Applicability of envelope model equations for simulation of narrow-spectrum unidirectional random wave field evolution: Experimental validation

Lev Shemer,1,a Anna Sergeeva,2 and Alexey Slunyaev2
1School of Mechanical Engineering, Tel-Aviv University, Tel-Aviv 69978, Israel
2Institute of Applied Physics, RAS, 46 Ulyanova Str., GSP-120, N. Novgorod 603950, Russia

(Received 9 August 2009; accepted 7 December 2009; published online 8 January 2010)

Combined experimental and numerical study of spatial evolution of unidirectional random water-waves is performed. Numerous realizations of wave fields all having identical initial narrow-banded Gaussian power spectrum but random phases for each harmonic were generated by a wavemaker in a 300 m long wave tank. The measured in the vicinity of the wavemaker temporal variation of the surface elevation was used to determine the initial conditions in the numerical simulations. The cubic Schrödinger equation (CSE) and the modified nonlinear Schrödinger (MNLS) set of equations due to Dysthe were used as the theoretical models. The detailed comparison of the evolution of the wave field along the tank in individual realizations, measured by wave gauges at different distances from the wavemaker and computed using the two theoretical models, was performed. Numerous statistical wave parameters were calculated based on the whole ensemble of realizations. Comparison of the spatial variation of the computed statistical characteristics of the random wave field with laboratory measurements indicates that contrary to the deterministic case, ensemble averaged statistical parameters derived from the CSE simulations compare reasonably well with the experiments and with the results derived from the MNLS model simulations. In particular, simulations based on both models indicate that the statistical characteristics of the random wave field depend on the local width of frequency spectrum and deviate from the Gaussian statistics: the probability of extremely large (the so-called freak) waves is highest when the local spectral width attains maximum. In view of the satisfactory agreement of the numerical results with the laboratory experiments, the computational domain was extended to distances exceeding twice the actual length of the tank and long-scale (at distances approaching 200 dominant wavelengths) variation of the statistical parameters is studied. © 2010 American Institute of Physics. [doi:10.1063/1.3290240]

I. INTRODUCTION

Evolution of random water-wave field constitutes a multifaceted problem that is of considerable interest from the scientific as well as from the practical points of view. A large number of harmonics with various frequencies exchange energy and transfer it by nonlinear interactions; at shorter scales wave energy is dissipated by breaking or otherwise. This phenomenon is sometimes called “wave” or “weak” turbulence, to acknowledge similarity to Kolmogorov energy cascade in fluid turbulence. Hasselmann1 was the first to apply a statistical approach and the kinetic theory to describe ocean wave turbulence. Only exact resonances were taken into account in this early study that resulted in Hasselmann’s kinetic equation. Recently, water wave turbulence theory was advanced considerably by Zakharov and his colleagues (see, e.g., Zakharov2 and references therein). Janssen3 suggested a modification of the kinetic equation in which the exact resonance condition is somewhat relaxed. The kinetic wave theory serves as a basis for modern wave climate prediction. The kinetic theory of random ocean waves is based on two fundamental assumptions: weak nonlinearity of waves and randomness of their phases. The random phase approximation is an essential assumption used for turbulent closures for all stochastic wave systems and even for a much broader range of turbulent systems.

Theoretical investigations aimed at describing the statistical properties of nonlinear wave fields were originated by Longuet Higgins.4 It was shown that for a narrow-banded wave field with random phases wave heights satisfy the Rayleigh distribution. An improved model that takes into account nonlinear effects has been suggested later.5 More recently several more advanced models were proposed in Refs. 6–10, see also additional references therein.

Numerous attempts have been made to explore the possibility to use deterministic nonlinear wave theories for forecasting the evolution of a random wave field. For example, Monte Carlo simulations based on the temporal Zakharov11 equation were used to investigate long-time behavior of nonlinear random wave field.12–15 These studies demonstrated the crucial role of nonresonant interactions in the evolution of nonlinear random water waves, contrary to the kinetic theory assumption. Dysthe et al.16 and Socquet-Juglard et al.17 carried out numerical simulations of the temporal...
evolution of a narrow-banded random wave field using the two-dimensional (2D) version of the MNLS equation. They observed the widening of the wave number spectrum and its evolution to an asymmetric shape. Dysthe et al. concluded that the initially narrow wave number spectrum evolves relatively fast toward a quasi-steady state, while the simulations of Socquet-Juglard et al. show that the second-order Tayfun distribution is a good approximation for the description of wave height probability distributions.

