DYNAMICAL MODELS FOR CROSS-SHORE TRANSPORT AND EQUILIBRIUM BOTTOM PROFILES

By Eliezer Kit1 and Efim Pelinovsky2

ABSTRACT: The long-term cross-shore dynamics of the surf zone affected by storms is investigated. The validity of the hypothesis of equilibrium bottom profiles in the coastal zone is examined using bathymetric measurements of some coastal regions in Israel for the period 1968–1995. It is shown that the power law of depth variation as a function of offshore coordinate (Dean’s profile) is satisfied on the average for nonbarred profiles. Bar development for the period of the last five years is analyzed. A mathematical model for description of time-dependent sea bottom deformation under the action of wind waves in the surf zone is developed. It is found that, besides the known static solutions of the balance equation for sediment transport, which correspond to the equilibrium bottom profile with zero net sediment transport, there exists a solution corresponding to the profiles of dynamic equilibrium when the surf zone is a “carrier” of sediment transport from the coast to the sea or in the opposite direction. Both equilibrium bottom profiles satisfy the power asymptotic law near shore. A class of steady-state solutions of the sediment balance equation, which correspond to the “traveling” bottom profiles, is studied also. Such “wave” solutions may be used for the description of the long-term processes of the beach migration. Data available on the deviation of bottom profiles from equilibrium are used to estimate an anticipated speed of the long-term beach deformation.

INTRODUCTION

Beach formation is an important factor of the nearshore dynamics, and because of its practical significance, it is treated in the framework of different approaches (geography, engineering, and geophysics). These approaches are described in numerous monographs (e.g., Komar (1983), Sleath (1984), Bird (1985), Rijn (1988), Fredsoe and Deigaard (1992), Nielsen (1992), Silvester and Hsu (1993)). Usually, a beach profile is considered on the average to be in equilibrium, when at each point the processes of erosion and accretion are balanced. This concept is of utmost importance for understanding of mechanisms of beach profile evolution and its stability.

Based on the analysis of beach profiles along the Danish North Sea coast and the United States’ Atlantic and Gulf coasts, the general equilibrium profile was found to satisfy the following simple empirical relation (Bruun 1954; Dean 1977, 1991):

\[ h(x) = Ax^{2/3} \]  

where \( h = \) local depth; \( x = \) distance from the shoreline; and \( A = \) an empirical constant, which depends on the properties of bottom sediments and, maybe, on the wave climate and coastal currents. In the simplified approach this can be expressed as a function of fall velocity of sediment particles \( w' \):

\[ A = 2.25 \left( \frac{w'}{g} \right)^{10/3} \]  

where \( g = \) acceleration due to gravity (Dean 1977, 1991). In particular, for a typical coast of Florida beach (fine sand, diameter \( d = 0.26 \text{ mm} \)), \( A = 0.13 \text{ m}^{0.67} \) (Kriebel and Dean 1985; Dean 1991).

The form of beach profile given by (1) was modified later by adding a small linear section in the vicinity of the shoreline, to eliminate the singularity at the coastline; this shape was called Dean-Moore-Wiegel profile (Pruszak 1993; Capobianco et al. 1993; Kriebel and Dean 1993; Foster et al. 1994; Work and Dean 1995). Dean’s profile in (1) and its generalizations are used widely in applied problems for estimation of stability of underwater slopes and in computations of balance of sediment transport.

In recent works various approximations of equilibrium profile, which do not lead to a singularity at the shoreline, are suggested. In particular, an exponential shape is suggested by Bodge (1992) and Komar and McDougal (1994)

\[ h = B [1 - \exp(-\mu x)] \]  

where \( B/\mu = \) beach slope at the shoreline \( x = 0 \). Logarithmic shape is suggested in the paper of Lee (1994)

\[ h = B \ln(1 + x/x_0) \]  

Constants in these distributions also are determined by the features of the bottom sediments and by the wind wave climate.

The generalized power approximation for an equilibrium bottom profile

\[ h = A_1 x^n \]  

is used also; e.g., for description of the Dutch dunes with \( A_1 = 0.14 \text{ m}^{0.72} \) and \( n = 0.78 \), see, Steetzel (1993). This profile represents an equilibrium solution in the theoretical model of dune erosion (Kobayashi 1987).

