ABSTRACT

The evolution along the tank of unidirectional nonlinear wave groups with narrow spectrum is studied both experimentally and numerically. Measurements of the instantaneous surface elevation within the tank are carried out using digital processing of video-recorded sequences of images of the contact line movement at the tank side wall. The accuracy of the video-derived results is verified by measurements performed by conventional resistance-type wave gauges. An experimental procedure is developed that enables processing of large volumes of video images and capturing the spatial structure of the instantaneous wave field in the whole tank. The experimentally obtained data are compared quantitatively with the solutions of the modified nonlinear Schrödinger (MNLS, or Dysthe) equation written in either temporal or spatial form. Results on the evolution along the tank of wave frequency spectra and on the temporal evolution of the wave number spectra are presented. It is demonstrated that accounting for the 2nd order bound (locked) waves is essential for getting a qualitative and quantitative agreement between the measured and the computed spectra.

INTRODUCTION

Rapid advancement in both theoretical and experimental studies of water waves that occurred in recent decades were prompted by the discovery by Benjamin and Feir [1] of the sideband instability of weakly nonlinear Stokes waves. Important theoretical model for studying the long time behavior of the nonlinear water waves was developed by Zakharov [2]. The Zakharov integral equation describes near-resonant interactions between waves and was originally derived for infinitely deep water. Zakharov also showed that under the assumption that the wave spectrum is narrow, the nonlinear Schrödinger (NLS) equation can be deduced from the Zakharov equation. The NLS equation, accurate to the 3rd order in the wave steepness \( \varepsilon \), therefore describes resonant interactions pertaining to a weakly nonlinear wave train with a narrow band of frequencies and wave lengths, and governs the slow modulation of the wave group envelope. Later the NLS equation for water gravity waves in water of finite depth was derived by using the multiple scales method [3] and by applying the averaged Lagrangian formulation [4]. An agreement between the experimentally found growth rates of the unstable sidebands with the theoretical predictions based on the NLS equations was obtained [5].

Shemer et al. [6] performed quantitative comparison of the numerical simulations based on the NLS equation with experiments in a laboratory wave flume. They demonstrated that while the NLS equation is adequate for qualitative description of the global properties of the group envelope evolution, such as focusing observed for water waves in sufficiently deep water, it is incapable of capturing more subtle features, for example the emerging front-tail asymmetry observed in experiments. Due to nonlinear interactions, considerable widening of the initially narrow spectrum can occur; therefore more advanced models are required for an accurate description of nonlinear wave group evolution. The modified nonlinear Schrödinger (MNLS) equation derived by Dysthe [7] is a higher (4th) order extension of the NLS equation, where the higher order terms account for finite spectrum width [8]. Further modification of the NLS equation appropriate for wider wave spectra was presented by Trulsen and Dysthe [9] and Trulsen et al. [10]. It was shown by Kit and Shemer [11] that this modification can be easily derived by expanding the dispersion term in the Zakharov equation into the Taylor series. The effect of each one of the 4th order terms in the Dysthe equation was studied in [12]. They demonstrated that for steep waves all these terms contribute significantly to the accuracy of the solution.

The theoretical models mentioned above were derived to describe the evolution of the wave field in time. Complete information on the surface elevation distribution along the tank at a prescribed instant constitutes the required for the solution of the problem initial condition. In laboratory experiments, however, waves are generated by a wavemaker usually placed at the end of the experimental facility. The experimental data are commonly accumulated using sensors
placed at fixed locations within the tank. Hence, to perform quantitative comparison of model predictions with results gained in those experiments, the governing equations have to be modified to a spatial form, to describe the evolution of the temporally varying wave field along the experimental facility. Such a modification of the Dysthe model was carried out by Lo and Mei [13] who obtained a version of the equation that describes the spatial evolution of the group envelope. Numerical computations based on the Dysthe model for wave groups propagating in a long wave tank indeed provided good agreement with experiments and exhibit the front-tail asymmetry [12]. The spatial version of the Dysthe equation was derived in [11] from the spatial form of the Zakharov equation, [14, 15] that is free of any restrictions on the spectrum width.