Tanaka argues that nonresonant interactions can be of some importance for one-dimensional waves only, where resonant interactions are not possible. He studied numerically the role of resonant and near-resonant interactions in the temporal evolution of a 2D random wave field consisting of a large number of harmonics by applying the high-order spectral method. JONSWAP spectrum with several versions of angular spreading function was selected as the initial condition. The simulations demonstrate that no significant changes of the wave field statistical parameters were observed after the initial interval in the simulation with duration of $O(1000\nu_0)$, in a qualitative agreement with the kinetic equation predictions.

The essential role of near-resonant interactions is already well documented in experimental and numerical studies of evolution of deterministic unidirectional waves. Comparison of the experimentally measured spatial evolution of narrow-banded nonlinear wave groups with different initial spectral shapes with the corresponding numerical simulations based on the cubic Schrödinger equation (CSE) (Ref. 21) clearly demonstrates that the CSE is only capable to describe correctly some qualitative features of the evolving wave groups, but fails to predict correctly the evolution of the group envelope shape and of the frequency spectrum. Consecutive studies established that more advanced theoretical models, such as Dysthe’s modified nonlinear Schrödinger (MNLS) equation, which is appropriate for narrow-banded waves, or the Zakharov equation which is valid for an arbitrary spectrum width, provide an adequate description of the experimentally observed evolution of the deterministic wave field. To enable accurate quantitative comparison of theoretical predictions with measurements in a wave tank, the spatial (as opposed to the temporal) version of the model equations should be applied. Spatial versions of the Dysthe equation were presented by Lo and Mei and by Kit and Shemer.

It should be stressed that the experimental validation of the accuracy of various nonlinear deterministic wave models has been carried out mostly in relatively short wave tanks, and the justification of application of those models to distances that exceed few tens of dominant wavelengths still remains somewhat questionable. These spatial extensions, however, are pertinent to the characteristic evolution scales of a random waves’ field. It is thus essential to confirm experimentally that the deterministic wave models indeed can be applied for computing wave field behavior over large distances.

Some experiments in a long wave tank have recently been performed on deep narrow-banded waves with random phases (Mori et al. and references therein). Results of these experiments indicate that in spite of lack of exact resonances in a unidirectional wave field, nonlinear effects are indeed essential and affect strongly the statistical properties of the wave field.

Recently, Shemer and Sergeeva reported on an experimental study of the evolution of narrow-banded nonlinear random waves in a long tank. Unidirectional random wavemaker-generated waves with initial narrow-banded Gaussian spectrum were recorded at numerous locations along the 300 m long large wave channel (GWK) in Hanover, Germany. Spatial evolution of numerous statistical wave field parameters was studied. The present study continues that of Ref. 30 in two directions. First, we take advantage of the accumulated experimental results to examine the accuracy of simulating the spatial evolution of a random narrow-banded unidirectional wave field by nonlinear envelope models over large scales unique to the GWK. The experimentally determined temporal variation of the surface elevation in the vicinity of the wavemaker in each random-phased realization of the given spectrum was used to derive the corresponding initial condition for numerical simulations. In addition, even in such a big facility as in Hanover, the relatively slow evolution process and the limited size of the tank can often make it difficult to arrive at decisive conclusions regarding the long-term evolution patterns of random wave fields. Carrying out carefully designed numerical simulations based on the proven accuracy of the theoretical models over large distances is therefore essential. The second goal of the study is thus extension of the simulations beyond the actual length of the wave tank to estimate the evolution of the random narrow-spectrum nonlinear wave field over distances larger that the GWK length.

II. EXPERIMENTAL FACILITY AND PROCEDURE

The experiments were carried out in the large wave channel (GWK) in Hanover, which is about 300 m long, 5.0 m wide, and 7.0 m deep (the water depth in the present experiments was set to be $h=5$ m). A sand beach with a slope of $30^\circ$ is located at the far end of the facility, starting at $x=270$ m. The computer-controlled piston-type wavemaker is equipped with a system to absorb the energy of reflected waves.

Spatial evolution of numerous realizations of a wave field all having identical initial free wave frequency power spectra with random phases is studied. The selected spectrum corresponds to that of a deterministic Gaussian-shaped unidirectional wave group with the surface elevation variation in time given by

$$\eta(t) = \eta_0 \exp(-t/mT_o^2)\cos(\omega_0 t),$$  \hspace{1cm} (1)

where $\omega_0=2\pi f_0=2\pi/T_0$ is the carrier wave circular frequency, $\eta_0$ is the maximum wave amplitude in the group, and the parameter $m$ defines the width of the group. The wave number $k$ is related to $\omega$ by

$$\omega^2 = kg \tanh(kh).$$  \hspace{1cm} (2)

The spectrum of Eq. (1) also has a Gaussian shape with the relative width at the energy level of half of the maximum,
\[ \Delta \omega / \omega_0 = \frac{1}{m \pi} \sqrt{\frac{\ln 2}{2}}. \]  

The value of the spectral width parameter in the present experiments was selected to be \( m = 3.5 \), yielding \( \Delta \omega / \omega_0 = 0.054 \). The total duration of a single group given by Eq. (1) is 51.2 s (2048 data points), thus the frequency resolution of the spectrum \( \Delta \Omega \) is better than 0.02 Hz.

