Experimental and Numerical Study of Long-time Evolution of Standing Waves in a Rectangular Tank

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Summary
Standing waves with their crests parallel to the side walls of a rectangular tank, were generated by a segmented wavemaker. The long-time evolution of these waves was investigated experimentally, and the results of the measurements were compared to the numerical solutions of the appropriate nonlinear Schrödinger equation. An agreement between theory and experiments was obtained when dissipation was taken into account in the model equations.

I. Introduction
We study resonant waves which appear in a rectangular tank, when excited by a wavemaker having a typical instantaneous length scale \( \lambda_n \approx 2b/n \), where \( b \) is the width of the tank and \( n \) is an integer representing the mode number. The frequency of forcing should be close to the linear cut-off value given by \( \omega_n^2 \approx 2g\pi/\lambda_n \tanh(2\pi h/\lambda_n) \), \( h \) being the mean water depth in the tank. These standing waves have their crests parallel to the tank side walls and are usually referred to as "sloshing waves" (Barnard et al. [1]). Two different steady sloshing wave regimes which may exist in the tank under identical flow conditions were observed in [1]. Aranha et al. [2] presented a derivation of a nonlinear Schrödinger (NLS) equation for the propagation of an acoustic wave in a duct with a wavemaker-like forcing. Their numerical results indicated that for a case which is analogous to deep water in a channel, no steady solution can be obtained. This contradiction between the existing steady experimental results and the numerical predictions triggered our interest in the problem.

In a recent investigation [3] we have presented some experimental results on sloshing waves in the vicinity of the 2nd cut-off frequency, \( \omega_2 \). These results were compared to the numerical solutions of the NLS equation which was derived for our experimental conditions. This comparison has supported the conclusion in [1] that dissipation has to be incorporated into the model equations in order to obtain qualitative agreement between the experiment and the numerical results. In the present work we investigate in

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greater detail the long time modulation of sloshing waves and draw some conclusions about the validity and the limitations of the theoretical model.

II. Experimental facility and procedure

The experiments were carried out in a rectangular tank which is 18m long, 1.2m wide and filled to a mean water depth of 0.6m. Waves were generated by a 4-segment paddle-type wavemaker. As it was demonstrated by [1] and [3], the stability of the forcing frequency is of crucial importance for the experiment, since the most interesting nonlinear phenomena are observed in the narrow range of frequencies in the vicinity of the cut-off value $\omega_c$. The wavemaker was therefore operated using the PDP 11/23 minicomputer analog output, which simulated the sine-wave generator and used 1MHz clock of the computer. The relative accuracy of tuning was as good as $3.6 \times 10^{-4}$. The sloshing waves of the 2nd mode were obtained by operating all wavemaker segments with identical amplitude and with 180° phase shift between the two inner and two outer segments. The instantaneous surface elevation was measured by 4 conductance-type wave gauges, which could be moved along the tank. This information, together with outputs of position potentiometers representing the instantaneous inclination angle of each segment of the wavemaker, and the forcing signal which served as a reference, were sampled by an A/D converter of the same minicomputer and recorded on magnetic tape for further processing. The general view of the experimental facility and the wave gauges is given in Fig. 1(a). Additional information about the experimental procedure was presented in [3] and [4].

III. Model equations

The derivation of the governing equations is based on full inviscous nonlinear boundary conditions at the free surface and the Laplace equation (see [3]). The equations are expanded to the 3rd order in a small parameter $\epsilon$ based on the dimensional stroke of the wavemaker $s/b$ at the mean water level. The calculations based on the actual geometry of our wavemaker when operated at the 2nd sloshing wave mode yield

$$\epsilon = 0.022s/b$$  \hspace{1cm} (1)

The scaled slow length variable along the tank $X$ and the slow time variable $\tau$ are defined for the case of deep water, which approximately holds for our experimental conditions ($\tanh 2\pi h/\lambda = 0.996$), as

$$X = (2\pi)^{1/2} \epsilon^{1/2} x/b; \quad \tau = 2\pi^3 \epsilon \omega_c t$$  \hspace{1cm} (2)