In the present work, the evolution of unidirectional nonlinear wave groups along the tank is studied using digital processing of video-recorded sequences of images of the contact line movement at the tank side wall. The technique allows accurate measurements of both the spatial variation of the instantaneous surface elevation along the whole tank, and of the temporal variation of the surface elevation at any prescribed location within the tank. The comparison of the experimentally obtained data thus can be carried out with the solutions of the model equations presented in their temporal or the spatial forms.

Narrow-spectra nonlinear deep-water wave groups excited by a wavemaker are studied. The Dysthe MNLS equation describes evolution of the complex nonlinear wave group envelope and constitutes an appropriate theoretical model for studying such wave groups. The advantages and disadvantages of the spatial and temporal forms of the model equation are discussed. The Dysthe equation in both temporal and spatial forms is solved numerically, and the results of both versions are compared quantitatively with the experimental data.

**THEORETICAL BACKGROUND**

Consider a narrow-banded unidirectional deep-water wave group with the dominant frequency \( \omega \) and wave number \( k_0 \) that are related by the deep-water dispersion relation for gravity waves:

\[
\omega^2 = k_0 g,
\]

where \( g \) is the acceleration due to gravity. Evolution of the wave group in a wave flume can be represented by variation in time and space of either the surface elevation \( \eta(x, t) \), or of the velocity potential \( \varphi \) at the free surface, \( \psi(x, t) = \varphi(x, z=\eta, t) \). For a narrow-banded wave group it is convenient to express the variation of \( \eta \) and \( \varphi \) at the leading order in terms of their complex envelope amplitudes:

\[
\eta(x, t) = \text{Re}[a \eta^* (x, t) \exp i(k_0 x - \omega_0 t)] \tag{2a}
\]

\[
\psi(x, t) = \text{Re}[i/2\omega_0 a \varphi(x, t) \exp i(k_0 x - \omega_0 t)] \tag{2b}
\]

The MNLS coupled system of equations, which describes the evolution of the complex envelope \( a(x, t) \) and of the potential of the induced mean current \( \varphi(x, z, t) \) was in fact derived by Dysthe for the surface velocity potential amplitude, \( a_\varphi \). It was demonstrated by Hogan [16], see also [11], that while at the 3rd order the governing equation for both amplitudes, of the surface elevation, \( a_\eta \), and of the free surface velocity potential, \( a_\varphi \), are identical, and thus there is no difference in the NLS equation for either of those amplitudes, at the 4th order the governing equations differ somewhat. For quantitative comparison of the model predictions with the experiment that directly provides data on the surface elevation variation, the equation describing the variation of \( a_\eta \) is applied in sequel, with the index “\( \eta \)” omitted. In fixed coordinates, the governing system of equations has the following form:

\[
\frac{\partial a}{\partial t} + \frac{a_\eta}{2k_0} \frac{\partial a}{\partial x} + ik_0 \frac{a_\varphi}{4} \frac{\partial^2 |a|^2}{\partial x^2} + \frac{i}{2} \epsilon k_0^2 \frac{\partial |a|^4}{\partial x^2} a - \frac{1}{16} k_0^4 \frac{\partial^4 a}{\partial x^4} = 0 \tag{3}
\]

\[
\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \quad (-h < z < 0). \tag{4}
\]

These equations are subject to the boundary conditions at the free surface

\[
\frac{\partial \varphi}{\partial z} = 0 \quad (z = 0) \tag{5}
\]

and at the bottom

\[
\frac{\partial \varphi}{\partial z} = 0 \quad (z \to -\infty) \tag{6}
\]

The first four terms in (3) constitute the cubic Schrödinger equation for deep water in the fixed frame of reference. Dysthe’s equations that can be derived from the Zakharov integral equation that is of the 3rd order in steepness by adding the narrow-band assumption with spectral width \( O(\varepsilon) \) [8].

The sign of the term \( a^2 \partial a^*/\partial x \) in (3) is positive, while in the velocity potential version used in [7] and [13] it is negative. The opposite signs of this term constitute the only difference between the two versions of the 4th order envelope evolution equation.