All experiments were carried out for the carrier wave period \( T_0 = 1.5 \) s, corresponding to the wavelength \( \lambda_0 = 3.51 \) m and the group velocity \( c_g = 1.17 \) m/s. The dimensionless water depth \( H_0 = 8.95 \) corresponds to the depth of 214 m from the wavemaker. The distance of 214 m from the wavemaker.

The total number of experimental runs is 46. For additional experimental details, see the reference by [Ref. 30].

The instantaneous water height is measured using 25 resistance type wave gauges distributed along the tank and attached to the tank wall. The output voltages of all wave gauges, as well as the wavemaker driving signal and the output of the wavemaker position potentiometer that provides information on the instantaneous wavemaker displacement were sampled at 40 Hz for the total sampling duration of 400 s, sufficient for the wave field excited by the wavemaker to propagate beyond the most distant gauge at the distance of 214 m from the wavemaker.

The maximum possible for a deterministic wave group (1) wave steepness defined as \( \varepsilon = \eta \beta \) is adopted as the measure of nonlinearity of the wave field. The experiments were carried out for \( \varepsilon = 0.25 \), for which no breaking was observed. The total number of experimental runs is 46. For additional details about the experimental facility and procedure see Ref. 30.

### III. THEORETICAL MODELS AND NUMERICAL SCHEMES

Evolution of the wave group in a wave flume can be represented by variation in time and space of the surface elevation \( \eta(x,t) \) of the velocity potential \( \varphi \) at the free surface, \( \varphi(x,t) = \varphi(x,z = \eta,t) \). For a narrow-banded wave group consisting of traveling waves it is convenient to express the variation of \( \eta \) and \( \varphi \) at the leading order in terms of their complex envelope amplitudes:

\[
\eta(x,t) = \text{Re}[a_g(x,t) \exp(i \omega_0 t - k_0 x)],
\]

\[
\varphi(x,t) = \text{Re}[-i (\varphi'/\omega_0) a_g(x,t) \exp(i \omega_0 t - k_0 x)].
\]

Variation of the complex envelopes along the tank can be described by either CSE, or by the Dysthe\textsuperscript{18} MNLS coupled system of equations, which is appropriate for the deep water conditions. The amplitudes \( a_g \) and \( a_\varphi \) are identical within the framework of the CSE; for the higher-order MNLS model, those amplitudes differ somewhat.\textsuperscript{31,27} While the CSE was shown to perform poorly when used to describe the evolution of deterministic wave groups (see, e.g., Refs. 21 and 23), it is still widely used as a theoretical model for the analysis of narrow spectrum random waves. For example, the CSE serves as a basis for Alber’s\textsuperscript{32} equation and the subsequent studies based on this equation. While the Dysthe MNLS equations were selected as the main theoretical model in this study, the limits of validity of the CSE to describe random wave field evolution were also examined. For quantitative comparison of the model predictions with the experiment that directly provides data on the surface elevation variation, the version of the equation describing the variation of \( a_g \) is applied, with the index “\( \eta \)” omitted in sequel.

In the fixed coordinate system, the governing equations are

\[
i \left( \frac{2 k_0}{\omega_0} a_g + \frac{k_0}{\omega_0} a_{g\varphi} + \frac{k_0}{\omega_0} |a|^2 a - 8 i \frac{k_0^3}{\omega_0} |a|^2 a_t - 2 i \frac{k_0^3}{\omega_0} a^2 a_t + \frac{4 k_0^3}{\omega_0^3} \frac{\partial \Phi}{\partial t} \right)_{z = 0} = 0, \quad (5)
\]

\[
\Phi_{zz} + \frac{4 k_0^2}{\omega_0^2} \Phi_{tt} = 0, \quad -h \leq z \leq 0, \quad \Phi_z = 0, \quad z = -h. \quad (6)
\]

In Eq. (5) \( \Phi(x,z,t) \) is the first approximation for the induced velocity potential. The group velocity, \( c_g \), is defined by \( c_g = \omega_0 / 2 k_0 \). Note that the four first terms of Eq. (5) constitute the CSE, while the remaining terms are of the higher (fourth) order in the nonlinearity parameter and stem from the finite spectral width.\textsuperscript{31,27} Formulation of the spatial model given by Eqs. (5)-(7) is completed by specifying the temporal variation of the envelope at the prescribed location \( a(x_0,t) \).

