NONLINEAR WAVE GROUP EVOLUTION IN SHALLOW WATER

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ABSTRACT: Wave group evolution in shallow water of constant depth is investigated experimentally in a laboratory wave tank and simulated numerically applying the Korteweg-de Vries equation. Good agreement between the experimental and numerical results is obtained. Decomposition into harmonics is performed, and the results are presented separately for the carrier wave frequency, second harmonic, and low harmonic. Experiments and simulations are performed for three values of the forcing amplitude at the wavemaker. The effect of the nonlinearity parameter and its ratio to the dispersion parameter on the evolution of the wave group along the tank is studied.

INTRODUCTION

Transformation of wave groups in the coastal region is of enormous significance for practical engineering, because it determines the design wave parameters required in construction of offshore marine structures. Evolution of wave groups is governed by a balance of dispersion, nonlinearity, and dissipation. Modeling of these processes has attracted considerable scientific interest for more than a century, and numerous theoretical approaches have been developed. In spite of that, the number of works dealing directly with transformations of wave groups is limited. Barnes and Peregrine (1995) demonstrated interesting details of the wave group evolution on the beach of constant slope within the full nonlinear potential theory. The wave packet with the soliton envelope in deep water attenuates after the “critical” depth \((kh = 1.36)\), where \(k\) is the carrier wave number and \(h\) is the local water depth, because the nonlinearity and dispersion cannot compensate each other in shallow water.

Recently, Shemer et al. (1998a) studied evolution of wave groups in intermediate water depth experimentally and numerically, applying the cubic Schrödinger equation (CSE). They have found that CSE can capture the main features of group envelope transformation. In particular, focusing of the wave group energy in deep water and demodulation in shallow water were shown. However, some more subtle details of group shape evolution measured in the experiments could not be described in the framework of the CSE. In particular, the observed experiments in shallow water strongly pronounced trough-crest and front-tail asymmetry in group shapes, as well as formation of secondary wave groups at higher harmonic, which were not reproduced in the numerical solutions of the CSE. Some of the effects noticed in the constant-depth experiments also were observed in a subsequent study by Shemer et al. (1998b) over a sloping beach in a laboratory tank. These results were compared with the modified CSE that accounts for variable water depth (Djordjevic and Redekopp 1978).

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Freeman and Davey (1975) demonstrated that, in water of limited depth, the CSE is valid only when the condition \((a/h)(k^2h^2) < 1\) is satisfied \((a\) is the wave amplitude). This condition restricts considerably the domain of applicability of the CSE in shallow water. Various modifications of Boussinesq’s equation (Peregrine 1967; Madsen et al. 1991; Nwogu 1993; Dingemans 1997; Kirby 1997) serve as alternative theoretical models, which can be successfully applied to analyze wave group evolution in shallow water. The simplest model adequately describing unidirectional wave evolution is the Korteweg-de Vries (KdV) equation \([\text{e.g., Mei (1989)}]\) and its generalizations for an uneven beach (Ostrovsky and Pelinovsky 1970, 1975; Kakinami 1971; Johnson 1973; Shuto 1974). Generation of long waves by a wave group was studied experimentally and simulated numerically applying Boussinesq’s equation by Mizuguchi and Toita (1996) and by Mizuguchi and Matsutate (1998) using a Schrödinger-type equation.

Kit et al. (1995) and Talipova et al. (1995) applied the generalized KdV equation to the study of wave groups. In these studies certain demodulation of the wave group in shallow water was observed. In the present work the analysis is restricted to a constant depth within the domain of applicability of the KdV equation. Detailed quantitative and qualitative comparison between the experimental results and the corresponding numerical solutions of the KdV equation is performed. The simple theoretical model makes it possible to delineate the effects of various terms, such as nonlinearity and dispersion, on the group evolution process.

