Generalized theory and simulation of spontaneous and super-radiant emissions in electron devices and free-electron lasers

Y. Pinhasi* and Yu. Lurie
Department of Electrical and Electronic Engineering—Faculty of Engineering, The College of Judea and Samaria, P.O. Box 3, Ariel 44837, Israel

(Received 3 August 2000; revised manuscript received 11 October 2001; published 22 January 2002)

A unified formulation of spontaneous (shot-noise) and super-radiant emissions in electron devices is presented. We consider an electron beam with an arbitrary temporal current modulation propagating through the interaction region of the electronic device. The total electromagnetic field is presented as a stochastic process and expanded in terms of transverse eigenmodes of the medium (free space or waveguide), in which the field is excited and propagates. Using the waveguide excitation equations, formulated in the frequency domain, an analytical expression for the power spectral density of the electromagnetic radiation is derived. The spectrum of the excited radiation is shown to be composed of two terms, which are the spontaneous and super-radiant emissions. For a continuous, unmodulated beam, the shot noise produces only incoherent spontaneous emission of a power proportional to the flux $eI_0$ (DC current) of the particles in the electron beam. When the beam is modulated or prebunched, a partially coherent super-radiant emission is also produced with power proportional to the current spectrum $|\tilde{I}(\omega)|^2$. Based on a three-dimensional model, a numerical particle simulation code was developed. A set of coupled-mode excitation equations in the frequency domain are solved self-consistently with the equations of particles motion. The simulation considers random distributions of density and energy in the electron beam and takes into account the statistical and spectral features of the excited radiation. At present, the code can simulate free-electron lasers (FELs) operation in various modes: spontaneous and self-amplified spontaneous emission, super-radiance and stimulated emission, in the linear and nonlinear Compton or Raman regimes. We employed the code to demonstrate spontaneous and super-radiant emission excited when a prebunched electron beam passes through a wiggler of an FEL.

DOI: 10.1103/PhysRevE.65.026501
PACS number(s): 41.60.Cr, 52.59.Px

I. INTRODUCTION

Electron devices such as microwave tubes and free-electron lasers (FELs) utilize distributed interaction between an electron beam and electromagnetic radiation. Random electron distribution in the $e$ beam due to its corpuscular nature causes fluctuations in current density, identified as shot noise in the beam current [1–4].

The shot noise current is characterized by a “white” power spectrum whose density is proportional to the average electron flux $eI_0$ of the beam ($e$ is the electron charge and $I_0$ is the DC current). The electromagnetic fields excited by each electron add incoherently, resulting in spontaneous emission noise in the radiation. If the electron beam is modulated or prebunched, the fields excited by electrons become correlated, and coherent summation of radiation fields from individual particles occurs. If all electrons radiate in phase with each other, the generated radiation becomes coherent (super-radiant emission). The terminology of super-radiance was suggested in [5] for radiation emitted in a quantum mechanical system during a transition between two energy levels of molecules in a gas of dimension small compared to a wavelength.

Electrons passing through a magnetic undulator emit a partially coherent radiation, which is called undulator synchrotron radiation [6]. In the classical analysis, each wig-
the (incoherent) spontaneous and of the coherent spontaneous (super-radiant) emission.

We employ the three-dimensional (3D) coupled-mode theory for an analytical derivation of spontaneous and super-radiant emissions in the linear regime of FEL operation, and also for the development of a three-dimensional particle simulation code. Unlike a previously developed steady-state numerical model [38], in which the interaction is assumed to be at a single frequency (or at its discrete harmonics), the present approach considers a continuum of frequencies, enabling the solution of nonstationary, wide-band interaction in radiation devices. Solution of the excitation equations in the space-frequency domain (and not in the space-time domain, as often carried out in numerical particle simulation codes), inherently takes into account dispersive effects arising from the cavity and the gain medium. Furthermore, it facilitates the consideration of the statistical and spectral features of the electromagnetic field excitation process, necessary in a study of noncoherent and partially coherent effects, such as spontaneous and super-radiant emissions, self-amplified emission and noise, in the linear and nonlinear regimes of the FEL operation.