The problem of wave field evolution in a tank admits two different formulations. In the so-called temporal formulation, the spatial distribution of the complex envelope \( a(x) \) is presumed to be known at a prescribed instant \( t_0 \), and its variation in time is obtained by numerical solution of the model equation. Alternatively, the variation of the complex envelope in time, \( a(t) \), can be specified at a prescribed location \( x = x_0 \), and the variation of \( a(t) \) along the tank is then studied in the spatial formulation using the appropriately modified model equations. It should be
stressed that the spatial formulation is routinely applied in the experiment-related studies [12, 13], since the wave gauges provide information on the temporal variation of the surface elevation at fixed locations. The experimental approach of the present study makes it possible to measure the variation with time of the instantaneous complex group envelope along the tank, as well as the variation of the surface elevation with time at any location within the tank. Both temporal and spatial formulations of the Dysthe equation are therefore employed.

Consider first the temporal model. In analogy to [13], in a coordinate system moving at the group velocity \( c_g = \omega_0 / 2k_0 \), the following dimensionless scaled variables are introduced:

\[
\tau = \omega_0 \phi, \quad \xi = \ell \phi(x-c_g t), \quad A = a/a_0, \quad \Phi = \omega_0 a_0^2 \phi, \quad Z = \varepsilon k_0 z \quad (7)
\]

In these variables, the equations for \( A \) and \( \Phi \) are:

\[
\frac{\partial A}{\partial \tau} + i \frac{\partial^2 A}{8 \partial \xi^2} + i \frac{1}{2} |A|^2 A = 0 \\
+ \varepsilon \left( \frac{1}{16} \frac{\partial^3 A}{\partial \xi^3} + \frac{1}{4} \frac{\partial^2 A^*}{\partial \xi^2} + \frac{3}{2} \frac{\partial A}{\partial \xi} \frac{\partial A}{\partial \xi z = 0} + i A \frac{\partial \Phi}{\partial \xi z = 0} \right) = 0
\]

(8)

\[
\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial Z^2} = 0 \quad (Z < 0)
\]

(9)

with \( \Phi \) satisfying the following boundary conditions:

\[
\frac{\partial \Phi}{\partial Z} = \frac{i}{2} |A|^2 \xi, \quad Z = 0, \quad \frac{\partial \Phi}{\partial Z} = 0, \quad Z \to -\infty
\]

(10)

The set of equations (7) - (10) and the appropriate initial conditions constitute the temporal version of the Dysthe model. The corresponding spatial version can be obtained either from (3) as in [13], or from the spatial version of the Zakharov equation [11]. The scaled dimensionless space and time variables in (7) are replaced for the spatial version by:

\[
\xi = \ell k_0 x, \quad \tau = \omega_0 (x/c_g t)
\]

(11)

The governing equations then assume the following form:

\[
\frac{\partial^2 A}{\partial \xi^2} + i \frac{\partial^2 A}{\partial \tau^2} + i |A|^2 A = 0 \\
+ \varepsilon \left( 8 |A|^2 \frac{\partial A}{\partial \tau} + 2 A^* \frac{\partial A}{\partial \xi} + 4 i A \frac{\partial \Phi}{\partial \xi z = 0} \right) = 0
\]

(12)

\[
\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial Z^2} = 0 \quad Z < 0
\]

(13)

\[
\frac{\partial \Phi}{\partial Z} \bigg|_{Z=0} = \frac{i}{2} |A|^2 \tau, \quad \frac{\partial \Phi}{\partial Z} = 0, \quad Z \to -\infty
\]

(14)

The formulation of the spatial model (11)-(14) is completed by specifying the temporal variation of the envelope at the prescribed location \( A(\xi_0, \tau) \).