THEORETICAL MODEL

The KdV equation is valid for weakly nonlinear and weakly dispersive long waves, which are characterized by two small parameters—nonlinearity \(\varepsilon = a/h\) (\(a\) is the wave amplitude and \(h\) is the water depth) and dispersion \(\mu' = (kh)^2\) (\(k\) is the wave number). The equation is valid when both small parameters of the problem, \(\varepsilon'\) and \(\mu'\), are of the same order. The canonical form of the KdV equation is \([\text{e.g., Whitham (1974), Mei (1989), Dingemans (1997)}]\)

\[
\frac{\partial \eta}{\partial t} + c \left( 1 + \frac{3}{2} \frac{\xi}{h} \right) \frac{\partial \eta}{\partial x} + \frac{ch^2}{6} \frac{\partial^3 \eta}{\partial x^3} = 0 \tag{1}
\]

where \(c = \sqrt{gh} \) is the limit value of the phase velocity for long linear waves.

Our aim is to compare the theoretical results with experiments in a wave flume, where slow variation of wave groups occurs along the tank. Eq. (1) thus should be modified to obtain evolution at a slow spatial scale. The following transformation to dimensionless variables is therefore employed:

\[
\tau = \omega(t - x/c); \quad X = x_0a/c \tag{2}
\]

where \(\omega\) is radian wave frequency. In the coordinates \((2)\), the wave group is almost frozen because the actual wave phase.
and group velocities in small but finite water depth differ somewhat from \((gh)^{1/2}\). Introducing the dimensionless surface elevation \(\eta^* = \eta/\eta_0\) (and dropping the \(*\)), (1) obtains the following form:

\[
\frac{\partial \eta^*}{\partial x} + \frac{3}{4} \varepsilon^{3/2} \frac{\partial^2 \eta^*}{\partial t^2} - \mu \frac{\partial^3 \eta^*}{\partial t^3} = 0 \tag{3a}
\]

In view of (3a), the small parameters of the problem are re-scaled, \(\varepsilon = 3\pi \varepsilon'\) and \(\mu = 1/6\mu'\). A slow spatial variable is now introduced, \(\xi = \mu x\). Eq. (3a) thus can be rewritten

\[
\frac{\partial \eta}{\partial \xi} - \frac{1}{4} \varepsilon \frac{\partial^2 \eta}{\partial \tau^2} - \mu \frac{\partial^3 \eta}{\partial \tau^3} = 0 \tag{3b}
\]

The linear terms in (3b), which represent the evolution and the dispersion, respectively, have the order of \(\mu\), while the nonlinear term is of the order of \(\varepsilon\). Because both small parameters \(\varepsilon\) and \(\mu\) are presumed to be of the same order, the equation is consistent. Eq. (3b) should be complemented by the initial condition at a certain location \(\xi_0\):

\[
\eta(\tau, \xi_0) = F(\Omega\tau) \cos \tau \tag{4a}
\]

and the periodic boundary condition

\[
\eta(\tau, \xi) = \eta(\tau + 2\pi/\Omega, \xi) \tag{4b}
\]

where \(\Omega = 2\pi/\omega T_{\text{gr}}, T_{\text{gr}} = \text{period of the wave group; and the periodic function} F(\Omega \tau) \text{ describes the shape of group envelope.}

Conservation laws for (3b) should be mentioned. It is known that the KdV equation describes an integrable system that has an infinite number of conservation laws. In the present study, the requirement of conservation of mass and energy fluxes results in

\[
\int \eta(\tau, x) \, d\tau = \text{const}; \quad \int \eta'(\tau, x) \, d\tau = \text{const} \tag{5a,b}
\]

These relations are verified in the numerical computations. For periodical conditions, the integrals are taken over the group period.

Eqs. (3b) and (4) are solved numerically using an explicit finite-difference scheme (Berezin 1987).

