Abstract: Spatial evolution of nonlinear deep-water wave groups is studied experimentally and numerically. Spatial two-dimensional version of the Zakharov equation describing evolution of deep-water gravity waves is used to derive two 4th order evolution equations, for the amplitudes of the surface elevation and of the velocity potential. The scaled form of the equations is presented. The experimental results for wave groups with initial narrow spectrum are compared with the computations based on the unidirectional Zakharov equation and the Dysthe model. The very good agreement between the computational results based on both models with the experiments prompted an attempt to perform simulations for a wider initial spectral width, that formally violate the assumptions adopted in the derivation of the Dysthe model. The accuracy of the results based on the Dysthe model is checked against the solutions of the Zakharov equation, which is free of restrictions on the spectral width. Conclusions regarding the domain of validity of the Dysthe model are drawn.

KEYWORDS –Nonlinear wave groups; Zakharov equation; Dysthe model; Wave evolution; Wave spectra
INTRODUCTION

Frequency spectra of waves observed in ocean are often quite narrow, and the waves therefore exhibit notable groupiness. As the wave groups propagate towards the shore in deep and intermediate-depth water, they are transformed by combined action of various factors, including dissipation and energy input due to wind. Of particular interest, however, is the wave envelope transformation as a result of action of energy-conserving factors, like nonlinear interactions and dispersion. Transformation of waves propagating toward the beach has significant practical consequences. It is also important for understanding the fundamental nonlinear dynamics of water waves. Among model equations that describe evolution of a conservative system of nonlinear water waves in the sea, the Zakharov (1968) equation and the modified nonlinear Schrödinger (MNLS) equation derived by Dysthe (1979) are considered to be the most general. The earlier versions of those equations described the temporal evolution of the wave field. For the coastal applications, however, of considerable importance is the possibility to describe the spatial evolution of the waves as they approach the beach. The first steps in this direction were undertaken by Lo and Mei (1985) for the MNLS equation and by Shemer et al. (2000) for the Zakharov-type model. Both these studies deal with unidirectionally propagating waves.

THEORY

A generalized, two-dimensional version of the Zakharov-type spatial model is suggested here. This equation describes the evolution of the amplitude of each spectral component $B(\omega, \theta, x)$:

$$
\frac{i}{c_g} \nabla_h B(\omega, x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{q=1}^{\infty} 2\pi^2 \frac{(\omega_{nq})^2}{\omega_n \omega_m \omega_q} \left( \omega_n g \right) \frac{1}{2} T_{n, n, n, n} B_1^* B_2 B_3 
$$

(1)

where $k$ is the wave vector with direction $\theta$, $c_g = [\omega/2k(\omega)]/k$ is the group velocity vector, $B_j = B(\omega_j, \theta_j, x)$ and $\nabla_h = (\partial/\partial x, \partial/\partial y)$ is the horizontal operator. The equation (1) describes slow spatial evolution of nonlinear waves satisfying conditions of near resonance: $\omega + \omega_1 - \omega_2 - \omega_3 = 0$; $k + k_1 - k_2 - k_3 = O(\varepsilon^2)$, where $\varepsilon$ is the nonlinearity parameter representing the wave steepness. The unidirectional discrete version of (1) for the discrete wave amplitudes $A_j(x)$ has the following form:

$$
i c_{g_n} A_{x_{n,m}} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} 2\pi^2 \frac{(\omega_{nq})^2}{\omega_n \omega_m \omega_q} \left( \omega_n g \right) \frac{1}{2} T_{n, n, n, n}^* A_{x_{n,m}}^* A_{x_{n,m}} e^{-i(k_n + k_{n-1} - k_{n-2} - k_{n-3})x} \delta(\omega_n + \omega_m - \omega_{n+1} - \omega_{n+2} - \omega_{n+3})
$$

(2)

