Cyclic Coded Integer-Forcing Equalization

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The Gaussian ISI channel

\[ y_k = x_k + \sum_{m \neq 0} h_m x_{k-m} + n_k \]

= \( x_k + \text{ISI}_k + n_k \)

- \( n_k \) is AWGN with unit power.
- Mutual Info: \( I(S_x(\cdot)) = \frac{1}{2\pi} \int_\omega \log \left(1 + S_x(e^{i\omega})|H(e^{i\omega})|^2\right) d\omega \)
- Assume (for simplicity) white input: \( S_x(e^{i\omega}) = \text{const} = \sigma_x^2 \)
- CSI@Rx only
We are interested in schemes where decoding is decoupled from equalization.

-Turbo equalization not considered.

- Multi-carrier (frequency domain) - OFDM/DMT
  - Transforms the ISI channel into parallel AWGN subchannels - simplifies equalization
  - Coding over a channel with varying SNR may incur an unbounded gap-to-capacity
  - PAPR
  - Non-applicable to channels with finite alphabet (magnetic etc.)
Single carrier (time domain)

- (Tomlinson-Harashima Precoding (THP) requires complete CSI@Tx - inapplicable...)
- Linear equalizers: ZF-LE, MMSE-LE
- Decision-feedback equalization (DFE)
Closer look at DFE

Equivalent channel:

\[ G(D) = (F + m \hat{m} H(D)) \]

or equivalently:

\[ G(D) = m \hat{m} G(D) \]

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Cyclic Coded Integer-forcing equalization
“Optimality” of DFE

- MMSE-DFE is known to be “optimal” assuming correct detection of past symbols (CDEF)

\[
\frac{1}{2} \log\left(1 + \frac{SNR_{DFE-MMSE-U}}{U}\right) = C
\]

- But how can one get error-free decisions?
- Must replace slicer with decoder
- Possible solution: Guess-Varanasi interleaving
- We pursue different solution: Move decoder before feedback loop
Equalize the channel to integer-valued impulse response
Add Zero-Padding/Cyclic prefix (as in OFDM) so that

\[ \text{Linear Convolution} \rightarrow \text{Cyclic Convolution} \]

Use linear cyclic code
\[ \Rightarrow \text{closed under integer-valued cyclic convolution} \]
Decode convolved codeword which is also a codeword
Apply DFE
Integer-Forcing Equalization

J. Zhan, B. Nazer, U. Erez, M. Gastpar ISIT 2010: Integer-Forcing Equalization proposed

⇒ $\text{FFE}(D) = \frac{I(D)}{H(D)}$ such that $I(D) = \text{FFE}(D)H(D)$ is a monic polynomial with integer coefficients

- More general than Zero-Forcing where $I(D) = 1$
- Less general than DFE since coefficients have to be integers
- FFE part is reminiscent of partial response equalization by lattice reduction (R. Fischer & C. Siegl 2005)
DFE- IF Equalization

\[ Y'(D) = \underbrace{\text{FFE}(D)H(D)X(D)}_{I(D)} + \text{FFE}(D)W(D) \]

\[ y'_n = x_n + \sum_{k=1}^{L} i_{n-k}x_k + z_n \]

- For DFE-IF choose \( I(D) \) so as to maximize

\[
\text{SNR}_{DFE-IF} = \frac{\sigma^2_X}{\sigma^2_Z} = \frac{\sigma^2_X}{\frac{1}{2\pi} \int_{-1/\pi}^{1/\pi} \frac{|I(e^{j\omega})|^2}{|H(e^{j\omega})|^2} d\omega}
\]
Cyclic Codes

- Let $\mathcal{C} = \{x_k\}_{k=1}^{2^{NR}}$ be a linear code over $\mathbb{Z}_q$
- $\mathcal{C}$ is cyclic if any cyclic shift of codeword is also a codeword
- $\Rightarrow$ Cyclic linear code is closed under integer-valued cyclic convolution with operations performed over $\mathbb{Z}_q$.

$$x \in \mathcal{C} \Rightarrow x' = [x \otimes i] \in \mathcal{C}$$

- Examples of cyclic codes:
  - "Most" algebraic codes - for example BCH
Finding $I(D)$

- We would like to maximize

$$\text{SNR}_{\text{DFE-IF}} = \frac{\sigma_x^2}{\sigma_z^2} = \frac{\sigma_x^2}{\frac{1}{2\pi} \int_{-1/\pi}^{1/\pi} \left| I(e^{j\omega}) \right|^2 \left| H(e^{j\omega}) \right|^2 d\omega}$$

- $\sigma_z^2$ can be written in matrix form

$$\sigma_z^2 = i \begin{bmatrix}
k_0 & k_{-1} & k_{-2} & \cdots & k_{-L} 
k_{1} & k_0 & k_{-1} & \cdots & k_{-(L-1)} 
\vdots & \vdots & \vdots & \ddots & \vdots 
k_{L} & k_{L-1} & k_{L-2} & \cdots & k_{0}
\end{bmatrix} i^T = i\tilde{K}i^T$$

where $k_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|H(e^{j\omega})|^2} e^{-jm\omega} d\omega$.

- A shortest lattice vector problem
- Can use LLL as an approximate solution
The noise enhancement caused by the IF equalizer is upper bounded by

$$\sigma_z^2 \leq \sigma_{ZF-DFE}^2 \cdot \min_{n \geq p+1} \left[ n \frac{2(1.4\pi n)^{\frac{1}{n}}}{\pi e} \left( \prod_{\mu,\nu} |z_{\mu}^* z_{\nu} - 1| \right)^{\frac{1}{n}} \right]$$

where $z_0, z_1, \ldots, z_{p-1}$ are the maximum-phase zeros of $H(D)H^*(D^{-*})$, and $p + 1$ is the channel’s length.

- Bound is based on Minkowski bound for shortest lattice vector
- Not tight in general
Simulation results

- 8-PAM constellation is used in a TCM-like manner
- cyclic LDPC $n=255$, $k=175$ ($R = 2.6862\frac{\text{bits}}{\text{channel use}}$) code for IF-DFE and MMSE-LE
- uncoded transmission for MMSE-DFE
- Channel is
  \[ 1 + 0.894D + 0.814D^2 + 0.239D^3 - 0.070D^4 + 0.036D^5 - 0.022D^6 \]
Summary, Extensions and Open Questions

- Integer-Forcing equalization allows channel decoding before applying the DFE loop.
  
  **Gains:**
  
  - No error propagation
  - Channel coding is much more effective

  **Penalties:**
  
  - DFE coefficients must be integers
  - Code must be cyclic

- Method is effective for channels of moderate lengths, and high SNR

- Extension to MMSE exists

- Explore specific channel models for which IF is advantageous