**EXPERIMENTAL SETUP**

Experiments were carried out in a wave tank that is 18-m long, 1.2-m wide, and 0.6-m deep. A false bottom made of a number of 1.18 × 1.25 m marine plywood plates, 1.8-cm thick, has been constructed. Each plywood plate is independently suspended on stainless steel rods from the steel frame of the tank. In the present study, these plates are arranged so that a constant effective water depth of 11.8 cm is attained. A paddle-type wavemaker hinged near the floor is located at one end of the tank. The wavemaker consists of four vertical modules, which in the present experiments are adjusted to move in phase with identical amplitudes and frequencies. The wavemaker is driven by a computer-generated signal. A wave-energy absorbing beach is located at the far end of the tank (starting at the distance of 12.5 m from the wavemaker). The instantaneous surface elevation is measured simultaneously by four resistance-type wave gauges. The probes are mounted on a bar parallel to the sidewalls of the tank and fixed to a carriage that can be moved along the tank. Measurements of the surface elevation are performed at eight carriage locations along the centerline of the tank. The spacing between the adjacent gauges on the bar is 0.4 m, the distance between the measuring stations being about 1.6 m. Information on the variation of the instantaneous surface elevation with time is thus obtained for 32 distances \(x\) from the wavemaker. Measurements are performed in the range 0.24 m ≤ \(x\) ≤ 11.32 m.

Probes are calibrated in situ using a stepping motor and a computerized static calibration procedure described in detail in Shemer et al. (1987). The calibration is performed at the beginning of each experimental run. The probe response is essentially linear for the range of surface elevations under consideration in the present study. The voltages of the four wave gauges, the signal driving the wavemaker, and the wavemaker position potentiometers outputs are sampled using an analog-to-digital converter and stored in the computer hard disk for further processing.

The periodic forcing signal with the wave group period \(T_{\text{gr}} = 32\tau\) (so that \(\Omega = 1/32\)) that drives the wavemaker, \(F_0(\Omega\tau)\), has a Gaussian shape and is given by

\[
s(\tau) = s_0 F_0(\Omega \tau) \cos \tau; \quad F_0(\Omega \tau) = \exp \left(\frac{3.2\Omega \tau}{\pi}\right)^2,
\]

\(-\pi < \Omega \tau \leq \pi\)

where \(s_0\) = forcing amplitude. This driving signal generates wave groups that are widely separated. The frequency spectrum of (6) also has a Gaussian shape. For the adopted value of the wave period and the effective water depth in the experiments, the wavelength corresponding to the shallow water dispersion relation \(\lambda = 2\pi/k = TV/gh = 0.97\) m so that the dispersion parameter \(\mu = 0.098\). Three values of the maximum driving amplitude \(s_0\) are used. The maximum driving amplitudes are selected so that, close to the wavemaker, the resulting carrier wave has the maximum wave amplitudes \(a_0\), corresponding to the nonlinearity parameter of about \(\varepsilon \approx 0.07\) (low amplitude), \(\varepsilon \approx 0.125\) (intermediate amplitude), and \(\varepsilon \approx 0.19\) (high amplitude).

To facilitate Fourier analysis of the recorded information, the sampling frequency is determined so that each data channel samples 128 data points during every carrier wave period. The duration of continuous sampling varies in the range of about 100s–200s.

**WAVE GROUP EVOLUTION ALONG TANK**

The measured temporal variation of the surface elevation at the closest to the wavemaker location \((x = 0.25\) m\) is chosen as the reference state and the initial condition for the numerical computations. This reference state is presented in Fig. 1 for the lowest (maximum value in the group of 1.1 cm) and the highest (maximum value of 3 cm) amplitudes. At low amplitude, the group shape is symmetric in Fig. 1(a), thus indicating that nonlinear effects are weak. At high amplitude, nonlinear effects are strongly pronounced and exhibit themselves in the asymmetry between crests and troughs of individual waves in a group. In Figs. 2–4 comparisons are carried out between temporal variations of the surface elevation recorded in the tank and computed using the KdV equation. This comparison is presented at three (out of 32 actually measured) representative locations along the tank, for the forcing amplitudes presented in Fig. 1.

The next measuring location \((x = 0.65\) m, Fig. 2), is separated by only about 1/2 carrier wavelength from the first one. Still, even at the low amplitude, wave shapes both in experiments [Fig. 2(a)] and in the numerical computations [Fig. 2(b)] differ notably from their reference state in Fig. 1(a). The trough-crest asymmetry, which was absent at this amplitude close to the wavemaker, becomes quite visible in both those figures. Reasonable quantitative agreement is observed between the experiments and the simulations. The nonlinear effects are apparently much stronger at high amplitude, both in experiments [Fig. 2(c)] and in computations [Fig. 2(d)].