The velocity potential $\phi$ for the sloshing waves of the 2nd mode is given by
\[ \phi = \sqrt{ebg} \cos(2\pi y/\lambda) \frac{\cosh(2\pi z/h)}{\cosh(2\pi \rho/\lambda)} \left[ C(x,t) e^{-i\omega t} + c.c. \right] \]  

(3)

with the bottom of the tank at \( z=-h \). The dimensionless complex amplitude of potential \( C \) satisfies the following nonlinear governing equation:

\[ i \frac{3C}{3t} + \frac{3C}{3x} + \lambda C + 2|C|^2C = 0 \]  

(4)

where the detuning parameter \( \lambda \) represents the ratio of the deviation of the forcing frequency \( \Omega \) from the linear cut-off value \( \omega_2 \) to the amplitude of forcing \( \epsilon \) and is given by

\[ \lambda = \frac{4\omega_2(\Omega - \omega_2)}{(2\pi)^2 \epsilon} \frac{b}{g} \]  

(5)

Equation (4) is subject to the boundary condition at the wavemaker

\[ \frac{3C}{3x}|_{x=0} = -1 \]  

(6)

Following [3], we now take into account the dissipation. One can distinguish between the dissipation at the wavemaker, which leads to the modification of the boundary condition (6), and the dissipation at the tank walls which results in an additional term in the governing equation (4):

\[ i \frac{3C}{3t} + \frac{3C}{3x} + \lambda C + a_1(x-1)C + 2|C|^2C = 0 \]  

(4a)

The boundary condition (6) is replaced by

\[ \frac{3C}{3x}|_{x=0} = -1 - a_2(x-1)C \]  

(6a)

In the present work we have chosen the following values of the dissipation coefficients: \( a_1=0.03 \) and \( a_2=0.7 \). The criterion for the choice of the dissipation coefficients will be discussed further. The equation (4a) with zero initial condition and the boundary condition (6a) was solved numerically. The numerical scheme was identical to that used by [3].

**IV. Experimental and numerical results**

In [3], measurements were performed at the values of \( \epsilon \) ranging from \( 0.42 \times 10^{-7} \) to \( 0.9 \times 10^{-7} \). In the present investigation, stronger forcing was used, and the amplitude of the wavemaker corresponded to \( \epsilon=1.29 \times 10^{-4} \) and \( \epsilon=1.56 \times 10^{-7} \). For all amplitudes of forcing, steady wave pattern was observed for low values of the detuning parameter \( \lambda \). In this regime, the wave field is basically decaying, and the surface elevation differs notably from zero only in the vicinity of the wavemaker. The decay is not necessarily monotonous, so that the maximum in the wave amplitude may be detached from the wavemaker. This mode may be considered as a trapped soliton, as discussed in [5]. When the frequency of forcing (and thus \( \lambda \)) is gradually increased, a sharp transition in the wave pattern occurs. The wave field becomes unsteady, and the wave amplitudes and the phases undergo a dramatic change.
At \( \lambda \) which exceeds slightly the transitional value, solitons are generated quasi-periodically at the wavemaker and then propagate away with a seemingly constant speed. Such propagating soliton can be seen in Fig. 1(a). The "snapshot" of the numerical solution of equation (4b) with the boundary condition (6b) is presented in Fig. 1(b).

When the frequency of forcing is gradually decreased again, the back transition to the steady state is attained. This transition occurred at lower frequencies, and thus at lower values of \( \lambda \), than the transition from steady to unsteady regime for increasing \( \lambda \) [3]. The hysteresis loop at \( \epsilon = 1.29 \cdot 10^{-3} \) is shown in Fig. 2. The transition between the steady and the modulated regimes manifests itself in this figure in the jump in the phase of the sloshing wave relative to that of the wavemaker, when measured close to the wavemaker. In the case of the modulated wave regime, the phase angles shown in the figure represent the averaged over long time values. The hysteresis loop covers the values of \( \lambda \) between -0.52 and -0.57. These values are close to the average transitional \( \lambda \) reported in [3] for weaker forcing. At strong forcing, \( \epsilon = 1.56 \cdot 10^{-3} \), the transition between the two wave patterns was observed at \( \lambda = -0.6 \), and no clear conclusions about the existence of hysteresis could be drawn from the experiments.

