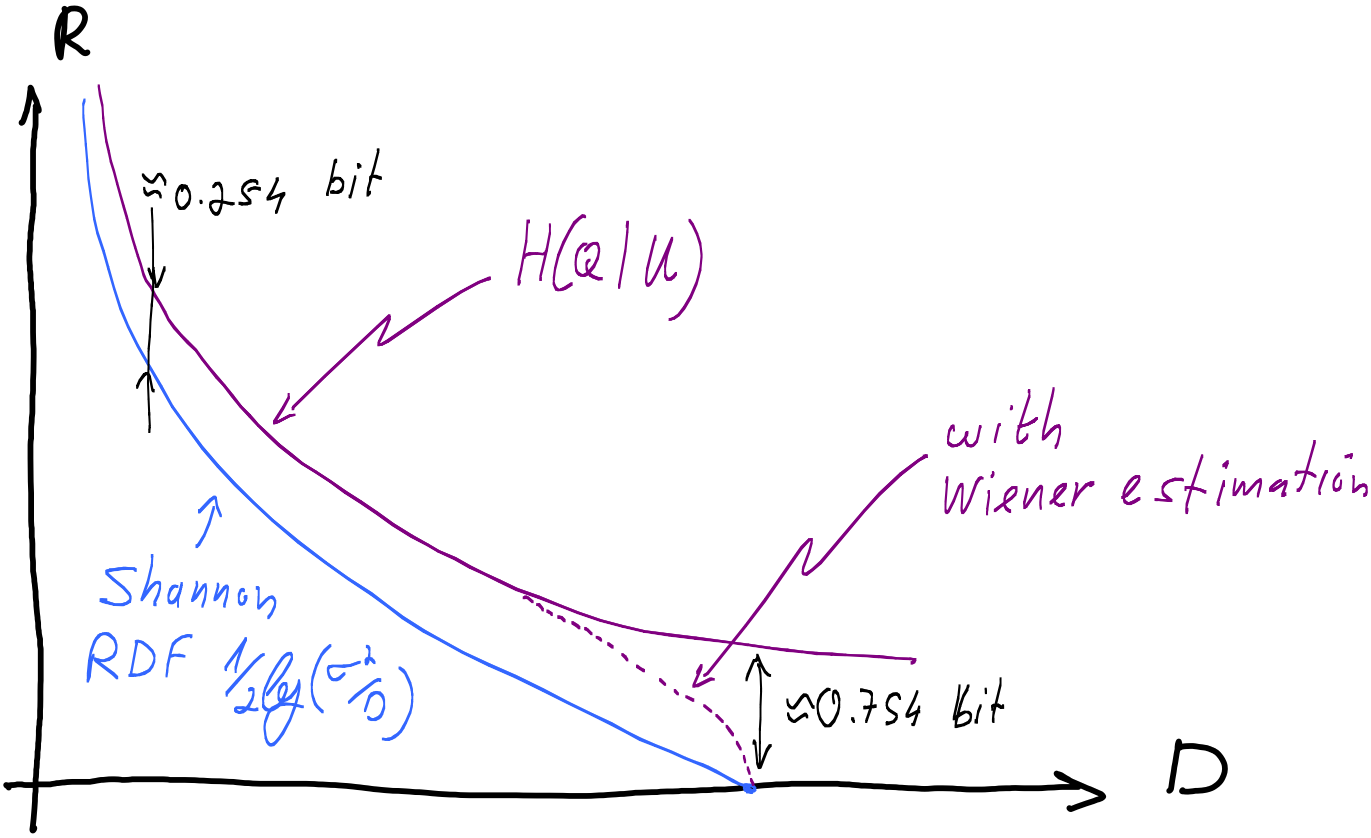


Equivalent Additive-Noise Wiener-Estimated Channel

$\Rightarrow$ distortion: $\sigma_\Lambda^2 \longrightarrow \dfrac{\sigma_x^2 \, \sigma_\Lambda^2}{\sigma_x^2 + \sigma_\Lambda^2}$

rate: $\underbrace{\frac{1}{2}\log\left(\frac{\sigma_x^2}{D}\right)}_{\text{saved extra "1" loss!}} + \frac{1}{2}\log\left(2\pi e G(\Lambda)\right)$