At a considerable distance from the wavemaker, \(x = 3.3\) m
FIG. 3), the trough-crest asymmetry at low amplitude of forcing nearly vanishes in experiments [Fig. 3(a)] as well as in simulations [Fig. 3(b)]. The level of noise in the experiments becomes notably higher at this location, indicating perhaps the appearance of higher harmonics between the groups. At high amplitude, in addition to a stronger trough-crest asymmetry of individual waves, the front-tail asymmetry of the whole group becomes visible both in experiments [Fig. 3(c)] and in the numerical results [Fig. 3(d)]. This asymmetry exhibits itself in higher wave amplitude at the group front as compared to the tail. At these distances and forcing amplitudes, higher harmonics appear not only in experiments, but in the numerical results as well.

Far from the wavemaker (x = 7.2 m, Fig. 4), the group shape at low amplitude of forcing seems symmetrical both with respect to the crest and trough of the carrier frequency and the front and tail of the group. This symmetry is obtained both in experiments [Fig. 4(a)] and in computations [Fig. 4(b)]. In both those figures, the appearance of higher harmonics at the group tail, which themselves resemble the group shape, is visible. In addition, it should be noted that in the experiments, Fig. 4(a), additional low amplitude wave “group” at the carrier frequency is clearly seen. This “group” most probably can be attributed to some reflection from the beach at the far end of the tank, which cannot be completely eliminated. To estimate this effect, a single wave group was generated by the wavemaker. These experiments demonstrated that the reflection coefficient of the carrier frequency wave from the far end of the tank does not exceed 20% and that of the second harmonic can be neglected. Similar estimate of the reflection coefficient of the carrier frequency can be obtained readily from Fig. 4(a).

All higher harmonics effects are even more pronounced at high amplitude of forcing. In contrast to Figs. 4(a and b), strong trough-crest asymmetry within the group exists both in the measurements [Fig. 4(c)] and in simulations [Fig. 4(d)]. A notable decay of the wave amplitude in the experiment is visible in Fig. 4(c) as compared to the KdV simulations in Fig. 4(d), which do not account for dissipation.

The amplitude spectra of the recorded wave shapes at the reference location, x = 0.25 m, and at the remote location, x = 7.2 m, are presented in Fig. 5 for two extreme amplitudes of forcing. The general agreement between the experimental and the numerical results is impressive. The spectral shapes in the region corresponding to the carrier frequency (in the vicinity of 1.1 Hz) are quite similar. The carrier frequency downshift, which is manifested in the location of the spectral peak, is practically nonexistent at low amplitude of forcing, whereas at high amplitudes it can be seen both in experiments and in...
computations. The other common feature of the numerical and the experimental results is the notable broadening with distance from the wavemaker of the peak corresponding to the carrier frequency. The nearly Gaussian shape of the second harmonic peak (around 2.2 Hz) in the vicinity of the wavemaker is strongly modified far away. Note the similarity in the spectral shape of this harmonic in the experiments and in the numerical simulations. The low frequency harmonic, corresponding to long waves with frequencies of the order of 0.1 Hz, appears at both amplitudes of forcing in experiments as
well as in computations. The rate of increase of this harmonic
with the distance is, however, much more pronounced in the
simulations as compared to the experiments.

**DISCUSSION**

The spectra presented in Fig. 5 for the low amplitude of
forcing \( \varepsilon = 0.07 \) indicate that the spectral shape in the vicin-
ity of the carrier wave frequency remains virtually permanent
along the tank, whereas the second and the low harmonics alter
considerably in the process of their evolution. It thus can be
concluded that for this low amplitude of forcing, the wave field
can be decomposed into a dominant wave group at the carrier
frequency \( \eta_1 \) (of order \( \varepsilon \)) and smaller contributions that rep-
resent the high and low harmonics components \( \eta_0 \) and \( \eta_2 \) of
orders \( \varepsilon_0 \) and \( \varepsilon_2 \), respectively

\[
\eta = \eta_1 + \eta_0 + \eta_2 
\]  