The dissipation coefficient along the tank \( a_s \) was estimated using the results of [6], where the correction factor suggested in [7] was taken into account. As was noticed in [3], the transition between the steady and the modulated wave regimes is governed mainly by the dissipation at the wavemaker. The value of \( a_s \) was chosen so that the transition between these two wave patterns in the numerical calculations and in the experiments occurs at approximately the same value of the detuning parameter \( \lambda \).

At sufficiently low frequencies (\( \lambda < -0.6 \)), in both the numerical solutions and in the experiments the steady wave pattern was obtained. The details of this steady wave field were discussed in [3]. In the numerical calculations, the transition to the modulated regime occurred at \( \lambda = -0.6 \). The evolution of the wave field at four locations along the tank at frequency slightly exceeding the transitional value (\( \lambda = -0.58 \)), is presented in Fig. 3(a). Periodic generation of solitons can be clearly seen. The dimensionless period of modulation estimated from the figure is \( T = 22.6 \). The experimental results obtained at the forcing frequency \( f = 1.1338 \) Hz, \( \lambda = -0.52 \), immediately beyond the transitional region, are presented in Fig. 3(b). The absolute values of the normalized velocity potential \( |G| \) in this figure were obtained from the experimentally measured surface elevation amplitudes, using eq. (3).
The results in Figs. 3(a) and 3(b) are shown at identical locations along the tank. The scaling of the vertical coordinate is also identical. The period of modulation in Fig.3(b) is about 402s, corresponding to 456 periods of forcing and giving the dimensionless value of T=22.8, indistinguishable from the numerically obtained modulation period. The general shape of the wave modulation pattern and the measured amplitudes are also very similar in both figures at all locations.

When the forcing frequency is increased, the period of modulation decreases both in experiments and in numerical calculations. The character of modulation also changes; periodic appearance of solitons is gradually replaced by a wave-train-like modulation. The results of numerical calculations at λ=0 are presented in Fig.4(a), and the experimental results obtained at identical value of λ (f=1.1393 Hz) are shown in Fig.4(b). Note that the total time span in this figure is half of that in Fig.3. The period of modulation at this λ from the numerical solution is T=6.4, while the experiments give the value of 133a, which corresponds to 151 periods of forcing and to the dimensionless value of T=7.6. While the general shapes of modulation in both figures are very similar, the quantitative agreement in the values of the period of modulation and the amplitudes is less impressive than in the previous figure. The notable disagreement in wave amplitudes in Fig. 4 may be attributed to possible reflection of the slowly propagating modulation wave train from the beach at the far end of the tank. This reflection apparently becomes more essential when the period of modulation and the modulation wave length become shorter.

V. Concluding remarks

The comparison of the numerical and experimental results presented in this paper allows us to conclude that the theoretical model, which includes dissipation both at the wavemaker and along the tank, provides a very good description of main features of the wave field. The model predicts correctly the existence of two qualitatively different wave patterns, steady and modulated, as a function of a single governing parameter, λ. More than that, with the dissipation coefficients properly chosen, the numerically obtained values of periods of long-time modulation, wave amplitudes and general modulation pattern, are in close quantitative agreement with the experiment, for the whole range of forcing frequencies and amplitudes. The only major physical feature which the model does not predict at the present stage, is the existence of hysteresis in the transition between the two wave regimes.
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References.


Figure 1(a) General view of the experimental facility with propagating soliton.

Figure 1(b) Numerical solution of the NLS equation (λ = -0.58).
Figure 2  The dependence of the phase angles of the surface elevation at the distance 10cm from the wavemaker on $\lambda$; $x$, for increasing frequency, $O$, for decreasing frequency; $\epsilon=1.2940^\circ$.

Figure 3  The dependence of the amplitude of the velocity potential on slow time at 4 locations along the tank: (a) numerical calculations, $\lambda=-0.58$; (b) experimental results, $\lambda=-0.51$. 
Figure 4. As in previous figure, $\lambda = 0$. 