In both the temporal and the spatial formulations, the normalized envelope shape \( A(\xi, \tau) \) determines the surface elevation at the leading order. With \( A(\xi_0, \tau) \) known, application of (2a) represents the so-called free waves only. The bound, or locked, waves can also be determined from \( A(\xi, \tau) \) using:

\[
B(A) = \frac{1}{2} \varepsilon A^2
\]

(15)

The surface elevation accurate to the 2nd order for both temporal and spatial formulation is thus obtained as

\[
\eta/a_0 = \Re \left( A e^{i(k_0 x - \omega_0 t)} + B(A) e^{2i(k_0 x - \omega_0 t)} \right)
\]

(16)

**EXPERIMENTAL FACILITY AND THE INITIAL CONDITIONS**

The experiments were performed in the wave tank that is 18 m long, 1.2 m wide and has transparent side walls and windows at the bottom which allow viewing of the flow from various directions. The tank is filled to mean water depth of 0.6 m. Waves are generated by a computer-controlled paddle-type wavemaker. The instantaneous surface elevation at any fixed location can be measured by resistance type wave gauges made of 0.3 mm platinum wire. The gauges are shifted along the tank using a carriage.

The wave gauges in this study were applied mainly for validation of the accuracy of surface elevation measurements by digital processing of video clips that record the contact line movement at the tank’s wall. The instantaneous contact line shapes were recorded by a single monochrome CCD video camera (Pixelink PL-A471) at a rate of 30 fps. The size of each frame is 640 by 480 pixels. The field of view of the camera located one meter from the tank wall spans 50 cm along the tank, yielding the pixel dimension of about 0.8 mm. Advantage was taken of extremely high repeatability of the wave field emanating from the prescribed wavemaker driving signal. The camera is also placed on the carriage to enable imaging of different regions of the tank. Each camera recording session is synchronized with the wavemaker driving signal using a common reference signal. A single wave group was generated for each recording session. For the consecutive recording session, the carriage is shifted along the tank, so that slightly overlapping images of the contact line movement along the whole experimental facility are obtained. Every frame of the recorded video clip at each camera location was processed separately.

Experiments were performed for a wave group with Gaussian envelope generated by the wave maker. The temporal variation of the surface elevation at the wavemaker has to the leading order the following form:

\[
\eta(t, x = 0) = a_0 A(t) \cos(\omega_0 t);
\]

(17a)

with the Gaussian envelope shape given by

\[
A(t, x = 0) = e^{-\left( \frac{t}{m_0} \right)^2}
\]

(17b)
The dominant wave period in the present experiments $T_0=0.7$ s corresponds to the wave length $\lambda_0=2\pi/k_0=0.76$ m. The shape of wave groups given by (17) with a significantly shorter dominant wave length can not be faithfully reproduced in the present facility. The initial width of the group is determined by the parameter $m$. The group becomes wider as the value of $m$ increases; correspondingly, the surface elevation frequency spectrum becomes narrower with $m$ increased. Based on our earlier studies [12], the value of $m=3.5$ was chosen.

The steepness of the wave group $\varepsilon=a_0k_0$ was selected according to the following considerations. To enable determination of the instantaneous spatial envelope shape of the wave group and to study its nonlinear temporal evolution, the entire group has to be present in the tank. Hence, on one hand, the group generation by the wavemaker has to be completed before initiation of the study of the temporal variation of the envelope shape, and on the other hand, spatial wave group structure measurements remain meaningful as long as the front of the group does not reach the beach. The spatial extension of the wave group for the adopted parameters does not exceed 6-7 m, the group velocity being $c_g=0.54$ m/s. When the generation of the group by the wavemaker is completed, the group front is about 5 m from the beach, leaving the duration of less than 7 s to study the wave group evolution before its front reaches the far end of the tank. According to (7), the time scale of the nonlinear effects is $O(\varepsilon^2)$. Hence, for the duration of the process prescribed by the group shape, the dominant frequency and the length of the tank, higher wave steepness increases the effective evolution time at the slow scale $\tau$. On the other hand, nonlinear waves with higher steepness tend to break. Since wave breaking cannot be accounted for by the Dysthe model, the wave steepness must remain below the value that can lead to wave breaking.

The maximum initial wave steepness of $\varepsilon=0.18$ ($a_0=22$ mm) adopted in this study was selected experimentally on the basis of visual observations of wave group propagation along the tank with different values of $a_0$. For this value of $\varepsilon$, $\varepsilon=1$ corresponds to dimensional duration $\tau=3.44$ s, or 4.9 dominant wave periods. This is well below the experiment duration limit of about 7 s imposed by the effective length of the tank. The spectral width of the signal given by (17), calculated as in [12], is $A\varepsilon_{fl}=0.054\varepsilon$, thus satisfying the narrow spectrum limit of the Dysthe model. For these experimental conditions, the dimensionless depth $k_0h=5$, so that the condition for the validity of the Dysthe model, $\kappa k_0=O(1/\varepsilon)$ is also satisfied.