The complex amplitude function, \( a(x,t) \), and the surface elevation data, \( \eta(x,t) \), are linked by virtue of the following relation:

\[
\eta = \text{Re} \left[ a \exp(i \omega_0 t - k_0 x) + \frac{k_0}{\omega_0} \frac{a}{2} - \frac{i k_0}{\omega_0} \frac{a_t}{2} \right] \times \exp(2i \omega_0 t - 2ik_0 x) + \frac{3k_0^2}{8} a^3 \exp(3i \omega_0 t - 3ik_0 x). \quad (8)
\]

The first term in Eq. (8) coincides with Eq. (4a), and constitutes the contribution of free waves; other terms in Eq. (8) are the contributions of the second and the third bound wave harmonics.

The time series \( \eta(x_0,t) \) measured by the first probe at \( x_0 = 3.59 \) m was used to define the initial condition \( a(x_0,t) \). To this end, the free-wave frequency domain was manually selected from the spectra of laboratory records, and the amplitudes of all harmonics beyond this domain were zeroed. The free-wave components are easily distinguished in the present experiments since the spectrum is narrow banded. In
each realization, the surface elevation corresponding to the wave field with bound waves filtered out is computed using the inverse Fourier transform, and the corresponding envelope \( a(x_j, t) \) is then easily obtained by Hilbert transform. The whole 400 s long records were used to compute the initial conditions in the simulations. These long records result in superfluous spectral resolution, so that the wave harmonics with significant energy occupied only a small part of the spectral domain. The resolution of every time record was therefore reduced, and in numerical simulations 4096 grid points were used in every 400 s long record. It is important to stress that long waves were present in certain time series in the experiments.\(^{30}\) To eliminate long waves from the initial conditions, the high-pass filter was applied to the recorded time series. Note also that long bound waves are not contained in Eq. (8).

Dysthe’s MNLS equations (5)–(7) are solved using the split-step Fourier method as described by Lo and Mei,\(^{20}\) with minor corrections and adjustment to the surface elevation amplitude version of the model. Results of two numerical codes, developed independently in our two institutions,\(^{23,33}\) were compared and showed very close agreement. The spatial integration step of 1 cm proved to be sufficiently small to ensure absence of numerical instabilities and energy conservation up to the fetch of 600 m.

The simulated wave fields represented by complex envelopes of free waves \( a(x_j, t) \) were stored at distances corresponding to the wave gauge positions in the experiment. Beyond the location of the most distant wave gauge at \( x=214 \) m, the simulated time series were stored for every 5 m up to the fetch of 600 m. The corresponding surface elevations obtained from Eq. (8) are compared with the laboratory measurements at the same distances and are used for statistical processing, as discussed in Sec. IV.

**IV. RESULTS**

The accuracy of the envelope model equations for prediction of the spatial evolution of individual wave groups is examined first. As mentioned above, the CSE has been shown to be inadequate to describe evolution of narrow-banded nonlinear deterministic wave groups with symmetric envelopes even for relatively short propagation distances.\(^{20,21,23,28}\) In Fig. 1 the accuracy of simulations based on the CSE, including the bound wave contribution computed according to Eq. (8), is examined on an example of a single realization in the GWK experiments. Due to random phases of free harmonics, the group envelope in each realization has an irregular shape. The measured and the computed surface elevations at two distances \( x \) from the wavemaker are presented. The results at \( x=52.2 \) m show that already at a relatively short distance corresponding to about 15 carrier wavelengths \( \lambda_0 \), notable differences appear between computations and measurements. At a larger distance corresponding to about \( 33\lambda_0 \), any quantitative agreement between the simulated and the measured wave fields practically ceases to exist. Nevertheless, some qualitative features of the group shapes, such as focusing of wave energy visible in the first group at \( x=116 \) m, as well the distribution of waves within individual groups, are retained to some extent in the CSE simulations. At even larger distances, any resemblance between the computations and the measurements fades away.

Figure 2 convincingly demonstrates that the performance of the MNLS model is far superior to that of the CSE. The measured and the computed temporal variation of the surface elevation is presented at three distances \( x \) from the wavemaker, all exceeding the values of \( x \) in Fig. 1. Excellent qualitative agreement between the simulations and the experiments is obtained for the whole extent of the measurements domain. The results at \( x=120 \) m (\( \approx 35\lambda_0 \)) exhibit good quantitative agreement between computations and measurements, although some deviations are visible. The quantitative agreement seems to deteriorate somewhat with the distance.