II. MODAL REPRESENTATION OF THE ELECTROMAGNETIC FIELD

The analysis is based on modal expansion of the total electromagnetic field in terms of transverse eigenmodes of the medium (free space or waveguide) in which the radiation is excited [33]. The field of each transverse mode $q$ in the angular frequency domain $\omega$ is given by

$$
\vec{E}_q(r, \omega) = \bar{C}_q(z, \omega) \vec{E}_q(x, y) e^{jk_{zq}(\omega)z},
$$

(1)

where $\vec{E}_q(x, y)$ is the transverse profile (Hermite-Gaussian free-space mode or waveguide mode) and $k_{zq}(\omega)$ is its wave number. (Although the form of mode presentation given in Eq. (1) is not valid in the far-field free-space propagation, it is still applicable to most electron devices in which the interaction takes place within a Rayleigh length of the Hermite-Gaussian modes, where the diffraction is small). $\bar{C}_q(z, \omega)$ is the propagating mode amplitude satisfying the excitation equation

$$
\frac{d}{dz}\bar{C}_q(z, \omega) = \frac{1}{2\mathcal{P}_q} e^{-jk_{zq}(\omega)z} \int \int J(r, \omega) \cdot \mathcal{E}_q^*(x, y) dx dy.
$$

(2)

$\mathcal{E}_q^*(x, y)$ is the Fourier transform of the current density defined in the positive frequency domain $\omega > 0$, and $\mathcal{P}_q = \int \int_{x,y} \left| \mathcal{E}_q \times \mathcal{E}_q^* \right|^2 dx dy$ is the normalization power of the propagating mode $q$.

$J(r, \omega)$ is the Fourier transform of the current density defined in the positive frequency domain $\omega > 0$, and $\mathcal{P}_q = 1/2 \text{Re}\int_{x,y} \left[ \mathcal{E}_q \times \mathcal{E}_q^* \right] \cdot \mathcal{E}_q \times \mathcal{E}_q^* dx dy$ is the normalization power of the propagating mode $q$.

According to the Wiener-Khinchine theorem, the power spectral density carried by the propagating mode during a temporal period $T$ is found by averaging the ensemble of the radiation fields emitted by the electrons in the beam pulse [34]

$$
\frac{dP_q(z)}{d\omega} = \frac{1}{2\pi} \frac{1}{T} |\bar{C}_q(z, \omega)|^2 \mathcal{P}_q
$$

(3)

defined for $\omega > 0$. The power spectrum is the Fourier transform of the space-time correlation function of the electromagnetic radiation, describing its coherence properties [35–37].

III. THE EXCITATION CURRENT

Consider an electron beam with a time-dependent current $i(t)$ entering the interaction region of an electron device. Figure 1 is an example illustrating an electron bunch passing through the undulator of a free-electron laser. The current density of $k$ electrons in the bunch, each moving at an instantaneous velocity $v_i$, is given in the space-time domain by

$$
J(r, t) = -e \sum_{i=1}^{k} v_i \delta(x-x_i) \delta(y-y_i) \delta(z-z_i(t)),
$$

(4)

where $e$ is the electron charge, $k$ is the number of particles in an electron beam pulse, and $(x_i, y_i, z_i)$ are the coordinates of the $i$th electron’s position at a time $t$.

The number $k$ of electrons in the pulse is a random variable satisfying the Poisson distribution. The probability that the $e$-beam pulse contains $k$ electrons is

$$
p(k) = \frac{1}{k!} \langle k \rangle^k e^{-\langle k \rangle},
$$

(5)

where

$$
\langle k \rangle = \frac{1}{e} \int_{-\infty}^{+\infty} i(t) dt
$$

(6)

is the expected number (statistical average) of electrons in the pulse with variance $\langle k \rangle^2 - (\langle k \rangle)^2 = \langle k \rangle$ (equal to the average).