The experimental results are compared with the theoretical predictions based on the numerical solution of the Dysthe equation in either temporal, equations (7)-(10), or spatial, equations (11)-(14), forms, together with the corresponding initial conditions. The initial envelope shape $A(0, \tau)$ for the spatial evolution case is given by (17b) and (11), while the initial condition for the temporal evolution case $A(\xi, 0)$ is obtained from (7) and (17b) with $t=x/c_g$.

The equations are solved using the pseudo-spectral split-step Fourier method based on [13]. The computed complex envelope is then translated into the physical coordinates $(x, t)$. The variation of the surface elevation at any fixed location $\eta(x_0, t)$ in the spatial formulation, or at the fixed instant $\eta(x, t_0)$ in the temporal formulation, is obtained from the complex envelope that contains the 2nd order correction using (16).

**VIDEO DATA PROCESSING**

An example of a recorded image is presented in Fig. 1. While the contact line can be clearly identified visually, the image contains numerous additional features such as the tank supporting beam, objects in the laboratory beyond the tank, reflections, etc. An effective algorithm was developed to extract quantitative information from the recorded video clips that contain thousands of images like that in Fig. 1.

![Figure 1. Contact line image that contains tank wall supporting beam and the viewing window.](image)

The images were first preprocessed using contrast enhancement and linear filtering (see, e.g. [17]). Search of the contact line coordinates is carried out in the vicinity of the contact line itself and is thus unaffected by other spurious curves that might appear in the image. A searching area is cropped from the whole image in the vicinity of the desired curve as a rectangular window that is small with respect to the entire image. The initial window built in the 1st image of the series around a point that constitutes the center of the searching area is chosen in a close vicinity of the desired curve. Since the slope of the interface is usually quite small, the window aspect ratio selected in most case is in the range of 2 to 3, the width of the window being about 50 pixels.

The vertical coordinate associated with every horizontal coordinate is defined as the weighted average of the pixel intensities along the vertical extent of the window. Once all vertical coordinates within the window are calculated, the least mean square fit is performed on the array of the detected points, so that contact line shape within the window is approximated by a second order polynomial. The vertical coordinate of the contact line at the center of the window is finally obtained from the polynomial value at the corresponding horizontal coordinate. The contact line...
coordinates determined by this procedure contain contribution of the pixel intensities in the vicinity of each point and are obtained at a resolution better than 0.5 pixel.

The window is then shifted forward by one pixel in the direction given by the slope of the contact line, and the process is repeated. At each step, the window is inclined by an angle corresponding to the window shift direction. This process continues until the whole image is covered. Once the coordinates of the contact line profile in a given frame have been determined, in the next frame the search is performed utilizing this information. At the 30 fps recording rate, the contact line shift between consecutive frames is quite small, making it advantageous to start the search in the next frame with the initial window built around the previous profile.

The applied procedure has an additional advantage of allowing processing of numerous clips captured during the experiment at different locations along the tank automatically. Despite the fact that the camera is moved between capturing consecutive clips, its vertical position and its distance from the opposite tank wall remain constant within a reasonable accuracy. Each clip was recorded after the camera has been shifted along the tank by the distance corresponding to the horizontal extent of the recorded image, and the wavemaker was activated after a delay sufficient for all waves from the previous run to dissipate. The initial search window in the consecutive clips is placed according to the coordinate of the interface determined in the last window of the clip recorded at the upstream location at the identical timing relative to the reference signal.