The rms value of the difference between the model predictions, \( \eta_{\text{mod}}(x, t) \) and the experimental results, \( \eta_{\text{exp}}(x, t) \) at each measuring station is characterized by
\[ \Delta = \left( \frac{\eta_{\text{mod}}(x, t) - \eta_{\text{exp}}(x, t)}{H_s} \right)^2, \tag{9} \]

where \( H_s \) is a significant wave height that is approximately conserved along the tank.\(^{30}\) The averaging in Eq. (9) is performed over the whole duration of the wave group passage at the given distance from the wavemaker, \( x \). The value of \( \Delta \) can be used as the quantitative parameter that characterizes the accuracy of the simulations. The increase in the relative error \( \Delta \) with \( x \) is close to linear, see Fig. 3; the values of \( \Delta \) remain within about 0.1 even at largest distances from the wavemaker employed in the experiments. Such a behavior suggests that the MNLS model provides a quantitatively accurate description of the spatial evolution of the deterministic nonlinear wave field up to distances of the order of \( O(10^3 \lambda_0) \). The accumulation of the error with \( x \) in Fig. 3 may be attributed mainly to inaccuracy in the initial conditions used in the model simulations. One can identify at least two sources of such inaccuracy. First, there is a certain error in the wave gauges calibration due to the large size of the experimental facility, as discussed in Ref. 24. Beyond that, the sampled wave gauge output also contains some noise. Second, the initial conditions are based on measurements performed at \( x = 3.59 \text{ m} \), about one carrier wavelength \( \lambda_0 \) from the wavemaker. At this distance, the wave field is still contaminated by the evanescent modes resulting from the wavemaker shape. These standing waves decay fully only at distances exceeding about two wave depths \( h \) (see, e.g., Ref. 34), i.e., for the GWK conditions, at \( x > 10 \text{ m} \). Unfortunately, for technical reasons no surface elevation measurements were possible at those distances. As a result, there is some mismatch between the initial free wave field in the tank at \( x = 3.59 \text{ m} \), and the adopted in the simulations initial shape of the free wave envelope.

The results presented in Figs. 1–3 thus indicate that evolution of an individual realization of a random wave field can be described adequately over large distances by the MNLS model, whereas the CSE is at best appropriate for revealing some qualitative properties of the evolution process. The relevance of those nonlinear envelope evolution equations to prediction of the statistical properties of the unidirectional random wave field is now considered.

The variation along the tank of the power spectrum of the surface elevation, which is one of the most important statistical parameters describing random wave field, is presented in Fig. 4. The experimental results are compared in this figure with simulations based on both CSE and MNLS models. The initial spectrum, as measured at \( x = 3.59 \text{ m} \), is also plotted for comparison. The free wave part of this initial spectrum, which is based on the initial group shape given by Eq. (1), is also symmetric relative to the carrier wave frequency and has a shape close to Gaussian, as expected. Since the initial spectrum is quite narrow, the frequency domains of the second and the third order bound waves are clearly separated.

The initial spectral shape undergoes fast variations and already at \( x = 52.2 \text{ m} \) (\( x = 15\lambda_0 \)) the free wave spectrum changes notably. The spectrum becomes wider and develops visible asymmetry. The MNLS model faithfully describes both these effects, and the agreement between the computed and the measured surface elevation spectrum in the free wave frequency domain remains good at all distances from the wavemaker. In simulations based on the CSE the widening of the free waves spectrum is obtained as well. However, the CSE is incapable to reflect the asymmetry of the developing spectral shape. It is well known (see, e.g., Ref. 23) that for the initially symmetric shape of the group envelope, this symmetry is retained by the CSE in the process of evolution. In the present work, while the generating group as given by Eq. (1) is symmetric in time, this symmetry of the envelope is lost once random phases are prescribed to the various harmonics. The power spectrum of Eq. (1), however, remains symmetric relative to the carrier wave frequency. It can be easily demonstrated that within the framework of the CSE, the initially symmetric free wave spectrum also retains its symmetry in the process of evolution. Since the experiments clearly show that the frequency spectrum loses its symmetry quite fast, the agreement of the spectra computed using the CSE with the experiments is imperfect, and at all distances from the wavemaker the free wave frequency spectra computed from the CSE are significantly wider than either those
derived from the MNLS simulations or the measured ones. The width of the free waves’ spectrum seems to attain maximum at distances about 100 m, and then decreases somewhat.