It is convenient to take $z$, the coordinate of axial propagation, as the independent variable, and write the time of particle arrival at $z$ as

$$
t_i(z) = t_{0i} + \int_{0}^{z} \frac{1}{v_{zi}(z')} dz'.
$$

(7)

$t_{0i}$ is the time that a particle $i$ entered at $z = 0$ and $v_{zi}(z)$ is its axial velocity along the path of motion. The electron arrival
times $t_{0i}$, at the entrance to the interaction region are independent random variables having a probability density function
\[ p(t_{0i}) = \frac{1}{\bar{k}e}. \] (8)

To find the electromagnetic field radiated from the moving charge, it is first required to calculate the Fourier transform of the current density as given in Eq. (4)
\[ \mathbf{J}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} J(t, \mathbf{r}) e^{+j\omega t} dt \]
\[ = -e \sum_{i=1}^{k} \frac{\mathbf{v}_i}{v_{zi}} \delta(x-x_i) \delta(y-y_i) e^{+j\omega t(z_i)}, \] (9)
and substitute it into the excitation Eq. (2), resulting in
\[ \frac{d}{dz} \tilde{C}_q(z, \omega) = \frac{e}{2\bar{p}_q} \sum_{i=1}^{k} \frac{1}{v_{zi}} \mathbf{v}_i \cdot \mathbf{E}_q^c(x_i, y_i) e^{+j[\omega t(z_i)-k_z z]}. \] (10)

IV. SPONTANEOUS AND SUPER-RADIANT EMISSIONS

The solution of the excitation equation at point $z$ in the interaction region is found by integration of Eq. (10), assuming $\tilde{C}_q(0, \omega) = 0$
\[ \tilde{C}_q(z, \omega) = e \sum_{i=1}^{k} \mathcal{H}_q(z, \omega) e^{+j\omega t_0}; \] (11)

\[ \frac{dP_q(z)}{d\omega} = \frac{1}{2\pi T} \frac{e^2}{\bar{p}_q} \sum_{i=1}^{k} |\mathcal{H}_q(z, \omega)|^2 + \sum_{i=1}^{k} \sum_{j=1}^{k} \mathcal{H}_q(z, \omega) \mathcal{H}_q^*(z, \omega) e^{+j[\omega t_0-\omega t_0]} \bar{p}_q. \] (14)

After performance of statistical averaging of Eq. (14)
\[ \frac{dP_q(z)}{d\omega} = \frac{1}{2\pi T} \frac{e^2}{\bar{p}_q} \left[ k|\mathcal{H}_q(z, \omega)|^2 + [\bar{k}(k-1)] \right] \]
\[ \times \left[ \mathcal{H}_q(z, \omega) \mathcal{H}_q^*(z, \omega) e^{+j[\omega t_0-\omega t_0]} \bar{p}_q \right]. \] (15)

The expected values
\[ e^{+j\omega t_0} = \langle e^{-j\omega t_0} \rangle^* = \int_{-\infty}^{\infty} e^{+j\omega t_0} p(t_0) dt_0 \]
\[ = \frac{1}{\bar{k}e} \int_{-\infty}^{\infty} i(t_0) e^{+j\omega t_0} dt_0 = \frac{1}{\bar{k}e} \mathcal{I}(\omega) \] (16)

are given in terms of the Fourier transform $\mathcal{I}(\omega) = \int_{-\infty}^{\infty} i(t) e^{+j\omega t} dt$ of the current. Note that according to Eq. (6), the statistical average of the number of electrons in the pulse can be written as $\bar{k} = 1/\mathcal{I}(0)$, where $\mathcal{I}(0) = \mathcal{I}(\omega = 0) = \int_{-\infty}^{\infty} i(t) dt$. Using these expressions in Eq. (15), the power spectrum of the radiation is given by
\[ \frac{dP_q(z)}{d\omega} = \frac{1}{2\pi T} \frac{1}{\bar{p}_q} \left[ e|\mathcal{H}_q(z, \omega)|^2 \mathcal{I}(0) \right. \]
\[ + \mathcal{H}_q(z, \omega) \mathcal{H}_q^*(z, \omega)|\mathcal{I}(\omega)|^2 \bar{p}_q \]. (17)