All these approximations assume that the bottom profile is stable, i.e., erosion and accretion compensate each other. Any other profile during storms must transform and eventually yields the equilibrium one. Its transition should be described by a time-dependent mathematical model, which accounts for wave transformation in the nearshore zone. In simplified models of long-term beach evolution (Kriebel and Dean (1993), Pruszak (1993), Pruszak and Aminti (1993), see also Roelvink and Boker (1993) and Work and Dean (1995)), the bottom profile is described by some functional form (e.g., Dean’s profile) and the variable parameter \( A(t) \) can be found by solving the ordinary differential equations or from analysis of observed variation using empirical orthogonal functions.

In more sophisticated models, which are convenient for description of the beach response on one storm, the partial differential equations (with defined or two coordinates) are used to describe the morphodynamics; these models were developed in the last decade (Mizuguchi and Mori 1981; Watan...
The wave crests are parallel to the straight coast, the long shore transport vanishes and the problem can be treated as one-dimensional in the horizontal plane. In the breaking zone, as a first approximation, the waves' parameters can be considered to be independent of their features in deep water (although the intensity of storms determines the width of the breaking zone). In particular, the waves can be considered as long ones and their heights are proportional to the local depth. In this case the balance equation of sediments becomes, essentially, independent of the wind wave equations and the correlation between them is only through the boundary condition at the breaker line and wave-induced setup. The balance equation of sediments is represented by a nonlinear diffusion equation with boundary conditions at movable boundaries (migrating breaker and shorelines). Such models are described in detail by Kriebel and Dean (1985), Kobayashi (1987), Kobayashi and Han (1988), and Lee et al. (1996). Analytical solutions of the corresponding governing equation are reported in Kobayashi (1987). The equilibrium profiles in (4) and (5) are steady (static) solutions of this nonlinear partial differential equation. Kobayashi also studied the erosion processes within simplified linearized version of this equation. In a recent work Nicholson and O'Connor (1997) report on development of a diffusion-based conceptual model of cross-shore transport. Their diffusion equation is actually a linear one because the diffusion coefficient in the numerical model depends on offshore distance and time only.

In the current work we study Kobayashi's theoretical model for description of the long-term evolution of the cross-shore profile. We present some new solutions corresponding to the dynamic equilibrium, when the bottom profile is steady or migrates as a whole with a constant speed. We found that the power asymptotic profile in (5) is typical for many non-static solutions. Also, we obtained rough relations to estimate the average speed of the beach erosion based on the observed empirical constant for the power asymptotic law of the beach profile. Some theoretical results obtained in the study are applied for the description of the beach dynamics in the surf zone at the Israeli coast. Data of bathymetric measurements for the period of 1968–1995 are used for analysis of equilibrium and time-dependent bottom profiles, including bar dynamics. Some data of the beach migration for other regions reported in the literature also are used for rough estimation of the speed of the beach translations.

**EXPERIMENTAL DATA**

General characteristics of Israeli beaches are described by Bird (1985), Goldsmith and Golik (1980), and Carmel et al. (1985). It is well known that this coast belongs to the Nil Littoral Cell where the transport of sediments is provided from the Nile Delta source to the Haifa Bay. With the exception of Mount Carmel, which protrudes into the sea forming Haifa Bay, the coastline is generally smooth with a gentle curvature expressed in the gradual change in orientation from 0°N8°E in the north near the Carmel Head to 0°N36°E in the south nearby Ashkelon. The knowledge of morphological regime and net sediment transport in this region (Ashkelon area) is essential because it determines the sediment flux entering the Mediterranean coastal region of Israel. Because of construction of several marine structures, a small harbor called Katza and a cooling basin of a power station, many geomorphologic and oceanographic measurements were conducted in this region including detailed bathymetric surveys (twice a year for the last five years, 1992–1996), frequent aerial photographs, offshore currents, and granulometric investigations.

The application of a one-dimensional beach deformation model seems to be justified for the Israeli coastal region in general: the shoreline is straight in many parts of the coast, and wind waves are normal to the beach in 33% of cases. It is especially suitable for the southern region of Ashkelon area, to the south of marine structures, where the beach is sandy and is not disturbed by rocks or cliffs. Therefore, we can analyze observed beach dynamics in the framework of the cross-shore sediment transport theory and the concept of the equilibrium bottom profile. Here we study the evolution of typical cross-shore bottom profile in the region of Ashkelon (in most cases sufficiently far from any marine structure) during the last 27 years (1968–1995). The results of the 15th field measurements (June 1968, December 1974, November 1976, November 1977, July 1980, July 1986, May 1990, November 1990, May 1991, December 1991, July 1992, December 1992, October 1993, July 1994, and March 1995) are presented on specific bathymetric charts. They were obtained by means of echosounder, which permits an accuracy of approximately ±0.5 m. Available bathymetry maps from 1967 up to now and digital data (from 1989) of the region were analyzed carefully and the maps 1968, 1974, 1975, 1977, 1980, and 1986 containing significant information were digitized.