The present experimental approach was validated extensively using conventional resistance wire gauges at a number of locations along the tank and comparing with data simultaneously acquired at same distance \( x \) by image processing technique. Measurements on the evolution of wide frequency spectra wave groups that vary significantly along the tank due to dispersion and nonlinearity [15] were performed for validation purposes. The spanwise uniformity of the surface elevation was also checked by placing the probes across the tank. The difference between the instantaneous values of the surface elevation measured by the wave gauge located close to the tank’s wall and by video image processing at various distances from the wavemaker always remains below 1 mm and does not exceed the deviation between the outputs of different probes. The spectra derived from those measurements exhibit a very good agreement for all frequencies in the spectra. More details on the experimental method employed are given in [18].

**EXPERIMENTAL AND NUMERICAL RESULTS**

The temporal variation of the surface elevation within a wave group with the initially Gaussian envelope at \( x=0 \) as given by (17) is studied first. The computed according to (11)–(14) variation of the surface elevation within the group at a number of locations along the tank is compared in Fig. 2 with the results of video image processing.

*Figure 2. Variation of the surface elevation within the group at different distances \( x \) from the wavemaker \((T_0=0.7s, a_0=0.22 m)\): experiments; simulations*
The computed values of $\eta(t)$ in Fig. 2 contain the contribution of the bound waves, see (15) and (16). The time is measured relative to the initiation of the wavemaker operation. The shift in the horizontal scale in the consecutive frames of Fig. 2 reflects the group traveling time between the measuring stations. Excellent agreement is obtained between the experimental results and the computations based on the spatial evolution equations (11)-(14). No measurements were performed in the immediate vicinity of the wavemaker due to the presence of evanescent standing waves. The sequence of frames in Fig. 2 clearly demonstrates that the duration of the group extends with $x$ and the initially symmetric Gaussian envelope shape gradually exhibits stronger left-right, with increasingly steep front and elongated tail. The maximum surface elevation within the group may exceed significantly the nominal value of $a_0$. This increase of the maximum amplitude is associated in part with the focusing properties of the nonlinear Schrödinger equation, as discussed in [6]. The NLS equation, however, is only capable of reflecting correctly some limited features of the solution, and the extension to the MNLS equation is required to get both qualitative and quantitative agreement between experiments and computations [11].

Apart of the focusing, additional reason for higher maximum values of the surface elevation notable in Fig. 2, as well as for the crest-trough asymmetry, is the contribution of 2nd order bound (locked) waves.

The notable variation of the group shape along the tank in Fig. 2 is due to both linear dispersion and nonlinear effects. To separate linear and nonlinear contributions, frequency spectra of surface elevation variation in time that vary only if nonlinear effects are essential, are presented in Fig. 3. The frequency spectra of Fig. 3 are plotted at the same locations along the tank as in Fig. 2. The spectra are computed for those parts of the surface elevation records that contain the whole group with duration of about 13 s (about $20T_0$). The spectra are thus discreet with the frequency increment of about 0.077 Hz. For demonstration purposes only, the amplitude spectra obtained for the computed temporal variation of the surface elevation that naturally are smoother than the results derived from the experimental data, are drawn as a solid line.

The agreement between experiments and computations in Fig. 3 is quite good. While the initial amplitude spectrum corresponding to (17) also has a symmetric Gaussian shape, the spectra of Fig. 3 are asymmetric and non-Gaussian. Note the existence of a kink in the spectral shape at the frequency slightly exceeding the dominant one, $f_0=1/T_0=1.43$ Hz, that is visible at $x=5.75$ m and becomes stronger at $x=6.85$ m. The kink is observable both in the measured and in the computed spectra.

The manifestation of the nonlinear effects is relatively weak over a limited span of evolution that barely exceeds 4 dominant wave lengths $\lambda_0$. Even for a relatively short extent of the spatial evolution, widening of the spectrum is visible, indicating that nonlinearity is essential in variation of the wave group shape along the tank.

Figure 3. Variation of the frequency spectra along the tank: symbols– experiments, line - simulations
The contribution of the 2nd order bound waves to the amplitude spectrum is quite significant at all locations. The measured spectrum of bound waves around the 2nd harmonic of the dominant frequency $f_0$ using the digital processing of the video images is in excellent agreement with the model predictions. The bound waves’ spectrum also becomes notably wider with the distance from the wavemaker, in agreement with variation of the free wave spectrum around the dominant frequency $f_0$.

