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Physics Letters A 344 (2005) 18–28
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Spectral and variational principles of electromagnetic field excitation in wave guides

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Received 21 April 2004; received in revised form 16 February 2005; accepted 17 June 2005

Available online 28 June 2005

Communicated by A.P. Fordy

Abstract

Possible variational principles for excitation of an electromagnetic field in a wave guide are discussed. Our emphasis is not on the calculation of the modal shapes, which is common in previous art, but rather on the calculation of modal amplitude evolution, which are important in electron devices such as free electron lasers and gyrotrons. Variational principles have considerable importance in theoretical physics and are used among other things to derive numerical solution schemes, conservation laws via the Noether theorem and correct boundary conditions for the derived equations including the important effects of the backward waves amplitudes.

PACS: 41.20.-q; 41.20.Jb; 41.60.-m; 41.60.Cr

1. Introduction

Interaction of radiation and plasma waves in many electron devices takes place inside an open or closed cylinder (wave guide) of some arbitrary cross-section (see Fig. 1 for a schematic illustration).

A well-known example is the free-electron laser, in which the electromagnetic field interacts with an electron beam in the presence of an undulator, generating high power coherent radiation. In order to achieve lasing, the radiation is being excited inside a resonator, dictating boundary conditions for both forward and backward waves (see Fig. 1).

Solution of the electromagnetic radiation field inside the resonator, requires simultaneous integration of the coupled excitation equations of forward and backward waves [10]. However, it becomes difficult to accommodate

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doi:10.1016/j.physleta.2005.06.054

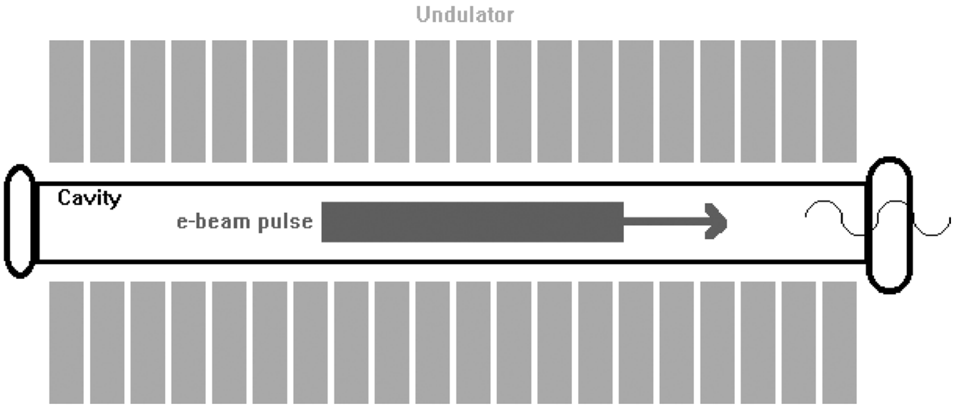

Fig. 1. The FEL scheme.

the different boundary conditions for both forward and backward modes in the same numerical integration scheme. Although the radiation power is built gradually in the direction of the electron beam propagation, the natural boundary conditions for the backward waves are given at the end of the interaction region. Thus it is desirable to develop a numerical procedure that allows non-local boundary conditions.

We suggest employing variational methods for calculating the total electromagnetic field, including excitation of forward and backward waves. Our developed variational principle is based on a modal representation of the total electromagnetic field in terms of the eigenmodes of the geometry in which the radiation is excited and formulation of the electromagnetic field action in the space–frequency domain.

Variational principles for electromagnetic field dynamics, including their interaction with matter are abundant in the literature [1–9]. Moreover, the behavior of the electromagnetic field inside a wave guide in terms of a variational principle was studied in many texts [4–9], most of the times in order to provide a basis for a numerical scheme. These works are concerned mainly with the derivations of eigenmodes for the case of non-trivial geometries or an inhomogeneous refraction index. In this work we are not concerned with the modal form rather we assume that it is known. Our main concern is the development of the modal amplitude inside the wave guide due to its interaction with propagating charge.

An additional difference between this work and previous art is the methodology we use for developing our variational principle. While previous workers initiated the development of their frequency domain variational principle from the relevant equations (usually defining the square of the equations as the variational functional). We start from the general 'canonical' form of the variational principle for electromagnetic fields [11] which is stated in terms of gauge fields, we than adapt it to a form which is relevant for the physical scenario taking place inside a wave-guide.