The agreement between the second order bound waves as measured in the experiment and as computed using the Dysthe MNLS model is reasonable at all distances from the wavemaker. Those waves have amplitudes that are smaller by an order of magnitude as compared to the free waves; their amplitudes are thus comparable with the sensitivity of the wave gauges. The simulations based on both CSE and MNLS models agree quite well with the measurements around the peak of the second order bound waves (at frequencies close to 1.3 Hz); away from the peak the MNLS simulations yield a much better agreement with the experiments than the CSE. The dip in the spectrum between the domains of free and second order bound waves is much deeper in simulations than in experiments. This can be attributed to the limited accuracy of the wave gauges. For the same reason, the agreement between simulations and experiment at frequencies corresponding to the third order bound waves (around 2 Hz) is only of qualitative nature.

The results of Figs. 1–4 clearly demonstrate good quantitative and qualitative agreement of the MNLS model simulations results with the experiments for all distances from the wavemaker covered in the GWK experiments, with deviations that can be plausibly attributed to experimental inaccuracies. Such an agreement suggests that the simulations can be extended to distances beyond the domain of the experiments. Carrying out simulations of evolution of a random wave field over large distances can hopefully shed some light on the evolution pattern at longer scales. In sequel, the results of simulations are presented for distances exceeding the actual length of the GWK by a factor of 2, whereas the experimental results apparently are only shown where available.

As can be seen from the spectra of Fig. 4, the free wave frequency domain \( \omega_{\text{min}} \leq \omega \leq \omega_{\text{max}} \) in the present study is well defined, with \( \omega_{\text{min}} \) and \( \omega_{\text{max}} \) representing the first minima in the spectrum \( S(f) \) at both sides of the peak at the carrier wave frequency. It is customary to define the free wave spectral width as

\[
\nu = \sqrt{\frac{m_0 m_2}{m_1^2}} - 1, \tag{10}
\]

where the \( j \)th spectral moment is defined as

\[
m_j = \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} \omega^j S(\omega) d\omega. \tag{11}
\]

The computed and the derived from the experiments variations with \( x \) of the spectral width of the free wave part of the spectrum, \( \nu \), are plotted in Fig. 5. The computed values of \( \nu \) for both model equations are in a reasonable agreement with the experimental results. The results obtained using the MNLS model generally yield underestimated spectral width in comparison with the measurements. Simulations based on the CSE show the same tendency, except for the distances closer to the wavemaker (\( x < 100 \text{ m} \)), where the computational results overestimate the measured spectral width. Both CSE and Dysthe models correctly predict the experimentally observed spectral widening, with \( \nu \) increasing nearly twice from its initial value \( \nu = 0.45 \) during the first 100 m of the evolution. The spectral width then decreases somewhat. The values of \( \nu \) obtained from the CSE simulations agree somewhat better with the experiments in comparison to the Dysthe model. One cannot claim, however, that the CSE is superior to the Dysthe model in description of the spectral evolution, since the CSE is incapable to reflect the varying asymmetry of the spectral shape along the tank. The simulations farther away from the wavemaker exhibit irregular, relatively weak oscillations of the spectral width (around the mean value of \( \nu \approx 0.065 \) for the Dysthe model and of \( \nu \approx 0.075 \) for the CSE model); the characteristic length scale of these oscillations exceeds 100 m, or about 30\( \lambda_0 \).

The higher statistical moments are usually presented by the skewness and the kurtosis coefficients of the surface elevation, defined, respectively, as

\[
\lambda_3 = \frac{\langle \eta^3 \rangle}{\sigma^3}; \quad \lambda_4 = \frac{\langle \eta^4 \rangle}{\sigma^4}, \tag{12}
\]

where \( \sigma \) is the rms surface elevation, defined at each location along the tank by the time-averaged value

\[
\sigma^2 = \langle \eta^2 \rangle \tag{13}
\]

and is a characteristic of wave amplitudes. For a Gaussian wave field, the skewness \( \lambda_3 = 0 \), while the kurtosis \( \lambda_4 = 3 \).

The vertical asymmetry of the wave field is characterized by \( \lambda_3 \). The variation of the skewness along the tank, as measured in the experiment, is compared in Fig. 6 with the results of simulations within both MNLS and the CSE models. Both models yield values of \( \lambda_3 \) that are quite close to the
measurements. The dependence of $\lambda_3$ on the distance $x$ in Fig. 6 bears some similarity to the behavior of $\nu(x)$ in Fig. 5. The skewness coefficient increases quite sharply initially, up to about $x = 100$ m, and then decreases somewhat, exhibiting relatively weak variability at length scales similar to those of Fig. 5. The values of $\lambda_3$ were also calculated for the surface elevation records that were band-pass filtered for the free waves' domain, $\omega_{\text{min}} < \omega < \omega_{\text{max}}$. The variation of the skewness coefficient $\lambda_3$ that represent free waves only is also plotted in Fig. 6. The values of $\lambda_3$ due to free waves are virtually zero at all locations, both in the experiments and in the simulations based on the Dysthe model (similar results obtained within CSE model are not presented in Fig. 6). The results of Fig. 6 thus demonstrate that the skewness, determined nearly solely by the contribution of the bound waves, is adequately described by both envelope equations.