The first term in Eq. (17) is identified as the expression for the incoherent shot-noise (spontaneous-emission) spectrum
\[ \frac{dP_q^s(z)}{df} = \frac{1}{T} e|\mathcal{H}_q(z, f)|^2 \mathcal{I}(0) \bar{p}_q \], (18)

while the second term corresponds to the spectral density of the super-radiant power.
where we define

\[
\frac{dP_q(z)}{df} = \frac{1}{T} \mathcal{H}_{q_i}(z,f) \mathcal{H}_{q_i}^*(z,f)|\mathcal{I}(f)|^2 P_q. \tag{19}
\]

In the linear, low-gain interaction regime \( \mathcal{H}_{q_i}(z,f) \mathcal{H}_{q_i}^*(z,f) = |\mathcal{H}_{q_i}(z,f)|^2 \) and the relation between the power spectral density of the spontaneous and super-radiant emissions can be written in the form

\[
e \mathcal{I}(0) \frac{dP_q(z)}{df} = |\mathcal{I}(f)|^2 \frac{dP_q^s(z)}{df}. \tag{20}
\]

V. SPONTANEOUS EMISSION AND SUPER-RADIANCE IN FREE-ELECTRON LASERS

In electron passage through the periodic field of an undulator (see Fig. 1), its total velocity vector \( \mathbf{v}_i \) consists of a transverse wiggling component of amplitude \( \mathcal{V}_w \), which is due to the Lorentz force, in addition to a longitudinal axial velocity \( v_{zi} \)

\[
\mathbf{v}_i(z) = \mathbf{\bar{v}}_w(z) + \text{Re}\{\mathcal{V}_w e^{-jk_wz}\}. \tag{21}
\]

\[ k_w = 2\pi/\lambda_w \]

\( \lambda_w \) is the wiggler period. The electron transverse trajectory in the wiggler is given by

\[
\mathbf{r}_{zi}(z) = \mathbf{\bar{r}}_{zi} + \text{Re}\{\mathcal{V}_w e^{-jk_wz}\}, \tag{22}
\]

where \( \mathbf{\bar{r}}_{zi} = (\mathbf{\bar{x}}_i, \mathbf{\bar{y}}_i) \) describes the average (over a wiggling period) transverse coordinates of the electron and \( \mathbf{\bar{r}}_{zw} = j\mathcal{V}_w/k_w v_{z0} \) is the amplitude of transverse displacement of the wiggler electron trajectory (“quiver”).

Substitution of the expressions for the electron velocity (21) and trajectory (22) in Eq. (12) results in

\[
\mathcal{H}_{q_i}(z,\omega) = \frac{\mathcal{V}_w}{4P_q v_{zi}} \sum_{\mathbf{\bar{x}}_i, \mathbf{\bar{y}}_i} \mathcal{V}_w e^{-j\mathbf{\bar{x}}_i, \mathbf{\bar{y}}_i} e^{j\int \theta_{q_i}(z',\omega) dz'}, \tag{23}
\]

where we define

\[
\xi_q = \begin{cases} 
1 & \text{for } TE \text{ modes,} \\
1 - \frac{k_{q}^2}{k_{q} k_w} & \text{for } TM \text{ modes,}
\end{cases} \tag{24}
\]

and

\[
\theta_{q_i}(z,\omega) = \frac{\omega}{v_{zi}} (k_{zi} + k_w) \tag{25}
\]

is the detuning parameter of the \( i \)th electron at position \( z \).

When the effect of electromagnetic radiation on electron motion is low, resulting in negligible amplification of the excited radiation (low-gain regime), it is assumed that all electrons in the beam move at a constant (averaged over wiggler period) axial velocity \( v_{zi}(z) = v_{z0} \) and that they maintain their initial detuning parameter \( \theta_{q_i} \) along the wiggler. Consequently, the solution of Eq. (12) at the exit of a wiggler of length \( L_w \) is found to be

\[
\mathcal{H}_{q_i}(L_w,\omega) = A_{q_i} \sin(c/\theta_{q_i} L_w) e^{j1/2\theta_{q_i} L_w} \tag{26}
\]

where \( A_{q_i} = (\xi_q/4P_q)(L_w/v_{z0}) \mathcal{V}_w \mathcal{V}_w e^{j\mathbf{\bar{x}}_i, \mathbf{\bar{y}}_i} \) and \( \sin(c) = \sin(\chi)/\chi \).