R-D Curves

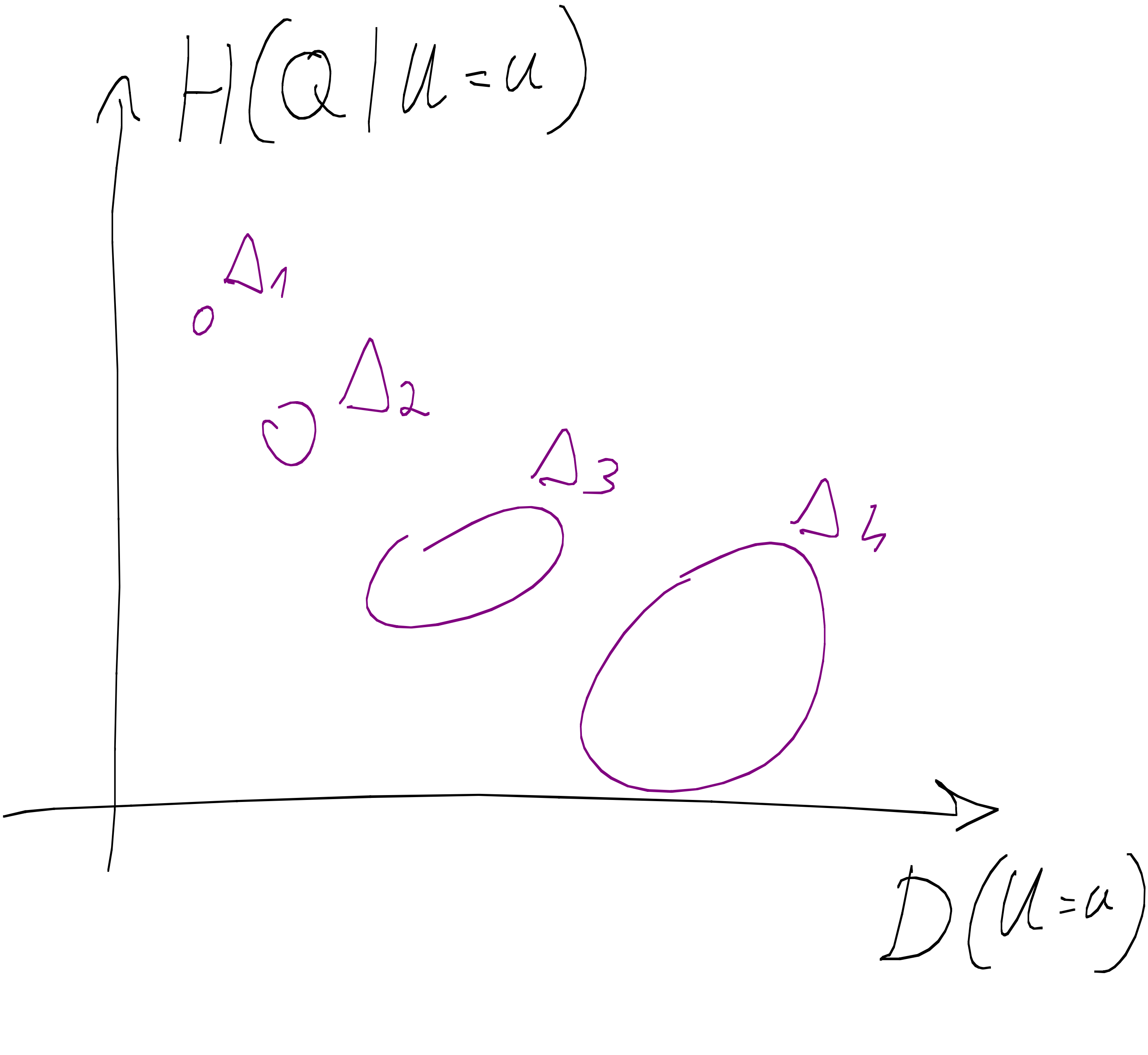


R
D
≈0.254 bit
H(Q|U)
with
Wiener estimation
≈0.754 bit
Shannon
RDF $\frac{1}{2}\log(\sigma^2/D)$

Open Question:

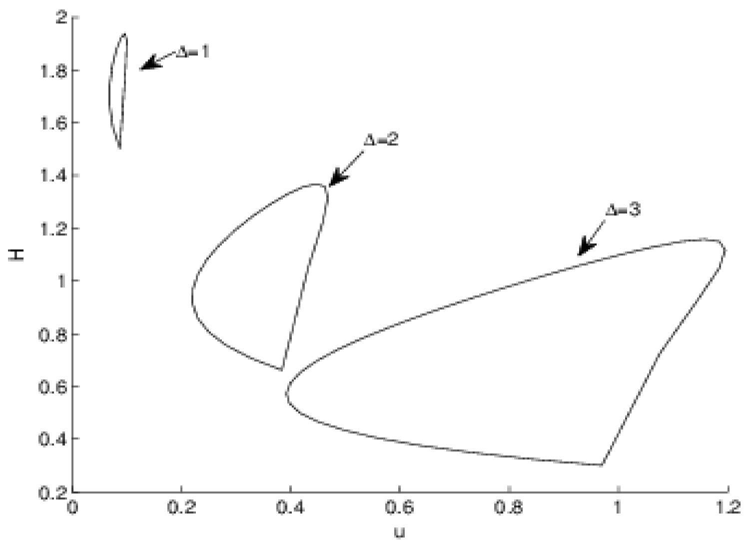
Is the dither really necessary?

R-D curves for non-random dither



$$\Delta_1 < \Delta_2 < \Delta_3 < \Delta_4$$

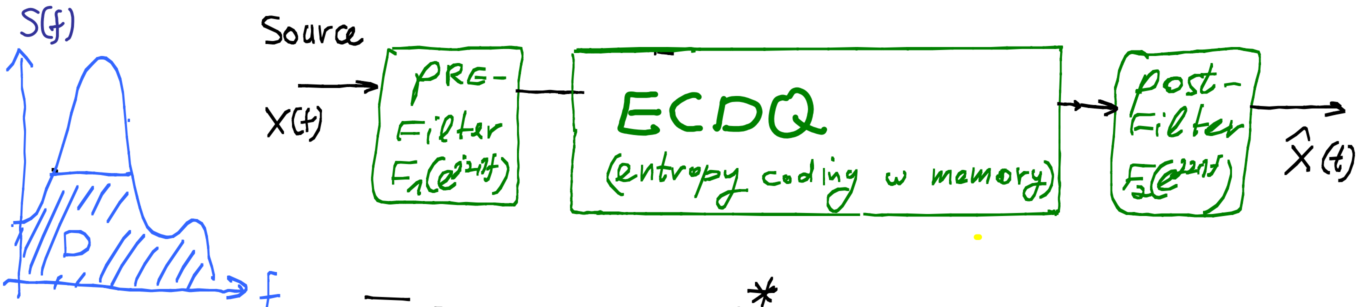
R-D curves for non-random dither



X ~ exponential

ECDQ Applications

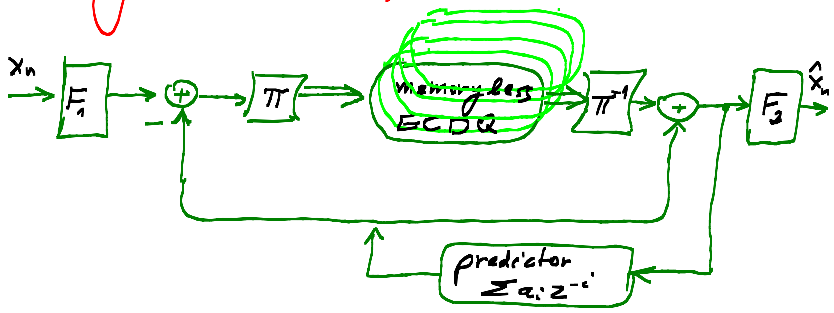
1. pre/post-filtered ECDQ: [Zamir - Feder]



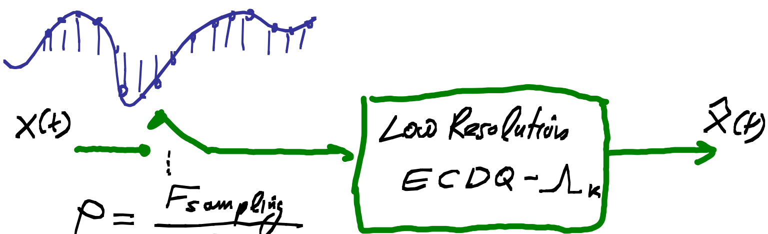
$$\bar{H}(Q_K|Z) = R^*(D) + \frac{1}{2}\log(2\pi e G_K)$$

2. Predictive Coding (DPCM):

[Z-Kochman-Erez]