As stressed above, the main motivation for developing the data acquisition method based on the processing of sequences of video images is in its capability to measure instantaneous spatial distribution of the surface elevation. Application of this method enables following the temporal evolution of the whole wave group as well. This information can be compared with the numerical solution of the system of equations (7)-(10) that constitute the Dysthe model in its temporal formulation. The initial conditions for the temporal evolution case $A(x, 0)$ are obtained from (17) using the group velocity to relate time and space, $t = x/c_g$.

To compare the numerical and the experimental results, the whole group at the selected instants has to be physically present within the wave tank boundaries; the instants when the spatial envelope obtained experimentally is compared with the numerical results therefore have to be properly selected. Analysis of the numerical solution of (7)-(10) indicates that at the dimensional time $t=12$ s (relative to the initiation of the wavemaker operation) the advancement of the group along the tank is sufficient for the tail of the computed instantaneous spatial envelope distribution to emerge within the tank. With this instant as a reference, the comparison of the numerical and the experimental results is carried out at three additional instants: $t_2 = 14$ s; $t_3 = 16$ s and $t_4 = 18$ s. Equations (15)-(16) are used again to account for the contribution of the 2nd order bound waves.

The spatial variation of the surface elevation as a result of the temporal evolution of the complex wave envelope is presented at the selected instances in Fig. 4. As in the spatial evolution case, good agreement is obtained between the numerical simulations and the experimental observations. At the initial instant, $t=12$ s, the formation of the group has just been completed and the group in its entirety emerges in the tank, while at the last instant presented, $t=18$ s, the front of the group approaches the far end of the wave tank.

Both left-right and trough-crest asymmetries observed in Fig. 2, as well as significant variations in the extreme values of the surface elevation within the group, are visible in Fig. 4 as well. Note, however, that the left-right asymmetry in Fig. 4 is opposite to that of Fig. 2, where the steeper part of the group appears at earlier sampling times.

Comparison of Figs. 2 and 4 also illustrates the well known fact that since the group velocity of deep water waves is a half of their phase velocity, the number of waves within the group in the temporal surface elevation variation records of Fig. 2 is twice larger than in the instantaneous spatial “snapshots” of the group plotted in Fig. 4.

The wave-number amplitude spectra corresponding to Fig. 4 are presented in Fig. 5.

![Figure 4. The instantaneous surface elevation at various instants: experiments; simulations](image-url)
The spectra based on the experimental data and on the numerical simulations were computed for instantaneous surface elevation that cover 12 m of the tank and contain the whole group. The longitudinal extent of the "snapshot" determines the wave number resolution of the resulting discrete spectra. As in Fig. 3, the spectra derived from the numerical solutions of the MNLS model are plotted as solid lines.

The agreement between the simulated and the experimental results is again quite good at all instances presented; the differences can be attributed in part to inaccuracy associated with choosing the initial condition.

There are similarities but also essential differences between the frequency spectra given in Fig. 3, and the wave number spectra of Fig. 5. In both Figures the spectra become wider in the course of the wave group evolution. The wave number spectra in all frames of Fig. 5 are however much wider than the frequency spectra in Fig. 3.

The larger width of the wave number spectra relative to the frequency spectra follows from the dispersion relation for deep water

\[
\frac{\Delta k}{k_0} = 2 \frac{\Delta \omega}{\omega_0}
\]

As a result, in all frequency spectra of temporal variation of the surface elevation for a narrow-band wave group moving along the tank, Fig. 3, the free waves and the bound waves are totally separated. Contrary to that, in the wave number spectra of the spatial variation along the tank of the instantaneous surface elevation the domains of the free and of the 2\textsuperscript{nd} harmonic bound waves in Fig. 5 overlap. Each measured spectrum apparently contains free as well as bound waves. In numerical simulations, complex group envelope that corresponds to free waves only is computed first. Bound waves are then obtained from the free wave field. The computed wave number spectra of free and bound waves are also plotted in Fig. 5.

The overlapping of free and bound waves domains in wave number spectra precludes straightforward filtering out of the free wave spectrum from the experimental results. This difficulty complicates significantly the determination of the spatial group envelope's shape that contains the free-wave part only from the experimental data. The initial conditions for solving the temporal evolution problem could not therefore be determined from experiment, and it was necessary to translate the temporal variation of the surface elevation at the wavemaker given by (17) into the spatial form.