Variational principles have importance in theoretical physics. And are used to:

- derive numerical methods for obtaining the modal amplitudes;
- obtain constants of motion using the Noether theorem;
- derive the differential equation of motion and the *correct* boundary conditions to those equations;
- quantize the system under investigation using the action in a path integral.

In this Letter we introduce three different variational principle describing the modal propagation inside a wave guide.

The structure of this Letter is as follows: first we discuss the fundamentals of electromagnetic field presentation in the frequency domain, followed by a short review of the modal representation in a wave guide. Then the vector potential modal representation is introduced. After that we review the classical variational principle of the electromagnetic field and its variational derivatives. Next the action is represented in the spectral–modal scheme and second order equations are obtained for the fields amplitudes. Finally the introduction of the quasi Hamiltonian allows us to obtain first order equations in terms of the field amplitudes, while the introduction of backward–forward waves puts the variational principle in a particular simple form which concludes our report.

2. Fundamental of electromagnetic field presentation in the frequency domain

The electromagnetic field in the time domain is described by the space–time electric $\mathbf{E}(\mathbf{r}, t)$ and magnetic $\mathbf{H}(\mathbf{r}, t)$ signal vectors. $\mathbf{r}$ stands for the (x, y, z) coordinates, where (x, y) are the transverse coordinates and z is the axis of propagation. The Fourier transform of the electric field is defined by

$$\mathbf{E}(\mathbf{r}, \omega) = \int_{-\infty}^{+\infty} \mathbf{E}(\mathbf{r}, t) e^{+j\omega t}\, dt, \tag{1}$$

where ω is the angular frequency and $j = \sqrt{-1}$. Similar expression is defined for the Fourier transform $\mathbf{H}(\mathbf{r}, \omega)$ of the magnetic field. Since the electromagnetic signal is real (i.e., $\mathbf{E}^*(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t)$), its Fourier transform satisfies $\mathbf{E}^*(\mathbf{r}, \omega) = \mathbf{E}(\mathbf{r}, -\omega)$.

Fourier transformation of the electric field results in a 'phasor-like' function $\tilde{\mathbf{E}}(\mathbf{r}, \omega)$ defined in the positive frequency domain and related to the Fourier transform by

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = 2\mathbf{E}(\mathbf{r}, \omega) u(\omega) \equiv \begin{cases} 2\mathbf{E}(\mathbf{r}, \omega), & \omega > 0, \\ 0, & \omega < 0. \end{cases} \tag{2}$$

The Fourier transform can decomposed in terms of the 'phasor-like' functions according to

$$\mathbf{E}(\mathbf{r}, \omega) = \frac{1}{2}\tilde{\mathbf{E}}(\mathbf{r}, \omega) + \frac{1}{2}\tilde{\mathbf{E}}^*(\mathbf{r}, -\omega) \tag{3}$$

and the inverse Fourier transform is then

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{E}(\mathbf{r}, \omega) e^{-j\omega t}\, d\omega = \Re\left\{ \frac{1}{2\pi} \int_{0}^{\infty} \tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-j\omega t}\, d\omega \right\}. \tag{4}$$

3. Modal presentation of electromagnetic field in the frequency domain

This section presents the formalism employed throughout this Letter for analyzing the excitation of electromagnetic fields by current sources distributed along a wave guide [10,12,13]. The approach taken here utilizes representation of the total electromagnetic fields and their sources in terms of vector functions, which are the eigenmodes solutions of the medium, free of charge or current sources. The 'phasor like' quantities defined in (2) can be expanded in terms of transverse eigenmodes of the medium in which the field is excited and propagates. The perpendicular component of the electric and magnetic fields are given in any cross-section as a linear superposition of a complete set of transverse eigenmodes

$$\tilde{\mathbf{E}}_\perp(\mathbf{r}, \omega) = \sum_q V_q(z, \omega) \tilde{\boldsymbol{\mathcal{E}}}_{q\perp}(x, y), \qquad \tilde{\mathbf{H}}_\perp(\mathbf{r}, \omega) = \sum_q I_q(z, \omega) \tilde{\boldsymbol{\mathcal{H}}}_{q\perp}(x, y). \tag{5}$$

The summations include propagating and cut-off TE and TM modes, for which $V_q(z,\omega)$ and $I_q(z,\omega)$ are the scalar amplitude of the electric and magnetic fields respectively and $\tilde{\boldsymbol{\mathcal{E}}}_{q\perp}(x,y)$ and $\tilde{\boldsymbol{\mathcal{H}}}_{q\perp}(x,y)$ are their respective profiles.