(7)

Under this restriction, it can be easily shown from the linear
approximation of the KdV equation \[(1)\] that the dominant
component of the wave group can be written

\[
\eta_1(\tau, \xi) = F(\Omega(\tau - \xi')) \cos(\tau - \xi') 
\]  

(8)

where \( \xi' = \xi - \xi_0 \). The joint equation for the low and the
second harmonics has a driving term resulting from the non-
linear interaction at the dominant frequency

\[
\mu \frac{\partial(\eta_0 + \eta_2)}{\partial \xi} - \mu \frac{\partial^2(\eta_0 + \eta_2)}{\partial \tau^2} = \varepsilon \frac{\partial \eta_1^2}{\partial \tau} 
\]  

(9)

It follows from \( (9) \) that the sufficient condition for consistency
of the decomposition \[(7)\] is \( \varepsilon/\mu << 1 \). This ratio can be seen
as the Ursell parameter of the problem \( \text{[e.g., Mei (1989)]} \). For
our experimental conditions, however, \( \mu = 0.098 \), whereas
even at the lowest amplitude of forcing \( \varepsilon = 0.07 \), so that this
sufficient condition for the validity of decomposition is not
satisfied. A closer analysis of the governing equations, triggered
by the experimental observations, provides justification
for the consistency of \( (7) \).

Eq. \( (9) \) allows determining the bound components \( \eta_1^b \) and
\( \eta_0^b \), which have the following form:

\[
\eta_1^b = \varepsilon_2 F^b(\Omega(\tau - 3\xi_0)) \cos(2(\tau - \xi_0)), \quad \eta_0^b = \varepsilon_2 F^b(\Omega(\tau - 3\xi_0)) 
\]  

(10)

resulting in the following scaling parameters:

\[
\varepsilon_0 = \varepsilon/8\mu; \quad \varepsilon_1 = \varepsilon/6\mu 
\]  

(11)

The coefficients in \( (11) \) provide the required justi\-fication for the decomposition \[(7)\].

For higher amplitudes of forcing in our experiments, the
value of \( \varepsilon \) increases by a factor of about 2 and 3, respectively,
snow the Ursell parameter \( \varepsilon/\mu > 1 \). Additional harmonics
are therefore generated, and the spectrum widens considerably
\( \text{[cf. Figs. 5(d) and 5(f)]} \). For a wavemaker driving signal such
as that given by \( (6) \), free long and second harmonic waves are
generated at the wavemaker \( \text{(Mizuguchi and Toita 1996; Flick
and Guza 1980)} \), so that the amplitudes of the harmonics \( \eta_0 \)
and \( \eta_2 \) are small in the vicinity of the wavemaker for arbitrary
forcing amplitude. The spectrum width at the dominant fre-
quency is thus narrow at \( x = 0.25 \) m at both amplitudes of
forcing, Figs. 5(a and b).

The gradual separation between the bound and the free sec-
ond harmonic waves, which propagate along the tank at dif-
ferent velocities, manifests itself in the alternatively construc-
tive and destructive interference, which in turn leads to notable
variation of the maximum and the minimum individual wave
amplitudes at various locations in the tank \( \text{[cf. Figs. 1–3]} \).

In contrast to the relatively slow second harmonic free
waves, free long waves propagate somewhat faster than the
group. Because the difference between those two velocities is

**FIG. 5.** Wave Amplitude Spectra: (a) Measured, \( x = 0.25 \) m, \( \varepsilon = 0.07 \); (b) Measured, \( x = 0.25 \) m, \( \varepsilon = 0.19 \); (c) Measured, \( x = 7.2 \) m, \( \varepsilon = 0.07 \); (d) Measured, \( x = 7.2 \) m, \( \varepsilon = 0.19 \); (e) KdV Simulations, \( x = 7.2 \) m, \( \varepsilon = 0.07 \); (f) KdV Simulations, \( x = 7.2 \) m, \( \varepsilon = 0.19 \)
small, the separation between the bound and the free low frequency wave components increases slowly and, for the length of our wave tank, this free wave remains embedded in the carrier wave group. The temporal-spatial distribution of the surface elevation in the tank due to the low frequency component \( \eta_0 \) can be presented

\[
\eta_0 = \frac{e}{6\mu} [F^2(\Omega - 3\Omega\xi) - F^2(\Omega\tau)]
\]