It is well known that the kurtosis coefficient $\lambda_4$ can be seen as a measure of deviation of the random wave field from Gaussianity. Variation of $\lambda_4$ along the tank is presented in Fig. 7. Again, the Dysthe model yields good agreement with the experiments, while the CSE significantly overestimates the maximum values of the kurtosis in the transitional domain at about 50 m $< x < 120$ m. At larger distances the predictions based on both those models seem to be closer, and the computed values of $\lambda_4$ vary somewhat (at the characteristic length scale of about 30 carrier wavelengths, similar to other statistical parameters), but remain in the vicinity of $\lambda_4 = 4$, indicating that the wave field departs significantly from the Gaussian distribution.

Recently, Annenkov and Shrira$^{35}$ considered kurtosis coefficient $\lambda_4$ based on free waves only, which they denote as “dynamic” following Janssen$^{36}$, see also Ref. 15. They stress that the dynamic part of kurtosis is nonlocal, i.e., it depends on the history of the spectral evolution. Contrary to that, contribution to the kurtosis due to bound waves, $\lambda_4^b$, depends exclusively on the local spectral shape. The total kurtosis coefficient $\lambda_4 = \lambda_4^d + \lambda_4^b$. Annenkov and Shrira$^{35}$ considered initially narrow-banded one-dimensional spectrum and performed 2D numerical simulations of the temporal evolution of a random wave field based on the Zakharov$^{11}$ equation. They demonstrated that kurtosis at the initial stage of evolution is positive and quite large, attaining values exceeding 3.9 at about 30 carrier wave periods $T_0$, and then decaying, in agreement with the present results. Annenkov and Shrira argue that such a behavior of the total kurtosis is mainly governed by its dynamic part, $\lambda_4^d$, while the bound waves' part, $\lambda_4^b$, decays monotonically. In their computations the dynamic part of the kurtosis falls below the Gaussian value of 3 at $t > 60 T_0$, while the total $\lambda_4$ approaches 3.1 thus indicating minor deviation from the Gaussianity. The present results at large distances from the wavemaker differ somewhat from those of Ref. 35; one possible reason for this being the total lack of 2D effects in our simulations.

Wave height exceedance distributions as measured in experiments and computed from MNLS and CSE simulations for different locations along the tank are presented in Fig. 8. The exceedance distributions in Fig. 8 are normalized by the standard deviation $\sigma$ of the surface elevation variation. The Rayleigh distribution for the scaled wave height is plotted as well. The comparison of experimentally determined probabilities with the second-order and the third-order nonlinear distributions suggested by Tayfun and Fedele$^{10}$ has been carried out in Shemer and Sergeeva.$^{30}$ The wave height exceedance distributions derived from both MNLS and CSE models agree well with the measured probabilities. In simulations based on both envelope models, as well as in the experimental results, the Rayleigh distribution overestimates the exceedance probability for $H < 4\sigma$ and underestimates it for higher values of wave heights. Moreover, the models are capable of providing adequate prediction of probability of appearance of waves with heights below $8\sigma$. Even higher waves are usually dubbed freak, or rogue waves. Both the simulation results and the experiment clearly indicate that the probability of those extremely high waves exceeds that corresponding to the Rayleigh distribution by an order of magnitude. Still, since the probability of such high waves is quite low, there is some spread in probability values for $H > 8\sigma$ evident in Fig. 8, both for the distributions based on the laboratory data and those obtained from the numerical simulations. This spread seems to be a result of insufficient size of the ensemble.

V. DISCUSSION AND CONCLUSIONS

In the present study, two deterministic envelope equations, the CSE and the Dysthe (MNLS) equation, were applied to carry out Monte Carlo simulations of random nonlinear wave trains. To assess the accuracy and the domain of
validity of those model equations, detailed comparison is carried out between the computations and the experimental results in a large wave tank. The initial variation of the surface elevation with time measured for each realization in the experiments reported in Ref. 30 was used to determine the initial conditions in the numerical simulations.

At the first stage of this study detailed comparison on a case-by-case basis was performed between the experimental results and numerical simulations based on the CSE and on the Dysthe models. These simulations clearly demonstrate that the Dysthe computations provide reasonable quantitative agreement between the computed individual wave group evolution along the tank and the experimental results for all distances from the wavemaker at which measurements were performed (more than 60 dominant wavelengths). The CSE, on the other hand, is incapable of reflecting the actual evolution of individual wave groups; the computed and the measured results diverge fast with the distance from the wavemaker. These results are in agreement with similar quantitative comparisons carried out in earlier studies of deterministic nonlinear wave groups.20,21,23,28

Next, the performance of both models in predicting the evolution of the statistical parameters of a random wave field was examined. In particular, the variation along the tank of the shape of the surface elevation spectrum and of its width, as well as of the higher moments of the surface elevation such as skewness and kurtosis coefficients, was investigated.