Expressions for the longitudinal component of the electric and magnetic fields are obtained after substituting the modal representation (5) of the fields into Maxwell's equations, where the Fourier transform of the current density $\mathbf{J}$, $\tilde{\mathbf{J}}(\mathbf{r},\omega)$ is introduced

$$\tilde{E}_z(\mathbf{r},\omega) = \sum_q I_q(z,\omega)\tilde{\boldsymbol{\mathcal{E}}}_{qz}(x,y) + \frac{1}{j\omega\epsilon}\tilde{J}_z(\mathbf{r},\omega), \qquad \tilde{H}_z(\mathbf{r},\omega)=\sum_q V_q(z,\omega)\tilde{\boldsymbol{\mathcal{H}}}_{qz}(x,y). \tag{6}$$

By imposing the appropriate boundary conditions, the Maxwell vector equations are transformed into scalar differential ('transmission line') equations, which describe the evolution of the equivalent electric and magnetic amplitudes $V_q(z,\omega)$ and $I_q(z,\omega)$:

$$-\frac{dV_q(z,\omega)}{dz} = -jk_{zq}I_q(z,\omega) + v_q(z,\omega), \qquad -\frac{dI_q(z,\omega)}{dz}=-jk_{zq}V_q(z,\omega) + i_q(z,\omega), \tag{7}$$

where

$$k_{zq} = \begin{cases} j\sqrt{k_{\perp q}^2 - k^2} = j|k_{zq}|, & k < k_{\perp q} \text{ (cut-off modes)}, \\ \sqrt{k^2 - k_{\perp q}^2} = |k_{zq}|, & k > k_{\perp q} \text{ (propagating modes)}, \end{cases} \tag{8}$$

is the axial wave number of mode q and

$$v_q(z,\omega) \equiv \frac{1}{\mathcal{N}_q}\iint \tilde{J}_z\tilde{\mathcal{E}}_{qz}^*\,dx\,dy, \qquad i_q(z,\omega) \equiv \frac{1}{\mathcal{N}_q}\frac{Z_q}{Z_q^*}\iint \tilde{\mathbf{J}}_\perp \cdot \tilde{\mathcal{E}}_{q\perp}^*\,dx\,dy. \tag{9}$$

The normalization of the field amplitudes of each mode is made via each mode's complex Poynting vector power

$$\mathcal{N}_q = \int\!\!\int_{\text{c.s.}} \left[\tilde{\mathcal{E}}_{q\perp}(x,y) \times \tilde{\mathcal{H}}_{q\perp}^*(x,y)\right] \cdot \hat{\mathbf{z}}\,dx\,dy \tag{10}$$

and the mode impedance is given by

$$Z_q = \begin{cases} \sqrt{\frac{\mu}{\epsilon}}\frac{k}{k_{zq}} = \frac{\omega\mu}{k_{zq}} & \text{for TE modes}, \\ \sqrt{\frac{\mu}{\epsilon}}\frac{k_{zq}}{k} = \frac{k_{zq}}{\omega\epsilon} & \text{for TM modes}, \end{cases} \tag{11}$$

ε is the electric susceptibility and μ is the magnetic permeability.

The transmission-line equations (7) can also be written in the form

$$\begin{aligned} &V_q''(z,\omega) + k_{zq}^2 V_q(z,\omega) = -v_q'(z,\omega) - jk_{zq}i_q(z,\omega), \\ &I_q''(z,\omega) + k_{zq}^2 I_q(z,\omega) = -jk_{zq}v_q(z,\omega) - i_q'(z,\omega), \end{aligned} \tag{12}$$

where ($'$) denotes a derivative in respect to z. Notice that only one of the equations in (12) needs to be solved, since solving for $V_q(z,\omega)$ we obtain immediately the solution for $I_q(z,\omega)$ through Eq. (7).