If the group envelope \( F \) has a Gaussian shape, the function \( F^2 \) has a Gaussian shape as well. Returning to the physical coordinates \((x, t)\) and expanding (12) into the Taylor series yields at any fixed location in the tank \( x_0 \)

\[
\eta_0(x_0, t) = \frac{e}{2\mu} x_0/c \frac{d(F^2)}{dt}
\]

thus indicating that it can be anticipated that the amplitude of the low frequency component should increase linearly with the distance from the wavemaker, and the form of this long wave should resemble the derivative of the Gaussian shape.

In view of these considerations, it becomes interesting to examine separately each one of the three harmonics, both in the experimental and in the numerical results. The wave frequency spectra presented in Fig. 5 clearly indicate that in most occasions the three harmonics are well separated. Band-pass filtering can therefore be applied to obtain temporal variation of each harmonic. Three frequency bands are selected: The carrier wave band, \( 0.4f_0 < f < 1.6f_0 \); the second harmonic band, \( 1.6f_0 < f < 2.4f_0 \); and the low frequency band, \( f < 0.4f_0 \).

The results representing the carrier frequency at the low amplitude of forcing at two locations along the tank are presented in Fig. 6. The numerical results [Fig. 6(c)] indeed show that in this band the wave group remains virtually permanent along the tank. The wave group reflected from the far end of the tank due to imperfect beach is clearly seen in Fig. 6(b).

The results for the second harmonic at two distances from the wavemaker, \( x = 0.25 \) m and \( x = 7.2 \) m, are presented in Fig. 7. At \( x = 7.2 \) m [Figs. 7(b and c)], the bound second harmonic wave group, which moves faster, leaves behind the free one and appears at earlier times. At this distance from the wavemaker, the bound and the free second harmonic wave groups are nearly separated both in experiments and in simulations. Interference of the bound and the free waves at the second harmonic results in strong variations of the maximum wave amplitudes in the group at the initial stages of their propagation along the tank. The comparison between the experimental and the numerical simulations indicates that the separation between the bound and the free second harmonic wave groups in the experiments is more pronounced than in the model results.

The results on the low frequency harmonic (the long waves) are presented in Fig. 8. The shape of those waves indeed corresponds to the derivative of the Gaussian distribution. In agreement with (13), the amplitude of the long waves in simulations increases approximately linearly with the distance \( x_0 \).
FIG. 8. As in Fig. 7, Long Waves

FIG. 9. Maximum Wave Amplitude Distribution Along Tank: (a) KdV Simulations; (b) Measurements

from the wavemaker. No quantitative agreement with the numerical results is observed in Fig. 8. Note that those long waves have the wavelength corresponding to the length of the carrier wave group, which is of the order of magnitude of the total length of our experimental facility. The reflection of such long waves from the far-end wave-energy-absorbing beach cannot be eliminated and remains significant. The actual distribution of those waves along the tank in the experiments is strongly influenced by the wave reflection. This effect is not accounted for in the model.

The maximum wave height $H_{\text{max}}$ in the group is computed as the maximum difference between the surface elevations in consequent crests and troughs defined by zero crossing. The computed $H_{\text{max}}$ using the KdV model distribution of $H_{\text{max}}/2$ along the tank is presented in Fig. 9(a) for all three amplitudes of forcing. At the lowest amplitude of forcing ($\varepsilon = 0.07$), the maximum wave height remains approximately constant along the tank, although some oscillations in the region adjacent to the wavemaker can be noticed. At intermediate forcing ($\varepsilon = 0.125$), those oscillations become stronger and certain decay of $H_{\text{max}}$ along the tank becomes visible. Both those effects become even more pronounced for high amplitude of forcing ($\varepsilon = 0.19$). The obtained oscillations of the maximum wave height close to the wavemaker result from the interference of the free and the bound second harmonic (Flick and Guza 1980).