According to Eq. (17), the spectral density of the radiation power emitted by the stream of electrons passing through the FEL undulator is

\[
\frac{dP_q(L_w)}{d\omega} = \frac{1}{2\pi} \frac{1}{|\mathcal{I}(\omega)|^2} \frac{dP_q^s(L_w)}{d\omega} \sin^2(\theta_{q_i} L_w). \tag{27}
\]

The first term in Eq. (27) is the spontaneous emission spectrum

\[
\frac{dP_q^s(L_w)}{d\omega} = \tau_{sp} \frac{dP_q^s(L_w)}{df} \sin^2(\theta_{q_i} L_w), \tag{28}
\]

where \( P_q^s(L_w) = 1/(T_{sp} e^{\mathcal{I}(0)} |\mathcal{A}_{q_i}|^2 P_q \) is the total spontaneous emission power carried by the transverse mode \( q \), and \( \tau_{sp} = |L_w/v_{z0} - L_w/v_{z} \) is the slippage time. The second term in Eq. (27) corresponds to the spectral density of the super-radiant power. Assuming that \( A_{q_i} A_{q_i} = |A_{q_i}|^2 \), it can be written as

\[
\frac{dP_q^s(L_w)}{df} = \frac{|\mathcal{I}(f)|^2}{|\mathcal{I}(f)|^2} \frac{dP_q^s(L_w)}{df} \frac{1}{e^{\mathcal{I}(0)}} \tau_{sp} \frac{dP_q^s(L_w)}{df} \sin^2(\theta_{q_i} L_w). \tag{29}
\]

VI. RADIATION FROM A SINGLE BUNCH

We consider an electron beam pulse having a temporal Gaussian current shape

\[
i(t) = \frac{I_0}{\sqrt{2\pi}} e^{-t^2/2T^2}, \tag{30}
\]

where \( T \) is the temporal “standard deviation” of the pulse. The Fourier transform of the current distribution is given by a Gaussian function in the frequency domain

\[
\mathcal{I}(f) = I_0 T e^{-1/2(2\pi ft)^2}. \tag{31}
\]

For \( f = 0 \), the Fourier transform results in \( \mathcal{I}(0) = I_0 T \). The power spectral density of the FEL spontaneous emission is given by Eq. (28), where the total spontaneous emission power is \( P_q^s(L_w) = 1/\tau_{sp} e^{\mathcal{I}(0)} |\mathcal{A}_{q_i}|^2 P_q \). In a FEL, utilizing a magnetostatic planar wiggler, the transverse wiggling amplitude is \( \mathcal{V}_w = a_w e/\gamma \) (where \( a_w = eB_w/mck_w \) is the wiggler parameter) and the total power of the spontaneous emission is given by
TABLE I. The operational parameters of millimeter wave free-electron maser.

<table>
<thead>
<tr>
<th>Accelerator</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Electron beam energy:</td>
<td>$E_{k} = 1.375 \text{ MeV}$</td>
<td></td>
</tr>
<tr>
<td>Electron beam current:</td>
<td>$I_0 = 1 \text{ A}$</td>
<td></td>
</tr>
<tr>
<td>Wiggler</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Magnetic induction:</td>
<td>$B_w = 2000 \text{ G}$</td>
<td></td>
</tr>
<tr>
<td>Period:</td>
<td>$\lambda_w = 4.444 \text{ cm}$</td>
<td></td>
</tr>
<tr>
<td>Number of periods:</td>
<td>$N_w = 20$</td>
<td></td>
</tr>
<tr>
<td>Waveguide</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rectangular waveguide:</td>
<td>$1.01 \text{ cm} \times 0.9005 \text{ cm}$</td>
<td></td>
</tr>
<tr>
<td>Mode:</td>
<td>$TE_{01}$</td>
<td></td>
</tr>
</tbody>
</table>