4. The vector potential

The scalar potential $\Phi(\mathbf{r},t)$ and vector potential $\mathbf{A}(\mathbf{r},t)$ are related to the electric and magnetic fields by

$$\mu\mathbf{H}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t), \qquad \mathbf{E}(\mathbf{r},t)=-\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) - \nabla\Phi(\mathbf{r},t). \tag{13}$$

Choosing a gauge transformation where $\Phi(\mathbf{r},t)=0$, the vector potential is given by the time integration

$$\mathbf{A}(\mathbf{r},t)=-\int_{-\infty}^{t}\mathbf{E}(\mathbf{r},t')\,dt'. \tag{14}$$

Consequently, in the frequency domain

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\frac{1}{j\omega}\tilde{\mathbf{E}}(\mathbf{r},\omega). \tag{15}$$

Using expansion (5), the perpendicular component of the vector potential can be presented by

$$\tilde{\mathbf{A}}_{\perp}(\mathbf{r},\omega)=\frac{1}{j\omega}\sum_q V_q(z,\omega)\tilde{\mathcal{E}}_{q\perp}(x,y) \tag{16}$$

and from (6), the longitudinal component of the vector potential is

$$\tilde{A}_z(\mathbf{r},\omega)=\frac{1}{j\omega}\left(\sum_q I_q(z,\omega)\tilde{\mathcal{E}}_{qz}(x,y)+\frac{1}{j\omega\epsilon}\tilde{J}_z(\mathbf{r},\omega)\right). \tag{17}$$

In terms of Eq. (9) the longitudinal component can be shown to have the form

$$\tilde{A}_z(\mathbf{r},\omega)=\frac{1}{j\omega}\sum_q\left(I_q^{\mathrm{TM}}(z,\omega)-\frac{jk_{z_q}}{k_{\perp_q}^2}v_q^{\mathrm{TM}}(z,\omega)\right)\tilde{\mathcal{E}}_{q_z}(x,y). \tag{18}$$

In this case Eq. (7) serve as a definition for I_q:

$$I_q\equiv\frac{\partial_z V_q+v_q}{jk_{z_q}}. \tag{19}$$

5. The classical variational principle

It is well known that the action of the electromagnetic field can be given in terms of the vector potential in the following form [11]:

$$\mathcal{A}=\int\left[\frac{1}{2}\left(\varepsilon(\partial_t\mathbf{A})^2-\frac{1}{\mu}(\nabla\times\mathbf{A})^2\right)+\mathbf{J}\cdot\mathbf{A}\right]d^3x\,dt. \tag{20}$$

Taking the variational derivative of $\mathcal{A}$ with respect the vector potential $\mathbf{A}$ we obtain

$$\delta\mathcal{A}=\delta\mathcal{A}_t+\delta\mathcal{A}_{\mathrm{boundary}}+\delta\mathcal{A}_{\mathrm{equations}}. \tag{21}$$

$\delta\mathcal{A}_t$ is the time condition term given by

$$\delta\mathcal{A}_t=\varepsilon\int\partial_t\mathbf{A}\cdot\delta\mathbf{A}\,d^3x\Big|_{t_1}^{t_2} \tag{22}$$

t_1 is the initial time of the system and t_2 is the final time. $\delta\mathcal{A}_{\mathrm{boundary}}$ is the boundary term

$$\delta\mathcal{A}_{\mathrm{boundary}}=\oint\mathbf{H}\times\delta\mathbf{A}\cdot d\vec{S}\,dt \tag{23}$$

the integral is taken over all the surface containing the volume under consideration. Finally, the equation part is given by

$$\delta\mathcal{A}_{\text{equations}} = \int \delta\mathbf{A}\cdot(\varepsilon\partial_t\mathbf{E} - \nabla\times\mathbf{H} + \mathbf{J})\, d^3x\, dt. \tag{24}$$

Provided that $\delta\mathcal{A}_t$ and $\delta\mathcal{A}_{\text{boundary}}$ vanish for otherwise arbitrary variations than $\delta\mathcal{A}_{\text{total}} = 0$ vanishes if and only if

$$\nabla\times\mathbf{H} = \varepsilon\partial_t\mathbf{E} + \mathbf{J}. \tag{25}$$

The other Maxwell equations are satisfied automatically by virtue of the vector potential representation Eq. (13) and by virtue of the continuity equation.

6. Spectral representation

In order to use the spectral representation in the variational technique one must assure in advance that the condition $\delta\mathcal{A}_t = 0$ (Eq. (22)) is taken care of. This can be done by choosing $t_1 = 0$ and $t_2 = T$ and demanding that $\mathbf{A}(0) = \mathbf{A}(T)$. Thus $\mathbf{A}$ is given (instead of Fourier transform of the type given in Eq. (1)) by the Fourier sum

$$\mathbf{A} = \sum_{n=-\infty}^{\infty} \mathbf{A}_n e^{j\omega_n t}, \quad \omega_n = \frac{2\pi n}{T} \tag{26}$$

for which

$$\mathbf{A}_n = \frac{1}{T}\int_0^T \mathbf{A} e^{-j\omega_n t}\, dt \tag{27}$$

and $\mathbf{A}_{-n} = \mathbf{A}_n^*$ since $\mathbf{A}$ is real.