Equation (1) is now applied to derive the narrow-spectrum approximation (MNLS equations) for the spatial wave evolution equation in two horizontal dimensions. It is shown that at the adopted 4th order of the Taylor series expansion in the spectral
width $\Delta \omega_c = O(\varepsilon)$, where $\omega_c$ is the dominant wave frequency in the spectrum, two different versions of the spatial equations are obtained, for the surface elevation amplitude $A_\eta$, and for the amplitude of the velocity potential at the free surface, $A_\psi$. The small parameter of the problem $\varepsilon = a_o k_o$, where $a_o$ is the maximum wave amplitude. Dimensionless scaled variables are now introduced:

$$a_\eta = a_o A_\eta, \quad a_\psi = a_o A_\psi, \quad \phi = \omega_o a_o^2 \Phi$$

$$\varepsilon \omega_o \left( \frac{2k_o}{\omega_o} x - t \right) = \tau, \quad \varepsilon k_o y = Y, \quad \varepsilon^2 k_o x = X, \quad \varepsilon k_o z = Z$$

The dimensionless scaled equations have the following form, respectively:

$$\frac{\partial A_\eta}{\partial X} + i \frac{\partial^2 A_\eta}{\partial \tau^2} - \frac{i}{2} \frac{\partial^2 A_\eta}{\partial Y^2} + \varepsilon \frac{\partial^3 A_\eta}{\partial \tau \partial Y^2} + |A_\eta|^2 A_\eta + 8\varepsilon |A_\eta|^2 \frac{\partial A_\eta}{\partial \tau} + 2\varepsilon A_\eta^* \frac{\partial A_\eta}{\partial \tau} |_{Z=0} = 0$$

(4a)

and

$$\frac{\partial A_\psi}{\partial X} + i \frac{\partial^2 A_\psi}{\partial \tau^2} - \frac{i}{2} \frac{\partial^2 A_\psi}{\partial Y^2} + \varepsilon \frac{\partial^3 A_\psi}{\partial \tau \partial Y^2} + |A_\psi|^2 A_\psi + 8\varepsilon |A_\psi|^2 \frac{\partial A_\psi}{\partial \tau} + 4i \varepsilon A_\psi \frac{\partial \phi}{\partial \tau} |_{Z=0} = 0$$

(4b)

In (4a, b), the velocity potential of the induced current $\phi(X,Y,Z,\tau)$ satisfies the Laplace equation, with the boundary conditions at the free surface and at the bottom are given by

$$\frac{\partial \phi}{\partial Z} |_{Z=0} = \frac{|A_\psi|^2}{\partial \tau}, \quad \frac{\partial \phi}{\partial Z} |_{Z=0} = 0$$

EXPERIMENTS

The experiments are performed in a wave tank, which is 18 m long and 1.2 m wide. The tank is filled to a mean water depth of 0.60 m. Waves are generated by a wavemaker located at one end of the tank. The wavemaker is driven by a computer-generated signal. In order to perform measurements for various water depths, a false bottom made of marine plywood plates has been constructed in the tank. In order to reduce the wave reflection from the end of the tank, a sloping energy absorbing beach is installed at the far end of the tank. The instantaneous surface elevation is measured by a set of four resistance wave gauges. The wave gauge wires are made of 0.3 mm stainless steel. The sensors are supported by bars mounted on a carriage, which can be moved along the tank.

Two kinds of repeated periodically driving signals with the carrier wave periods $T_0 = 0.7$ s and $T_0 = 0.7$ s are employed. The first driving signal is given by:
\[ s(t) = s_0 \left| \cos(\Omega t) \right| \cos(\omega_0 t) \quad \Omega = \omega_0/20, \]

where \( s_0 \) is the forcing amplitude, \( \omega_0 = 2\pi/T_0 \), and \( \Omega \) is the modulation frequency. The period of this forcing signal is \( 10T_0 \). Its spectrum has a maximum peak at the carrier frequency \( \omega_0 \) and consists of a set of discrete frequencies spaced by \( \omega_0/10 \), with only the two closest to the carrier frequency sidebands being significant. The second kind of the driving signal is defined as:

\[ s(t) = s_0 \exp\left(\frac{-(t/mT_0)^2}{m} \right) \cos(\omega_0 t) \quad -16T_0 < t < 16T_0 \]  

The parameter \( m \) in (7) determines the width of the initial group. The value of \( m = 3.5 \) was selected in the present experiments. The driving signal is repeated periodically with the period of \( 32T_0 \) and generates wave groups that are widely separated. The discrete frequency spectrum of (7) has a Gaussian shape with the maximum peak at \( \omega_0 \) and a larger number of significant peaks. The longer period of the forcing signal results in a finer spectral resolution of \( \omega_0/32 \).