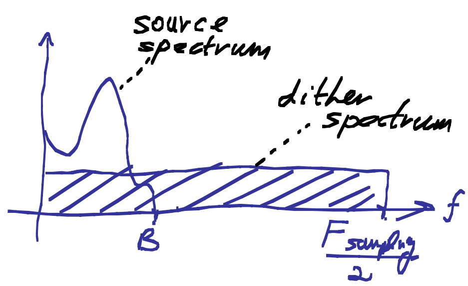


3. Oversampled ECDQ:



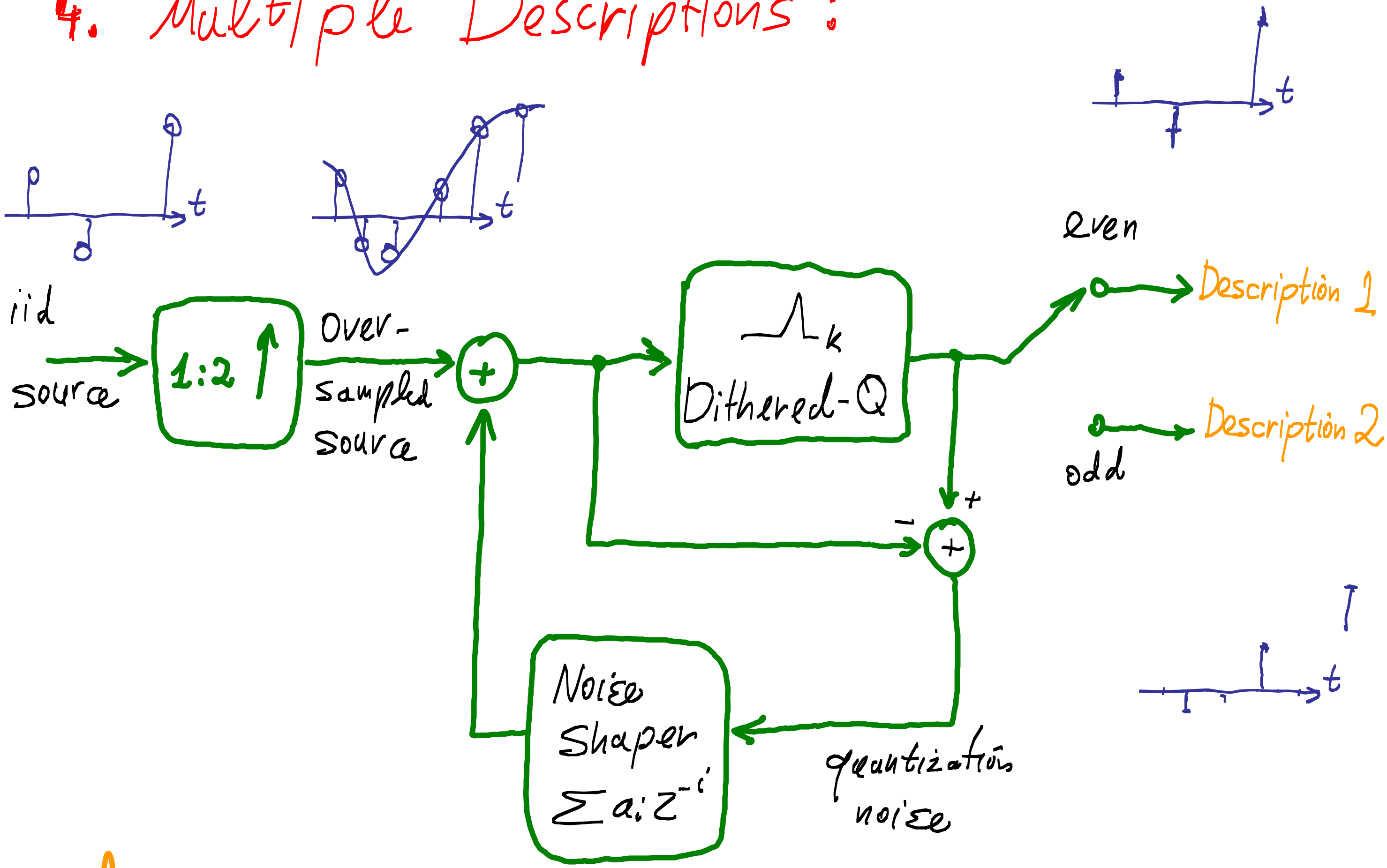
$$\rho = \frac{F_{sampling}}{2B}$$

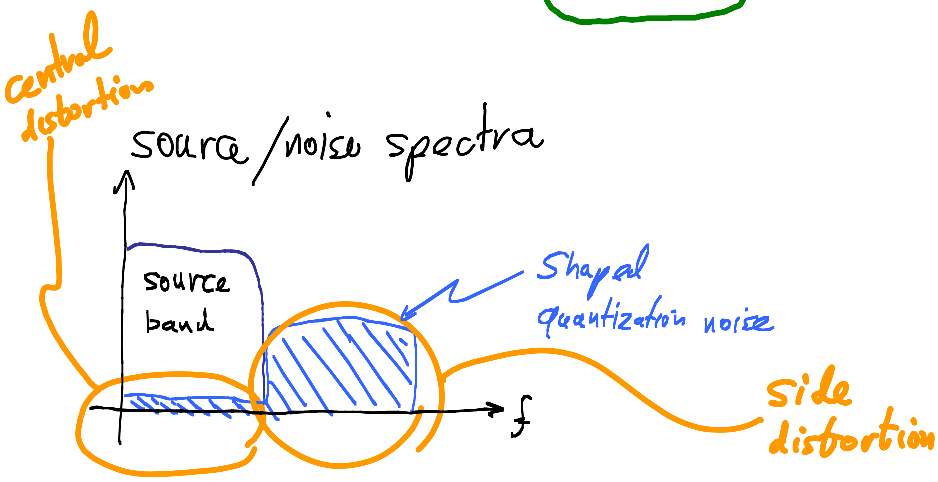
= oversampling ratio



ECDQ Applications (cont.)

4. Multiple Descriptions:





[Ostergaard-Z]

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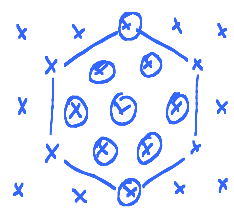


10. Side-information problems $\quad$ $Modulo^2(\Lambda)$
11. Gaussian networks $\quad$ $Modulo^n(\Lambda)$

Tutorial - Part A Outline

5. Infinite constellation

$$P_e(\Lambda)$$

... and Non-uniform signaling

Unconstrained Channels [Poltyrev 1994]

$R_\infty \triangleq$ rate per unit volume

$=$ normalized logarithmic point density