In terms of this notation Eq. (20) takes the form

$$\mathcal{A} = T\left[\sum_{n=1}^{\infty}\mathcal{A}_n + \mathcal{A}_{\text{DC}}\right]. \tag{28}$$

In which

$$\begin{aligned}\mathcal{A}_n &= \int\left[\varepsilon\omega_n^2|\mathbf{A}_n|^2 - \frac{1}{\mu}|\nabla\times\mathbf{A}_n|^2 + \mathbf{J}_n^*\cdot\mathbf{A}_n + \mathbf{J}_n\cdot\mathbf{A}_n^*\right] d^3x \\ &= \int\left[\varepsilon\omega_n^2|\mathbf{A}_n|^2 - \frac{1}{\mu}|\nabla\times\mathbf{A}_n|^2 + 2\Re\big(\mathbf{J}_n^*\cdot\mathbf{A}_n\big)\right] d^3x, \\ \mathcal{A}_{\text{DC}} &= \int\left[-\frac{1}{2\mu}|\nabla\times\mathbf{A}_0|^2 + \mathbf{J}_0\cdot\mathbf{A}_0\right] d^3x. \end{aligned} \tag{29}$$

$\mathbf{J}_n$ is the Fourier component of the current density, $\Re(x)$ stands for the real part of x, $|x|$ stands for the absolute value of x and x^* stands for the complex conjugate of x. The action given in Eq. (28) and Eq. (29) is obviously real (despite its apparent complex nature). From now on we disregard $\mathcal{A}_{\text{DC}}$ which is not relevant for time dependent source currents.

7. The action in a wave guide

Inserting the modal representation Eqs. (16) and (17) into Eqs. (28) and (29) we obtain

$$\mathcal{A} = \frac{T^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sum_{q}\frac{\mathcal{N}^*_{q,n}}{k_{z_{q,n}}}L_{q,n}, \qquad L_{q,n} \equiv \int \mathcal{L}_{q,n}\,dz,$$

$$\mathcal{L}_{q,n} \equiv \frac{1}{2}k^2_{z_{q,n}}|V_{q,n}|^2 - \frac{1}{2}|\partial_z V_{q,n}|^2 - \frac{1}{2}\frac{\omega_n^2}{c^2k^2_{\perp_q}}|v_{q,n}|^2 - \Re\bigl(v^*_{q,n}\partial_z V_{q,n}\bigr) - \Im\bigl(k_{z_{q,n}}i_{q,n}V^*_{q,n}\bigr) \tag{30}$$

in which $\Im(x)$ stand for the imaginary part x. From Appendix A (see also [13]) it is obvious that $\frac{\mathcal{N}^*_{q,n}}{k_{z_{q,n}}}$ is real and so are the Lagrangian $L_{q,n}$ and the Lagrangian density $\mathcal{L}_{q,n}$. The quantities $i_{q,n}, v_{q,n}$ are defined in Eq. (9). From now on we will suppress the indices q, n.

In terms of the amplitude V and its complex conjugate V^*, the Lagrangian density $\mathcal{L}$ can be written as

$$\mathcal{L} = \frac{1}{2}\left\{k_z^2 V V^* - \partial_z V\partial_z V^* - \frac{\omega^2}{c^2k_\perp^2}vv^* - \partial_z V v^* - \partial_z V^* v + jk_z i V^* - jk_z^* i^* V\right\}. \tag{31}$$

It should be noted that using the modal representation given in Eq. (16) the boundary $\delta\mathcal{A}_{\text{boundary}}$ term given in Eq. (23) vanishes on the circumference of the wave-guide in the modal representation and the only contributions come from the planes $z = 0$ and $z = L_w$ which amount to the condition

$$\Im\bigl(k_z I\delta V^*\bigr)\big|_0^{L_w} = 0 \tag{32}$$

for each mode.

Taking the variational derivative of Eq. (31) we obtain

$$\delta L = \frac{1}{2}\left\{\int\bigl[jk_z i + k_z^2 V + \partial_z^2 V + \partial_z v\bigr]\delta V^*\,dz - (\partial_z V + v)\delta V^*\bigg|_0^{L_w} + \text{c.c.}\right\} \tag{33}$$

in which c.c. stands for complex conjugate. The boundary conditions given in Eq. (33) are the same as in Eq. (32) by virtue of Eq. (19). The equations obtained are the same as in Eq. (12) and its complex conjugate.