\[ P_{q}^{sp}(L_w) = \frac{1}{8} \frac{e I_0}{\tau_{sp}} \left( \frac{a_w}{\gamma \beta_{zq}} \right)^2 \frac{\xi_q^2 Z_q}{A_{em_q}} L_w^2 \]  

where $Z_q$ is the mode impedance and 

\[ A_{em_q} = \frac{\int |\vec{E}_q(x,y)|^2 \, dx \, dy}{|\vec{E}_q(0,0)|^2} \]

is its effective area. The spectrum of the super-radiant power radiated from the electron bunch is 

\[ \frac{dP_{q}^{sr}(L_w)}{df} = \bar{k} \tau_{sp} P_{q}^{sp}(L_w) e^{-(2 \pi f T)^2} |\vec{E}_q(0,0)|^2 \]  

where $\bar{k} = I_0 T/e$ is the expected number of electrons in the pulse.

We shall investigate spontaneous and super-radiant emissions radiated when an electron pulse passes through a wiggler of a FEL having operational parameters as given in Table I. Figure 2 shows that the beam and waveguide dispersion curves intercept at two points corresponding to the upper- and lower-synchronism frequencies—100 and 29 GHz, respectively. In such FEL schemes, the electron bunch is emitted when a photocathode is illuminated by a pulsed UV laser radiation [23–28]. Utilizing state-of-art femtosecond UV laser system, enables the generation of ultra-short e-beam bunches with duration of less than a period of the FEL radiation at millimeter wavelength and thus demonstrate super-radiant emission at this regime.

Analytical and numerical calculations of the resulting spontaneous emission spectrum are drawn in Fig. 3. The dashed line curves results from a particle simulation WB3D code, which is based on a three-dimensional, space-frequency model, utilizing an expansion of the total electromagnetic field (radiation and space-charge waves) in terms of transverse eigenmodes of the waveguide. Since shot noise is proportional to the particle charge [see Eq. (18)], the spontaneous emission spectrum obtained from $N$ particles simulation is $\bar{k}/N$ times that of the spontaneous emission resulting from $\bar{k}$ expected electrons in the $e$-beam pulse.

In the following, we shall focus our attention on radiation near the upper-synchronism frequency $f_0 = 100 \text{ GHz}$ and calculate the power spectrum of the radiation emitted when a pulse of electrons of temporal length $T$ passes through the wiggler. Figure 4 shows a curve of the spectral distribution of radiation energy $dW_q^{sr}(L_w)/df = |\bar{k} P_{q}^{sr}(L_w)/df|$ emitted when the bunch period $T$ is smaller than the temporal period $1/f_0$ of the signal. For $T f_0 = 0.1$, the super-radiant emission power is observed to be much higher than the power of spon-
The number of electrons expected in the pulse fluctuation peaks at the pulse duration $T_f$ of the signal and super-radiant emissions for an intermediate case, where the number of electrons is increased. Figure 5 shows the power spectra of spontaneous emissions from $T = 3 \text{ ps}$ electron bunch ($T_f_0 = 0.3$). Analytical calculations (solid line) and numerical simulation (dashed line).

The super-radiant power decreases as compared to the spontaneous emission power as the pulse duration $T$ is increased. Figure 5 shows the power spectra of spontaneous and super-radiant emissions for an intermediate case, where the $e$-beam pulse duration is slightly smaller than the temporal period of the signal $T_f_0 = 0.3$ ($k = 1.875 \times 10^7$ electrons in the pulse). For long $e$-beam pulses, the power of the super-radiant emission is reduced below the spontaneous (shot-noise) power and diminishes as $T \to \infty$. The signal-to-noise ratio (SNR) is the relation between the super-radiant and spontaneous emissions power spectra. Using Eq. (20) for the case of a Gaussian pulsed electron beam, the signal-to-noise ratio can be written as

$$\frac{dP_q^{\text{sp}}(z)/df}{dP_q^{\text{sr}}(z)/df} = \bar{k} e^{- (2\pi T_f)^2}.$$  \hspace{1cm} (34)

The graph of the signal-to-noise ratio is shown in Fig. 6. The triangles correspond to the results of numerical simulations with particles numbers $N = 100$ (empty symbols) and $N = 1000$ (solid symbols) at synchronism frequencies $f_0 = 29 \text{ GHz}$ (triangles down) and $f_0 = 100 \text{ GHz}$ (triangles up).