$M(B) \triangleq$ # codewords in a body B in $\mathbb{R}^n$ $\triangleq V(B) \cdot 2^{n R_\infty}$

$$\Longrightarrow \quad R_\infty(\Lambda) = \frac{1}{n} \cdot \log\left(\frac{1}{V(\Lambda)}\right)$$

$$= -\frac{1}{2} \log(\mu \cdot \sigma^2)$$

AWGN channel

$\mu = VNR = \frac{V^{2/n}}{\sigma^2}$

Poltyrev Capacity: $R_\infty \leq C_\infty$

$$C_\infty \triangleq \lim_{\text{power} \to \infty} \max_{P(X)} \left[I(X; X+Z) - h(X) \right]$$

$$= -h(Z)$$

AWGN

$$= -\frac{1}{2} \log(2\pi e\, \sigma_z^2)$$

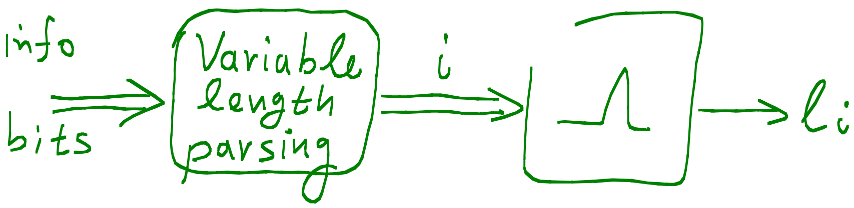
$\Longrightarrow$ **Lattice gap to capacity:**

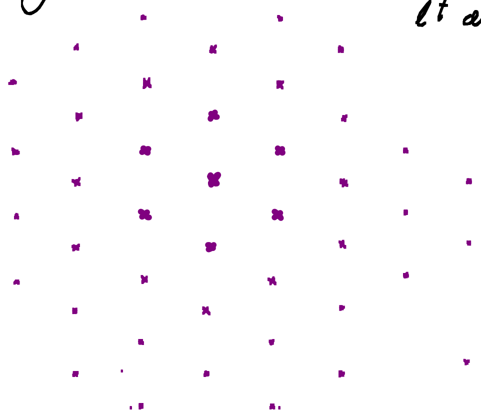
$$C_\infty - R_\infty(\Lambda)\big|_{P_e} = \frac{1}{2} \log\left(\frac{\mu(\Lambda, P_e)}{2\pi e}\right)$$