8. Some numerical aspects

At this stage it is already clear that the Lagrangian appearing in Eq. (30) is real. However, in order to understand its mathematical structure we shall write it in terms of real quantities. Every complex number C can be written as $C = C_r + jC_i$ in which C_r, C_i are real numbers and $C_r = \Re(C)$ is the real part of C while $C_i = \Im(C)$ is the imaginary part of C. Representing all the quantities appearing in $\mathcal{L}$ (Eq. (30)) in terms of their real and imaginary parts we arrive at the result

$$\begin{aligned}\mathcal{L} &\equiv \frac{1}{2}k_z^2|V|^2 - \frac{1}{2}|\partial_z V|^2 - \frac{1}{2}\frac{\omega^2}{c^2k_\perp^2}|v|^2 - \Re\bigl(v^*\partial_z V\bigr) - \Im\bigl(k_z i V^*\bigr)\\ &= \frac{1}{2}\left[k_z^2\bigl(V_r^2 + V_i^2\bigr) - \bigl(\partial_z V_r^2 + \partial_z V_i^2\bigr) - \frac{\omega^2}{c^2k_\perp^2}\bigl(v_r^2 + v_i^2\bigr) - \partial_z V_r v_r - \partial_z V_i v_i\right]\\ &\quad - \frac{1}{2}\begin{cases} k_z(i_i V_r - i_r V_i) & \text{(propagating modes)},\\ |k_z|(i_r V_r - i_i V_i) & \text{(cut-off modes)}.\end{cases}\end{aligned} \tag{34}$$

Notice that the cut-off modes Lagrangian density decouples into two separate Lagrangian densities

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_r + \mathcal{L}_i, \\
\mathcal{L}_r &= \frac{1}{2}\left[k_z^2 V_r^2 - (\partial_z V_r)^2 - \frac{\omega^2}{c^2 k_\perp^2} v_r^2 - \partial_z V_r v_r - |k_z| i_r V_r\right], \\
\mathcal{L}_i &= \frac{1}{2}\left[k_z^2 V_r i^2 - (\partial_z V_i)^2 - \frac{\omega^2}{c^2 k_\perp^2} v_i^2 - \partial_z V_i v_i + |k_z| i_i V_i\right]
\end{aligned} \tag{35}$$

while the propagating modes Lagrangian density cannot decouple. Using any type of discretization the Lagrangian density given in Eq. (34) will become a real bilinear form. For cut-off modes the form of $-\mathcal{L}$ appears to be positive since $k_z^2 = -|k_z^2|$ according to Eq. (8). Thus the solution will correspond to the minimum of the bilinear form which can be found by standard numerical techniques such as the conjugate gradient method [15]. For propagating modes $k_z^2 = |k_z^2|$ the solution will correspond to a saddle point of the linear form and can be found using techniques such as the ones described in [16–18].

9. The quasi-Hamiltonian

In certain cases it is desirable to obtain first order equations instead of the second order Eq. (12). In analytical mechanics [14] their is a well-known technique to reach this goal using the Hamiltonian construction. Since L given in Eq. (30) is not strictly speaking a Lagrangian (time which appears in proper Lagrangians is replaced here by the longitudinal coordinate z) we will denote the analogue construction of the Hamiltonian a 'quasi-Hamiltonian'. For convenience we introduce the Lagrangian density $\bar{\mathcal{L}}$:

$$\bar{\mathcal{L}} = -2\mathcal{L} = \partial_z V \partial_z V^* - k_z^2 V V^* + \frac{\omega^2}{c^2 k_\perp^2} v v^* + \partial_z V v^* + \partial_z V^* v - j k_z i V^* + j k_z^* i^* V \tag{36}$$

in which we utilized Eq. (31). Next we define the quasi canonical momentums of $V' \equiv \partial_z V$ and $V'^* \equiv \partial_z V^*$:

$$\Pi \equiv \frac{\partial \bar{\mathcal{L}}}{\partial V'} = V'^* + v^* = -j k_z^* I^*, \qquad \Pi^* \equiv \frac{\partial \bar{\mathcal{L}}}{\partial V'^*} = V' + v = j k_z I \tag{37}$$

in which Eq. (19) is used. Notice that the quasi canonical momentums are proportional to I^* and I, respectively. Having done this we are in a position to define the quasi-Hamiltonian density