The maximum driving amplitudes \( s_0 \) for each shape of the driving signal are selected so that close to the wavemaker, the resulting carrier wave has the maximum wave amplitudes required in each particular experimental run.
RESULTS

Fig. 1. Measured (left column) and simulated using the Zakharov equation (2) (right column) amplitude spectra for the driving signal (6), $T_0 = 0.9$ s and $\varepsilon = 0.21$ at the distances from the wavemaker of 0.24 m ($a$, $b$), 6.58 m ($c$, $d$) and 9.47 ($e$, $f$).

In the system of equations (2), each equation determines the spatial evolution along the tank of a given free wave. The results presented in fig. 1 for the amplitude spectra and in fig. 2 for the temporal variation of the surface elevation are given at 3 locations along the tank. Both the simulated results and the experimental observations demonstrate similar energy spreading in the course of the propagation of waves away from the wavemaker. This spreading can be observed both in the amplitude spectra and in the surface elevation, which becomes especially prominent at the 3rd location ($x = 9.47$ m).

The higher spectral modes observed in experiments correspond to bound waves, as demonstrated by Jiao (1999).
Fig. 2. Measured and simulated surface elevation variations for conditions and locations of fig. 1.

Comparison of the experimental results with the numerical simulations based on the Zakharov equation (2) and the modified nonlinear Schrödinger (MNLS) model (4a, 5) was carried out for the Gaussian driving signal (7) with $m = 3.5$, carrier wave period $T_0 = 0.7$ s and the maximum wave steepness $\varepsilon = 0.21$. The experimental results at $x = 2.89$ m and $x = 8.67$ m are presented in figs. 3a, b, respectively. The corresponding MNLS model simulations at these two locations are presented in figs. 3c, d, while the Zakharov (2) simulations are given in figs. 3e, f. At $x = 2.89$ m, both these simulations are quite similar to the experimental result, the agreement being slightly better for the Zakharov model. This similarity between the simulated surface elevations in both models and the experimental observations is retained away from wavemaker, at $x = 8.67$ m.
Fig. 3. (a) and (b) Experimental results for the Gaussian initial shape with $m=3.5$, $\varepsilon=0.21$ at $x=2.89$ m and $x=8.67$ m; (c) and (d) MNLS (eqs. 4a, 5) simulations at the corresponding locations; (e) and (f) Zakharov simulations (eq. 2).

The results presented in fig. 3 clearly demonstrate that the 4th order system of the MNLS equations (4a, 5) is capable of describing wave fields with reasonably wide spectra. The energy spectrum of (7) also has a Gaussian shape with the relative width at the energy level of $\frac{1}{2}$ of the spectrum maximum given by

$$\frac{\Delta \omega}{\omega_0} = \frac{1}{m\pi} \sqrt{\frac{1}{2} \ln 2}$$  \hspace{1cm} (8)

For $m = 3.5$, the relative spectrum width $\Delta \omega / \omega_0 = 0.054 < \varepsilon$, thus satisfying the narrow spectrum assumption of the Dysthe model.
This agreement prompted an attempt to check the domain of validity of the Dysthe MNLS models for wider spectra. Generation of narrow initial wave group envelopes (with correspondingly wide initial spectrum) represents not an easy task in a relatively short experimental wave tank, in part due to the presence of long waves in spectrum. On the other hand, the validity of the Zakharov model for wave groups with narrow initial spectra was confirmed by extensive experiments. Since this model is free of any restrictions on the spectral width, the numerical solutions based on the Zakharov equation can serve as a basis for determining the domain of applicability of the Dysthe model.

Fig. 4. Simulations of the wave field evolution for the Gaussian initial shape with $m = 1.0$ and $\varepsilon = 0.24$. 
Spectrum of desired width is obtained in the present study by varying the value of the parameter $m$ in the Gaussian envelope shape given by (7). Simulations are carried out for the carrier wave period $T_0 = 0.7\text{s}$, maximum wave steepness $\varepsilon = 0.24$ and two values of the coefficient $m$, 1.0 and 0.6. For $m = 1.0$, the effective relative spectral width obtained from (8) is 0.19, just below the value of $\varepsilon$. For $m = 0.6$, $\Delta \omega / \omega_0 = 0.31$, higher than the maximum wave steepness. Both variation of the surface elevation with time and the amplitude spectra are presented at three distances from the location of wave generation, at $x = 5\text{ m}$, $x = 10\text{ m}$ and $x = 20\text{ m}$.