Accounting for the 2\textsuperscript{nd} harmonic bound waves is essential to get a better agreement with the measured spectra at high wave numbers. At low wave numbers, the differences in spectra of Fig. 5 seem to be more pronounced than those at low frequencies in Fig. 3. The disagreements between computations and measurements in the low wave number region of the spectrum may partially stem from the fact that for longer waves, the depth of the experimental
facility of 0.6 m is not large enough for those wave components to be considered deep. The low wave number bound waves may become significant and can constitute a significant contribution to the spectral shape. The effect of long bound waves was considered in the framework of the Zakharov equation in [15]. The validity of Dysthe equation that served as the theoretical model in the present study, however, is restricted to deep waves. The long bound waves were therefore not considered in the current study.

DISCUSSION

Two forms of the MNLS equation are considered here: The first is based on the original formulation of Dysthe [7], equations (7) to (10), that describes evolution in time of a unidirectional narrow-spectrum wave group with prescribed initial spatial distribution of the complex group envelope. Lo and Mei [13] were the first to notice that in order to carry out comparison with experimental data provided by point sensors, a version of the MNLS equation that describes evolution of the wave group envelope in space is required. The spatial version introduced in [13] requires prescribed temporal variation of the complex wave envelope at a given location, usually at the wavemaker, as the initial condition.

The solution of either the spatial or the temporal version of the MNLS equation yields variation of the wave field both in time and in space. The derivation of the spatial MNLS version by Lo and Mei was based on the temporal Dysthe equation and the appropriate change of variables. The two version of the MNLS equation can also be derived from the corresponding temporal [8, 16] or spatial [11] versions of the Zakharov equation.

These derivations shed light on two important facts. First, the evolution equations for complex envelopes of the surface elevation variation and for the velocity potential are somewhat different at the 4th order appropriate for the MNLS model. Since the surface elevation is the parameter measured directly, the surface elevation version of the MNLS equation is used here for carrying out quantitative comparison of model predictions with the experimental results.

The second comment is related to the inclusion of additional linear terms in the temporal version of the MNLS equation [9, 10]. Derivation of the Dysthe model from the Zakharov equation clearly demonstrates that these additional terms appear due to expansion of the interaction coefficient in the Zakharov equation around the carrier wave frequency $\omega_0$ in terms of the wave number deviation from $k_0$. For unidirectional waves in deep water this expansion has an infinite number of terms and therefore has to be truncated. In the spatial evolution case the situation is different and the expansion is of the wave numbers around the frequency $\omega_0$. For the dispersion relation (1), $k=\omega^2/g$, hence, the expansion does not contain terms beyond quadratic. In the spatial version of the unidirectional MNLS the dispersion is thus presented exactly. The spatial evolution equation is therefore more accurate in this sense than the temporal version.

CONCLUSIONS

The experimental approach adopted in the present study that is based on digital processing of synchronized sequences of video images depicting the contact line movement enables to measure both the spatial and the temporal evolution of narrow-banded unidirectional wave groups. The experimental results are compared in detail with the solutions of the appropriate version of the MNLS equation.

The present experimental and numerical study demonstrates that the envelopes of deep-water unidirectional wave groups with narrow spectrum have certain similarities in their evolution pattern in both time and space. Good quantitative and qualitative agreement between measurements and computations is obtained for both the spatial and the temporal evolution formulations. The most visible feature of the evolution process is the gradual transformation of the initially symmetric envelope shape into a strongly asymmetric one. This feature can not be reproduced by the cubic Schrödinger equation in which the initially symmetric envelope can not become asymmetric, and requires the extension to the MNLS equation for its proper description.

Both the spatial and the temporal version of the model describe correctly the widening of the spectrum in the course of evolution. The shapes of the spectra are, however, quite different in these formulations, the wave number spectrum being twice wider than the corresponding frequency spectrum.

ACKNOWLEDGEMENTS

The support of this study by a grant # 964/05 from the Israel Science Foundation is gratefully acknowledged.

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