Infinite versus Finite Constellations

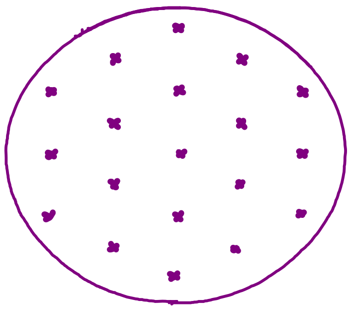
1. Unbounded lattice : capacity per unit volume [Poltyrev 93]

2. Probabilistic Shaping : [Gallager, Forney et al., Kschischang et al.]

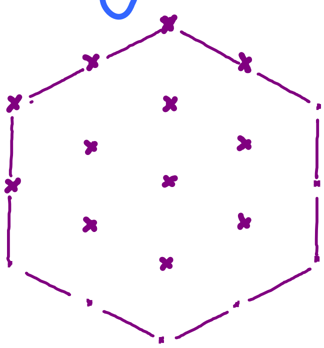




3. Deterministic Shaping - Spherical : [De Buda 89]

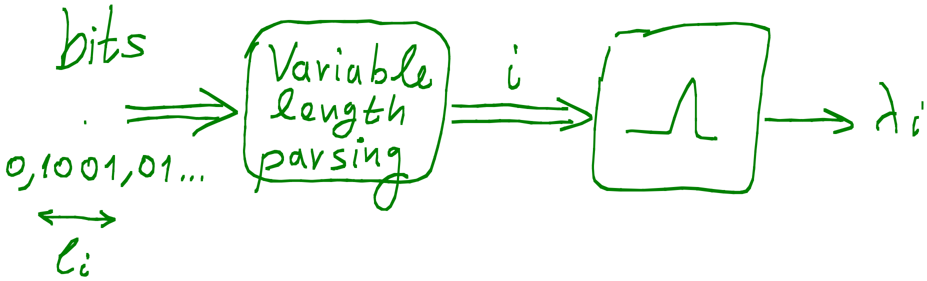


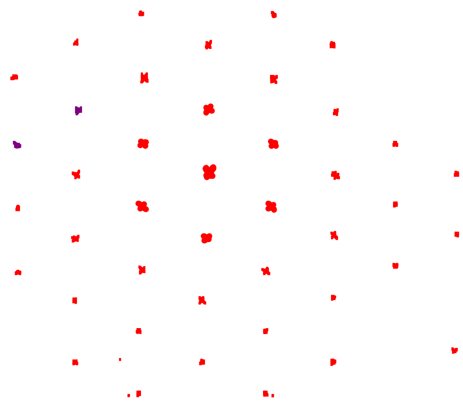
4. Deterministic Shaping - Voronoi Code : [Sloane, Forney 89]



The Channel-Dual of ECDQ: Non-Uniform Signaling

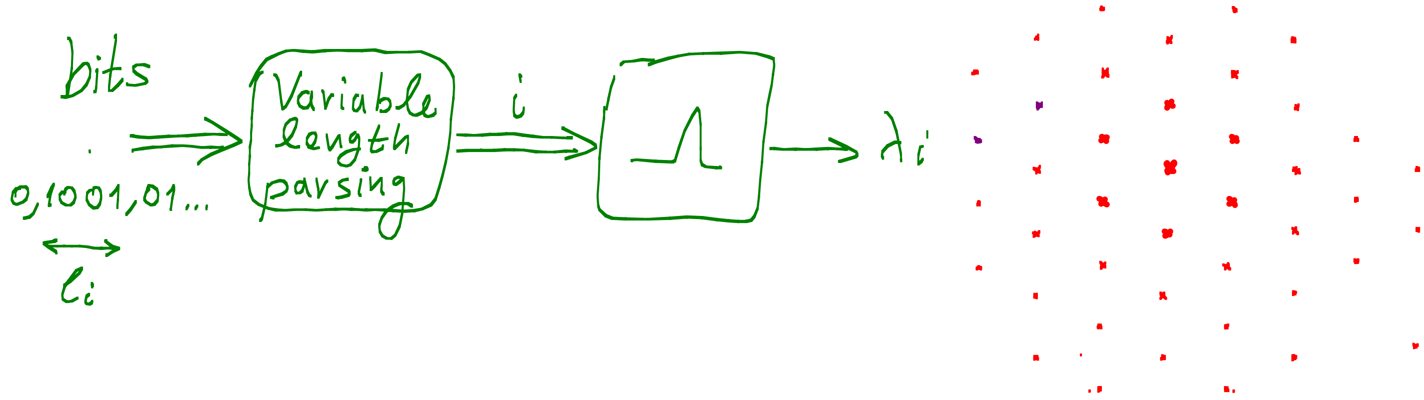
1. Transmission:



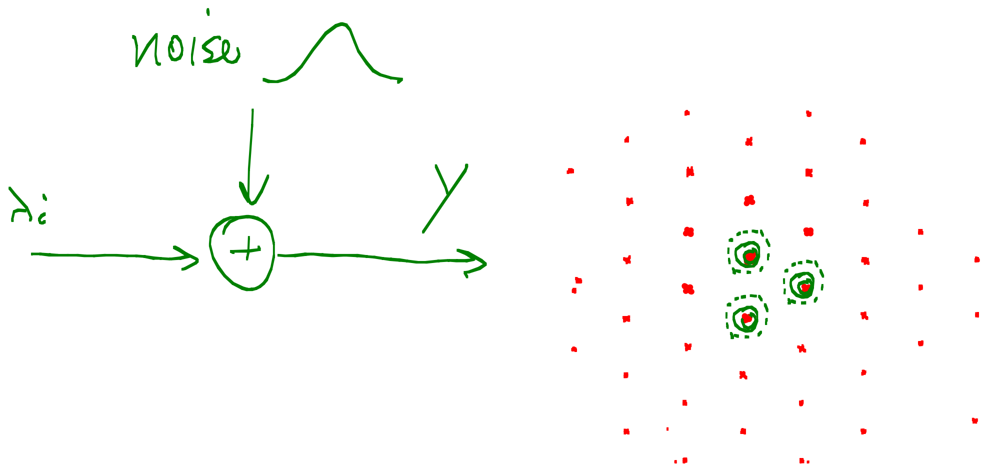


The Channel-Dual of ECDQ: Non-Uniform Signaling

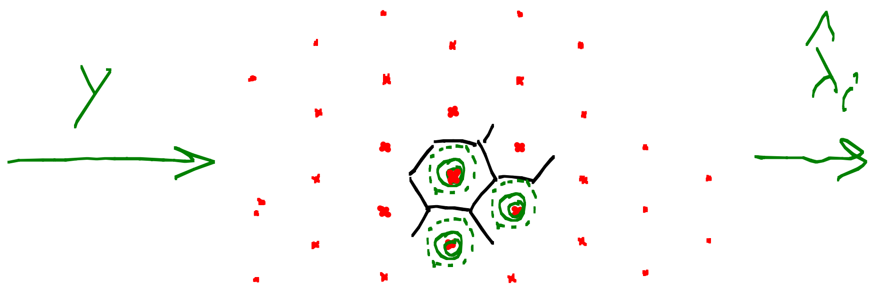
1. Transmission:



2. Noisy Channel:



3. Decoding:



$$P_e = \Pr\{\hat{\lambda}_i \neq \lambda_i\}$$

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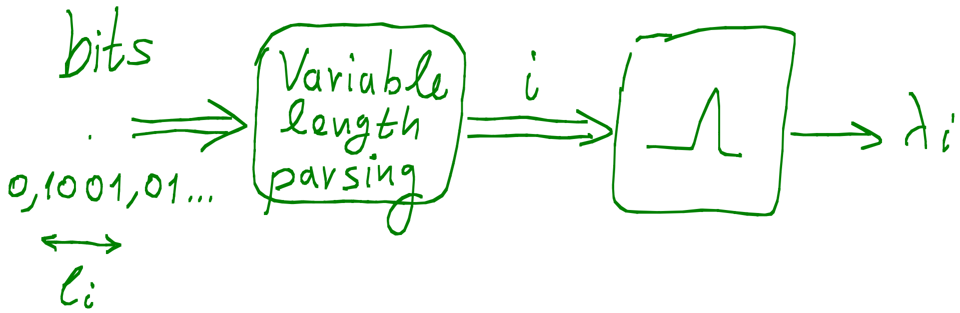