The decrease in the maximum wave height in the group propagating along the tank can be seen as group demodulation. The KdV model employed here is essentially Hamiltonian, and the total wave energy in the group is indeed conserved in the present simulations. The decrease of the maximum wave height in the group is therefore accompanied by group broadening. As can be seen from Fig. 9(a), this demodulation process is of an essentially nonlinear nature. The effect of the wave group demodulation in shallow water has been discussed in relation with the numerical solution of the CSE by Shemer et al. (1998a).

Experimental results on the variation of the maximum wave height along the tank are given in Fig. 9(b). Strong oscillations in $H_{\text{max}}$ observed in this figure also can be attributed to the interference of the bound and the free components of the second harmonic. In contrast to the theoretical model, certain dissipation is always present, particularly in shallow water experiments. The decay of the maximum wave height even for weak forcing is most probably related to these effects.

CONCLUDING REMARKS

Generation of the second and the low harmonics, each comprising both free and bound components, is observed in the experiments and obtained in the numerical solutions of the KdV equation. Due to different propagation velocities of these bound and free waves, they gradually become separated while traveling along the tank. For that reason, new groups at the second harmonic appear between the groups at the carrier frequency at larger distances from the wavemaker.

For small values of the nonlinearity parameter $\varepsilon$, the dominant harmonic is decoupled from the second and the low frequency components. In the vicinity of the wavemaker, this decoupling holds for arbitrary forcing amplitudes. Away from the wavemaker, the wave spectra around the carrier wave frequency remain narrow and sharp for low driving amplitude, whereas for stronger forcing, notable broadening of the spectra is obtained both in experiments and in the numerical simula-
tions. This spectrum broadening is accompanied by spreading of the wave energy along the group (in the time domain), which can be interpreted as the group demodulation.

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APPENDIX I. REFERENCES


APPENDIX II. NOTATION

The following symbols are used in this paper:

\[ a = \text{wave amplitude}; \]
\[ c = \text{velocity of long linear waves}; \]
\[ g = \text{gravitational acceleration}; \]
\[ F = \text{wave group envelope}; \]
\[ F_w = \text{envelope of forcing signal}; \]
\[ f = \text{wave frequency}; \]
\[ f_o = \text{carrier wave frequency}; \]
\[ H_{max} = \text{maximum wave height in group}; \]
\[ h = \text{water depth}; \]
\[ k = \text{carrier wave number}; \]
\[ s = \text{wavemaker displacement}; \]
\[ x_0 = \text{wavemaker displacement amplitude}; \]
\[ T = \text{carrier wave period}; \]
\[ T_r = \text{period of wave group}; \]
\[ t = \text{time}; \]
\[ X = \text{distance from wavemaker (dimensionless)}; \]
\[ x = \text{distance from wavemaker (dimensional)}; \]
\[ \varepsilon = \text{nonlinearity parameter (scaled)}; \]
\[ \varepsilon' = \text{nonlinearity parameter}; \]
\[ \eta = \text{surface elevation}; \]
\[ \eta' = \text{surface elevation due to bound waves}; \]
\[ \eta_0 = \text{surface elevation due to long waves}; \]
\[ \eta_1 = \text{surface elevation due to dominant waves}; \]
\[ \eta_2 = \text{surface elevation due to second harmonic waves}; \]
\[ \lambda = \text{carrier wavelength}; \]
\[ \mu = \text{dispersion parameter (scaled)}; \]
\[ \mu' = \text{dispersion parameter}; \]
\[ \xi = \text{scaled dimensionless distance from wavemaker}; \]
\[ \xi_0 = \text{scaled dimensionless reference location}; \]
\[ \xi_0' = \text{scaled dimensionless distance from reference location}; \]
\[ \tau = \text{dimensionless time}; \]
\[ \omega = \text{radian wave frequency}; \]
\[ \Omega = \text{radian group frequency}. \]