Characteristics of the sediment particles are as follows: the mean diameter dₐ₀ of the particles weakly depends on the cross section and varies mostly with the depth of the location, from 0.25 mm at the shoreline to 0.18 mm at the depth of 10 m; the particle size distributions vary from location to location, the very rough estimate of geometric standard deviation σₑ = \sqrt{dₐ₀/dₐ₁} indicates a value of 1.6.

For analysis of the equilibrium bottom profile we have chosen a set of bottom profiles with no bars or weak bars (1974, 1980, 1986, 1990, and May 1991), these profiles are presented in Fig. 1. The mean profile (solid line) for this set is monotonic and significantly smoother. This profile is in good agreement with Dean's equilibrium profile in (1) providing A = 0.13 m³ (see Fig. 2). Therefore, the Dean's profile describes satisfactorily on the average the measured bottom profiles without bars.

It follows from Fig. 1 that the beach slightly varies from one year to another. Following ideas of Kriebel and Dean (1993), Pruszak (1993), and Pruszak and Aminti (1993) we can present the local bottom profile for each time moment in the form of (1) with time-dependent parameters. It was found

**FIG. 1.** Cross-Shore Profiles with No Bars (Solid Line Corresponds to Mean Bottom Profile)
that Dean's parameter calculated separately for the nonbarred bottom profiles (Fig. 1) varies insignificantly, less than 5% (from 0.128 to 0.133). The exponent $n$ of the power equations (see (5)) for the instantaneous bottom profile also varies insignificantly (from 0.73 to 1.1). Therefore, the concept of the equilibrium bottom profile is effective for Israeli conditions.

Sandbars in the coastal zone of Israel can be noticed in the profiles measured in 1968, 1977, 1980, and 1991–1995. The last period was characterized by one season of large winter storms (1991–1992). Unfortunately, all measuring instruments in the area near Ashkelon were malfunctioning during this period and only evidences and observations allowed the estimation of the occurrence of several lengthy storms with a significant wave height of approximately 6 m. Although, according to the wave statistics in the region, the frequency of occurrence of such storms is quite high, once in five years, the simultaneous appearance of several strong storms during one season and their duration are quite unusual.

Evolution of the bar for the period of 1991–1993 years is shown in Fig. 3(a). This bar appears at smaller depth and then moves down on the slope. The bar developed fast during the large winter storms in the beginning of 1992 and shifted 250 m over a period of six months or less. From available data three apparently independent bars can be selected: in 1968, 1977–1980, and 1991–1995. The positions of the top of all bars are presented in Fig. 3(b). For the last two years the location of the bar remains practically unchanged. It might be worth noting that only mild storms with wave heights slightly exceeding 4 m took place during these two seasons. Silvester and Hsu (1993) suggested an empirical relation of the water depth at the bar crest versus the distance from the shoreline

$$ h = 0.111x^{0.795} \quad (6) $$

where $h$ and $x$ are expressed in meters. This relation (as well as the Dean profile) is presented in Fig. 3(a) [the actual coordinate $x$ in (1) and (6) is calculated from the shoreline position at $x = 50$ m] and is in agreement with observed data.

The following conclusions can be made from the analysis of observed bottom profiles:

1. For 27 years the bottom profiles in the coastal zone of Israel demonstrate complex dynamics: bar development and movement during strong storms and the existence of profiles with no bars.

2. For no-barred beaches Dean's profile is in good agreement with the averaged observed bottom profile. Local bottom profiles can be expressed by Dean's profile [see

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**THEORETICAL MODEL FORMULATION**

For description of the bottom deformation, the well-known model of the cross-shore transport (Kriebel and Dean 1985; Kobayashi 1987) can be used. The balance equation of sediment transport (conservation of the volume of sediment) for "pure" cross-shore processes (straight shoreline, no alongshore current, and wave crests are parallel to shoreline) can be written as

$$ \frac{\partial z}{\partial t} = \frac{\partial q}{\partial x} $$

where the $x$-axis is directed seaward; $z_0$ = vertical displacement of the developing beach profile from the initial profile denoted as $h_0(x)$ [Fig. 4, similar to that of Kobayashi (1987)]; and $q$ = specific depth-integrated sediment transport rate. For convenience, the effects of porosity of the upper sediment layer are included in $q$ [thus, this definition differs from that normally used by a factor of $(1 - n_p)$, where $n_p$ is the porosity of marine sediments]. It is convenient in the following analysis to con-
who suggested the following two-parameter sediment transport rate equation, which is a generalization of (13)

$$q = \delta(h) \frac{\partial h}{\partial t} - K(h)$$  \hspace{1cm} (14)

where $\delta$ and $K$ are empirical functions of water depth expressed in the form

$$\delta = \epsilon\sqrt{h}, \quad K = \frac{2}{3} \epsilon A^{1/3}$$  \hspace{1cm} (15)