$$\begin{aligned}
\mathcal{H} &\equiv V'\Pi + V'^*\Pi^* - \bar{\mathcal{L}} \\
&= |k_z|^2 |I|^2 + k_z^2 |V|^2 - j k_z I v^* + j k_z^* I^* v - \frac{k_z^2}{k_\perp^2}|v|^2 + j k_z i V^* - j k_z^* i^* V \\
&= k_z\left[k_z\big(|V|^2 \pm |I|^2\big) - j I v^* \pm j I^* v - \frac{k_z}{k_\perp^2}|v|^2 + j i V^* \mp j i^* V\right]
\end{aligned} \tag{38}$$

the upper sign should be attributed to propagating modes while the lower signs should be attributed to decaying modes. Thus $\bar{\mathcal{L}}$ can be written as

$$\begin{aligned}
\bar{\mathcal{L}} &= V'\Pi + V'^*\Pi^* - \mathcal{H} \\
&= k_z\left[\mp j I^* V' + j I V'^* - k_z\big(|V|^2 \pm |I|^2\big) + j I v^* \mp j I^* v - j i V^* \pm j i^* V + \frac{k_z}{k_\perp^2}|v|^2\right].
\end{aligned} \tag{39}$$

Our next step will be to take the variational derivative with respect to I and V and their complex conjugates of $\bar{L}$ which is defined as

$$\bar{L} = \int \bar{\mathcal{L}}\, dz = -2L. \tag{40}$$

This will lead to the expression

$$\delta\bar{L} = k_z \int dz \big[\delta I\big(jV'^* \mp k_z I^* + jv^*\big) \mp \delta I^*(jV' + k_z I + jv) + \delta V\big(\pm jI'^* - k_z V^* \pm ji^*\big) - \delta V^*(jI' + k_z V + ji)\big] + jk_z\big(I\delta V^* \mp I^*\delta V\big)\Big|_0^{L_w}. \tag{41}$$

The boundary conditions term given in Eq. (41) is the same as in Eq. (33) by virtue of Eq. (19). The equations obtained are the same as Eq. (7) and their complex conjugates.

10. The forward–backward formulation

In terms of V and I one can define the following new variables [13]:

$$C_+ \equiv \frac{1}{2}(V + I)e^{-jk_z z}, \qquad C_- \equiv \frac{1}{2}(V - I)e^{jk_z z}. \tag{42}$$

Or vice versa as

$$V = C_+ e^{jk_z z} + C_- e^{-jk_z z}, \qquad I = C_+ e^{jk_z z} - C_- e^{-jk_z z}. \tag{43}$$

Thus C_+ and C_- appear as the amplitudes of forward and backward waves respectfully (see Fig. 2) in the case of propagating modes. Inserting the above variables into $\bar{L}$ given in Eq. (40) we obtain

$$\bar{L} = jk_z\left\{2\int dz\left[C_-^* C_-' + C_+'^* C_+ - C_+^*\beta + C_+\beta^* + C_-^*\alpha - C_-\alpha^* - j\frac{k_z}{2k_\perp^2}|v|^2\right] + \left(C_-^* C_+ e^{2jk_z z} - C_- C_+^* e^{-2jk_z z} - |C_-|^2 - |C_+|^2\right)\Big|_0^{L_w}\right\}. \tag{44}$$

In which

$$\alpha = \frac{1}{2}(v - i)e^{jk_z z}, \qquad \beta = \frac{1}{2}(v + i)e^{-jk_z z}. \tag{45}$$

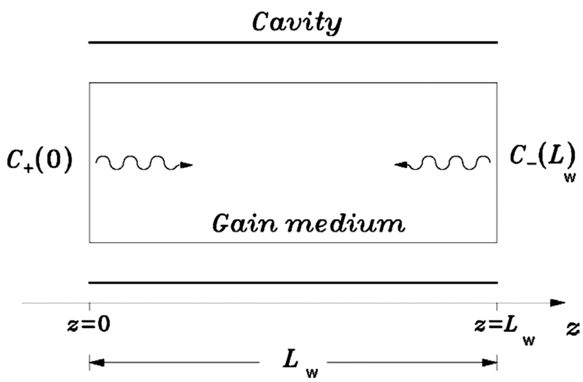

Fig. 2. Interaction of the electromagnetic field in a gain medium.