Fig. 5 Amplitude spectra for the Gaussian initial shape with $m = 1.0$ and $\varepsilon = 0.24$, a) Zakharov equation, b) Dysthe model.
Results for the surface elevation variation with the distance for \( m = 1.0 \) are presented in fig. 4. Strong dispersion leads to dramatic variation of the group shape, which is obvious in both computations. The resemblance between the results of both models seems to indicate that the Dysthe model is adequate for these parameters even at relatively large distances. Additional insight into the nonlinear physics of the wave group transformation along the tank is obtained by analyzing the discrete amplitude spectra of the surface elevation. Smoothed lines representing those spectra at the three selected locations are compared with the initial spectral Gaussian shape in fig. 5. The similarity between the two simulations is impressive, excluding the low frequency range. The Dysthe model correctly represents the gradual narrowing of the spectrum with the distance, with the corresponding growth of the peak value.

Fig. 6. As in Fig. 4, for \( m = 0.6 \).
For $m = 0.6$, the Dysthe model-based simulated temporal variation of the surface elevation at different locations along the tank still compare favorably with the computations based on the Zakharov equations, although in this case the agreement becomes less impressive at larger distance of 20 m. The groups retain their identity up to the distance of about $x = 10$ m, and farther away the faster moving longer waves penetrate to the slower moving shorter waves of the previous group.

Fig. 7. As in Fig. 5, for $m = 0.6$.

The corresponding amplitude spectra obtained from the Dysthe model retain definite similarity to those obtained using the Zakharov equation, especially at higher frequencies. As in fig. 5, in both simulations the spectrum becomes narrower with the distance, although for $m = 0.6$ this effect is less pronounced than for $m = 1.0$. The rate of variation of the spectrum in fig. 7 is quite fast at the first stages of evolution, but at larger distances from the location of wave generation the spectral shape
remains nearly constant. This can be attributed to the spreading of the wave energy over the computational domain visible in fig. 6, which results in gradual linearization of the problem with increasing distance. At the lower end of the spectrum, however, the Dysthe model exhibits notable noise, in contrast to the smooth behavior of the spectrum obtained from the Zakharov equation at those frequencies.

CONCLUSIONS

A very good agreement between the experimental results and the simulations based on both Zakharov and Dysthe models for unidirectional spatial evolution of nonlinear wave groups with narrow initial spectrum was obtained. The validity of both those nonlinear theoretical models is confirmed. The application of the Dysthe model is advantageous since it requires substantially less computer resources. Moreover, the Dysthe model is relatively simple and the evolution, the linear dispersion and the non-linear terms are clearly identified.

The Zakharov equation that is free of any restriction on the spectral width was used in this study as a basis for determining the limits of the Dysthe model applicability for wider initial spectra. It is demonstrated that for the relative spectral width of the order of c the relatively simple deep-water Dysthe model yields both surface elevation variation and amplitude spectra that are in a reasonable agreement with the solution of the Zakharov equation. The accuracy of the Dysthe model reduces significantly when the initial spectral width is substantially higher than the maximum initial wave steepness.

This study illustrates that a better understanding of the complex problem of the nonlinear wave evolution can be achieved by analyzing both surface elevation history and amplitude spectra at various locations. The surface elevation history plots make apparent the effects of dispersion, while the spectra clearly show the contribution of nonlinearity.

The present simulations for wider initial spectra demonstrate that the wave energy from the high-frequency components in the spectrum is shifted in the course of unidirectional evolution process towards lower frequencies, thus changing substantially the spectral shape. The frequency of the peak in an initially wide spectrum, though, shows a trend towards higher frequencies. No significant energy exchange is observed in the low-frequency range of the spectrum.

ACKNOWLEDGEMENT

Partial support of this study by INTAS is gratefully acknowledged.
REFERENCES