Good agreement between the MNLS Dysthe model predictions and measurements was expected on the basis of the very good performance of the Dysthe equation in describing evolution of individual nonlinear waves groups (see Figs. 2 and 3). Comparison of the statistical parameters of a unidirectional random wave field computed using the MNLS equation with the experimental results, presented in Figs. 4–8, indeed demonstrates excellent agreement between calculations and measurements for all statistical quantities.

In view of the poor agreement between CSE-based computations and measurements of evolution of individual nonlinear wave field realizations, it is somewhat surprising that the CSE seems to be capable to reflect adequately the basic qualitative features of the evolution of the statistical parameters of the wave field along the tank, even though the quantitative agreement between the computational results and the experiments for some parameters remains inferior to that obtained using the MNLS model. It is well known that the CSE is deficient as compared with the more advanced models in accounting for the dispersion of deep-water waves. Of additional importance to the present study is the lack of capability of CSE to change the symmetry of the initial surface elevation spectrum. While the initially symmetric with respect to the carrier frequency spectrum was employed in this investigation, both experiments and the Dysthe model show that the spectral symmetry is lost in the process of evolution. This inability of the CSE to reflect qualitatively correctly the emerging in the evolutions process lack of spectral symmetry contributes significantly to the quantitative disagreement between the numerical and experimental results.

Contrary to CSE, in the spatial version of the Dysthe equation the dispersion relation is accounted for exactly; it describes four-wave nonlinear interactions among free wave harmonics that satisfy the conditions of near resonance.27 The results on the spectra variation presented in Figs. 4 and 5 demonstrate that the initially narrow spectrum undergoes widening during the initial stages of the evolution, but then becomes quite narrow again farther away from the wavemaker. These results indicate that narrow spectrum assumption is satisfied throughout the whole evolution process, and the spatial MNLS equation thus remains applicable for prediction of evolution of nonlinear wave groups over large distances. Annenkov and Shrira35 in their 2D Monte Carlo simulations of a random wave field based on the Zakharov equation observed that in the initially unidirectional random wave field the spectrum remains one-dimensional for the first few hundred carrier wave periods. This observation provides additional justification for extending the application of the Dysthe equation for larger distances.

Since the computations based on the Dysthe model yielded good qualitative and quantitative agreement with the experiments for the whole length of the measurements domain, while the CSE simulations provided reasonable agreement of qualitative nature, it was decided to extend the numerical computations to distances well beyond the physical dimensions of the tank. In these simulations an attempt was made to shed some light on the long-range evolution pattern of a unidirectional random wave field.

As stressed above, random waves' propagation poses questions of fundamental scientific importance. It still remains unclear whether the spectrum and other statistical parameters of a random water-wave field with a given initial spectrum eventually reach a steady state, as suggested by computations,17,19,35 or the characteristic parameters of the field undergo periodic, or quasiperiodic, modulation at a slow scale.36,37

The present results do not provide a clear-cut answer to this question. Important variations of the statistical parameters of the wave field occur initially at the length scales of the order of \( \varepsilon^{-2} \lambda_0 \), as can be expected from the scaling of both MNLS and CSE models, see Eq. (5). This length scale is much shorter than that predicted by the kinetic equation. The discrepancy can be in part attributed to the lack of exacts resonances in a unidirectional wave field. While certain variability of the statistical parameters is observed also at larger distances (again, at a characteristic scale of \( \varepsilon^{-2} \lambda_0 \)), it remains unclear whether those variations can be indeed seen as quasiperiodic oscillations, or rather they exhibit slowly decaying variability of each one of those parameters, which gradually approach some steady value.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the European Community support under the Access to Research Infrastructures Action of the Human Potential Programme (Contract No. HPRI-CT-2001-00157) that made possible experiments in the Large Wave Channel (GWK) of the Coastal Research Center (FZK) in Hanover. The research is supported by Grant Nos. 964/05 and 1194/07 from the Israeli Science Foundation (LS) and for the Russian partners by Grant Nos.
06-05-72011, 08-02-00039, and 08-05-00069 of the Russian Foundation for Basic Research, by the Russian State Programme “World Ocean,” and by the European Community’s Seventh Framework Programme No. FP7-SST-2008-RTD-1 under Grant No. 234175.