At this stage one is tempted to discard the boundary term in the above equation since it appears to have no effect on the resulting equations, however, this will lead to unphysical boundary conditions and thus should be avoided. Taking the variational derivative we obtain

$$\delta\bar{L} = \delta\bar{L}_{\text{equation}} + \delta\bar{L}_{\text{boundary}} \tag{46}$$

in which

$$\delta\bar{L}_{\text{equation}} = 2jk_z \int \left[(\alpha + C'_-)\delta C^*_- - (\beta + C'_+)\delta C^*_+ - \text{c.c.}\right] dz \tag{47}$$

in which c.c. stands for complex conjugate and

$$\delta\bar{L}_{\text{boundary}} = -jk_z\left[\delta C^*_+\left(C_- e^{-2jk_z z} - C_+\right) - \delta C^*_-\left(C_+ e^{2jk_z z} - C_-\right) - \text{c.c.}\right]\Big|_0^{L_w}. \tag{48}$$

The boundary term given in Eq. (48) are the same as in Eq. (32) by virtue of Eq. (43). The equations obtained are

$$C'_- = -\alpha, \qquad C'_+ = -\beta \tag{49}$$

and their complex conjugates which provides a truly elegant way to compute the field dynamics.

For decaying modes $\bar{L}$ given in Eq. (40) takes the form

$$\begin{aligned}\bar{L} = jk_z\Bigg\{&2\int dz\left[C^*_+ C'_- + C_+ C'^*_- + C^*_+\alpha + C_+\alpha^* - C^*_-\beta - C_-\beta^* - j\frac{k_z}{2k_\perp^2}|v|^2\right] \\ &- \left[|C_-|^2 e^{-2jk_z z} - |C_+|^2 e^{2jk_z z} + C_- C^*_+ + C^*_- C_+\right]\Bigg|_0^{L_w}\Bigg\}.\end{aligned} \tag{50}$$

At this stage one is tempted to discard the boundary term in the above equation since it appears to have no effect on the resulting equations, however, this will lead to unphysical boundary conditions and thus should be avoided. Taking the variational derivative we obtain

$$\delta\bar{L} = \delta\bar{L}_{\text{equation}} + \delta\bar{L}_{\text{boundary}} \tag{51}$$

in which

$$\delta\bar{L}_{\text{equation}} = 2jk_z \int \left[(\alpha + C'_-)\delta C^*_+ - (\beta + C'_+)\delta C^*_- + \text{c.c.}\right] dz \tag{52}$$

in which c.c. stands for complex conjugate and

$$\delta\bar{L}_{\text{boundary}} = -jk_z\left[\delta C^*_+\left(C_- - C_+ e^{2jk_z z}\right) + \delta C^*_-\left(C_- e^{-2jk_z z} - C_+\right) + \text{c.c.}\right]\Big|_0^{L_w}. \tag{53}$$

The boundary term given in Eq. (53) are the same as in Eq. (32) by virtue of Eq. (43). The equations obtained are the same as in Eq. (49) and their complex conjugates which provides a truly elegant way to compute the field dynamics.

11. Conclusions

Three different action principles were obtained in this work: one in terms of the V modal amplitude leading to second order equations. Another principle was formulated in terms of the V, I amplitudes through the quasi-Hamiltonian concept leading to first order equations. And finally an action principle in terms of the forward and backward modes were derived including the correct boundary conditions for those equations. The action can be used as a basis for a numerical scheme as outlined in Section 8. It was observed that different numerical techniques should be used for propagating and cut-off modes. Additional possible applications of the above variational principles include derivation of constants of motion using the Noether theorem and quantization of the electromagnetic field in a wave-guide using the action in a path integral technique.

Acknowledgements

This research was carried under grant # 134/01 of the Israel Science Foundation.

Appendix A. The complex nature of the electro-magnetic profiles in the wave guide

Table 1
Parameters of propagating and cut-off modes

Mode type		Propagating	Cut-off
Axial wavenumber	k_{zq}	real	imaginary
Impedance:	Z_q	real	imaginary
TE mode:			
Longitudinal magnetic field component:	$\tilde{\mathcal{H}}_{qz}$	real	real
Transverse magnetic field component:	$\tilde{\mathcal{H}}_{q\perp}$	imaginary	real
Transverse electric field component:	$\tilde{\mathcal{E}}_{q\perp}$	imaginary	imaginary
TM mode:			
Longitudinal electric field component:	$\tilde{\mathcal{E}}_{qz}$	real	real
Transverse electric field component:	$\tilde{\mathcal{E}}_{q\perp}$	imaginary	real
Transverse magnetic field component:	$\tilde{\mathcal{H}}_{q\perp}$	imaginary	imaginary
Power normalization	$\mathcal{N}_q$	real	imaginary

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