We use the model to investigate the evolution of the total spontaneous emission power generated when a long $e$-beam pulse ($T_f_0 = 10$) is passing through a wiggler. Figure 7 shows the power growth along the wiggler as a function of the wiggling periods $N_w$. In the first few periods, the spontaneous radiation is excited from short noise in the electron beam and its power increases proportional to $N_w^2$ [see Eq. (32)]. Within this stage, the mutual interaction between the electromagnetic radiation and the electron beam is small, the power amplification is low (low-gain regime) and the power growth follows the analytical solution given in Eq. (32). An exponential growth of SASE is inspected later after passing a sufficient number of periods, revealing that the interaction enters to the high gain regime, until saturation occurs when arriving to the nonlinear regime of the FEL operation.

VII. PERIODIC BUNCHING

Assume a continuous electron beam with sinusoidal modulated current at frequency $f_0$ \cite{39}

$$i(t) = I_0 [1 + m \cos(\omega_0 t)],$$  \hspace{1cm} (35)

where $I_0$ is the average DC current and $m$ is the modulation depth. In that case,

$$\lim_{T \to \infty} \frac{1}{T} \mathcal{I}(f) = \mathcal{I}_0$$

and

$$\lim_{T \to \infty} \frac{1}{T} |\mathcal{I}(f)|^2 = I_0 \left[ \delta(f) + \frac{m^2}{4} \delta(f-f_0) + \frac{m^2}{4} \delta(f+f_0) \right].$$

The evolution of spontaneous emission power along the wiggler (statistical distribution is shown by dashes and the average by bullets).
Using these relations, the power spectral density of the spontaneous emission is found to satisfy Eq. (28), where the total spontaneous emission power is $P_{q}^{sp}(L_w) = 1/\tau_{sp} \mu_0 A_q J^2_q$. The power spectral density of the super-radiant emission is found from (29) to be

$$\frac{dP_{q}^{sr}(L_w)}{df} = \frac{m^2 I_0}{4} e^{\tau_{sp}} P_{q}^{sp}(L_w) \sin^2 \left( \frac{1}{2} \theta_q(f_0) L_w \right) \delta(f-f_0).$$

(36)

The total power of the super-radiant emission is given by

$$P_{q}^{sr}(L_w) = \frac{m^2 I_0}{4} e^{\tau_{sp}} P_{q}^{sp}(L_w) \sin^2 \left( \frac{1}{2} \theta_q(f_0) L_w \right).$$

(37)

Figure 8 shows the super-radiant power as a function of the fundamental modulation frequency $f_0$ for various modulation levels. A comparison is made with simulation results. Super-radiant power emitted by an infinite series of ultra-short bunches (impulses) is also shown. In this case, the current can be expanded in a Fourier series

$$i(t) = \sum_{n=-\infty}^{+\infty} I_0 T \delta(t-nT) = I_0 \left[ 1 + 2 \sum_{n=-\infty}^{+\infty} \cos \left( \frac{2\pi n}{T} t \right) \right].$$

(38)

The resulting spectrum of super-radiant emission contains all harmonics of the prebunching frequency $f_0 = 1/T$ each having a sinusoidal current modulation with modulation index $m = 2$. Figure 8 shows a curve of the super-radiant power emitted by a series of impulses as a function of the fundamental modulation frequency $f_0$. The discrepancy between analytical calculations and numerical simulations at high-modulation levels is due to stimulated emission effects that arise in the simulations, but not taken into account in the analytical calculations (where the effect of the radiation on electrons in not considered).

ACKNOWLEDGMENTS

The research was supported in part by the Israel Science Foundation and Ministry of Science. The authors would like to thank A. L. Eichenbaum for his help and useful remarks. The research of the second author (Yu.L.) was supported in part by the Center of Scientific Absorption of the Ministry of Absorption, State of Israel.