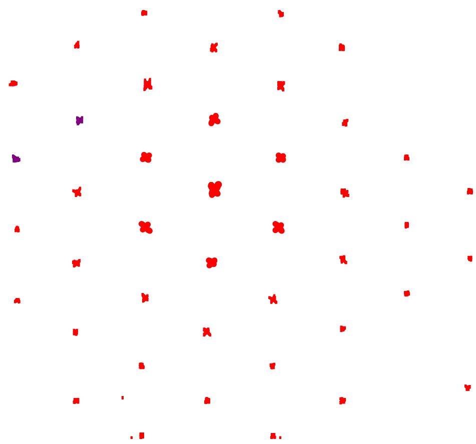

$$p_i = 2^{-\ell_i}$$

$$\text{Rate} = \frac{\text{average \# bits}}{\text{dimension}} = \frac{1}{n} \sum_i p_i \ell_i = H(\underline{p})$$

$$\text{power} = \frac{1}{n} \sum_i p_i \|\lambda_i\|^2$$

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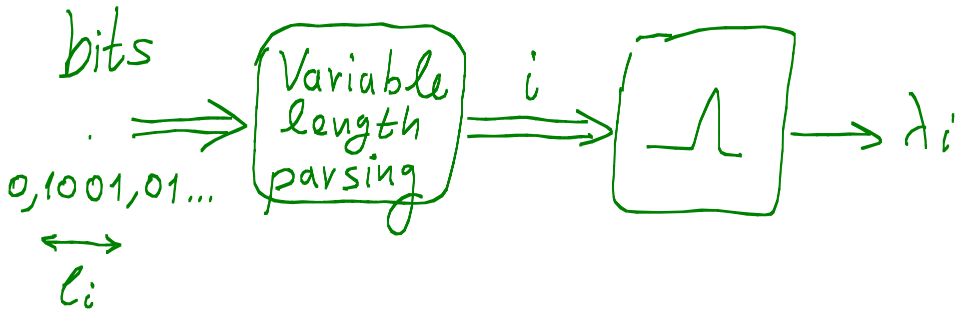
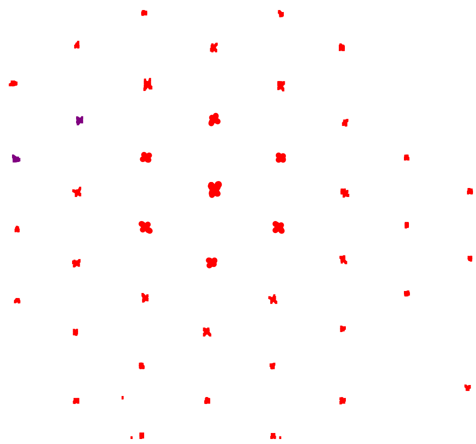




$$p_i = 2^{-l_i}$$

$$\text{Rate} = \frac{\text{average \# bits}}{\text{dimension}} = \frac{1}{n}\sum_i p_i l_i = H(\underline{p})$$

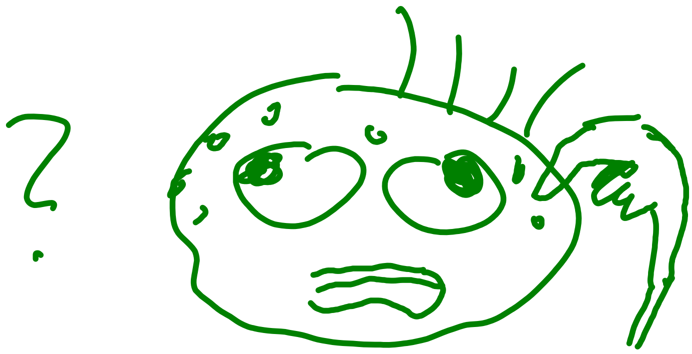
$$\text{power} = \frac{1}{n}\sum_i p_i \|\lambda_i\|^2$$

$$\boxed{\text{Rate} = \underbrace{\tfrac{1}{2}\log(SNR)}_{\text{AWGN channel capacity up to loss of "1"}} - \underbrace{\tfrac{1}{2}\log\left(\frac{\mu(\Lambda, P_e)}{2\pi e}\right)}_{\text{lattice gap to capacity}}}$$

Open Question :

Can we get the "1" in $\log(1+SNR)$ by MAP decoding ?

(preliminary work by Palgy-Z)



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