Such form of presentation enables one to obtain the power equilibrium bottom profile in (5) as a steady solution of (9). For $n = 2/3$ the expression in (15) becomes

$$\delta = \epsilon\sqrt{h}, \quad K = \frac{2}{3} \epsilon A^{1/3}$$  \hspace{1cm} (16)

according to (13) and recommended values for fine grain sand are $\epsilon = 0.013$ m$^{3}$/s and $A = 0.13$ m$^{10}$ (Kobayashi 1987). Eq. (14) describes a nonlinear diffusion with moving boundaries and "external forces" (wave setup and sediment fluxes at the boundaries).

When $n = 1$, the linear bottom equilibrium profile can be obtained and then

$$\delta = \epsilon_1, \quad K = \epsilon_1 A_1$$  \hspace{1cm} (17)

where $A_1 = 0.14$ and $\epsilon_1$ is of the order of 0.03 m to be concurrent with the observed averaged profile at the Israeli coast. Hence, for the linear equilibrium bottom profile ($n = 1$) the nonlinear diffusion in (9) becomes linear and therefore is convenient for analytical treatment. Kobayashi (1987) studied the case $n = 2/3$ analytically using the linearized parametrization of the diffusion coefficient. Numerical model of on-shore offshore sediment transport with moving boundaries, which realized Kobayashi’s theory and is capable to take into account the nonlinearity of the problem, was presented recently by Lee et al. (1996).

Here we will present a new set of analytical solutions of the considered nonlinear problem for different approximations of the equilibrium bottom profile. They can be used for an analysis of the long-term variation of the erosion processes as well as for testing of numerical models.

The difficulties of determination of functions $\delta(h)$ and $K(h)$ for different approximations of the equilibrium bottom profile were discussed in Kit and Pelinsonsky (1995). Following Kobayashi (1987) the steady solution of (9), when the sediment fluxes at the breaker and shoreline are absent ($Q_w = Q_r = 0$), can be presented in an integral form

$$x = \int_0^h \frac{\delta(h)}{K(h)} \, dh$$  \hspace{1cm} (18)

To obtain the equilibrium bottom profile in the observed form from this solution, it is necessary to select at least one of the functions in (14). If the linear relationship between the sediment flux and wave energy dissipation in the surf zone suggested by Kriebel and Dean (1985) is used, the function $\delta(h)$ has the form of (16). It follows from (18) that $K(h) = \delta(h) \partial h/\partial x$ and thus, for the power equilibrium profile in (5) the following approximation for functions $\delta$ and $K$ can be obtained (Kit and Pelinsonsky 1995):

$$\delta = \epsilon\sqrt{h}, \quad K = \epsilon A_1^{1/3} h^{1/2}$$  \hspace{1cm} (19)

In this model, the function $K$ is bounded if $n$ is greater than 2/3, and this limitation seems to be realistic. Similar calculations can be performed for equilibrium bottom profiles of nonpolynomial shape. Therefore, there is a set of approximations for $\delta$ and $K$, which can be selected to fit the models describing the transformation of bottom profiles under the ac-
tion of storms, which enables the account for regional features. Choice of optimal expressions for $\delta$ and $K$, if the equilibrium profile does not coincide with the Dean's profile, should be made after verification of the mathematical model at the beaches. Bird (1985) describes the possibility of evolution of quasi-equilibrium beach profiles through dynamic regimes will be demonstrated. We will use Kobayashi's expressions for the sediment transport components [see (15)] resulting in a simple power law approximation [see (5)] for an equilibrium bottom profile (mostly, Dean's profile, $n = 2/3$).

**DYNAMIC REGIMES OF QUASI-EQUILIBRIUM BEACHES**

The theoretical model described earlier (or its modification) has been applied for the numerical simulations of the bottom profile evolution under the action of wave wind in Kriebel and Dean (1985) and Lee et al. (1996). The possibilities of analytical methods are limited because of nonlinearity of the problem and moving boundary. To study the dune erosion analytically Kobayashi (1987) reduced the nonlinear diffusion equation to a linear diffusion equation by replacing $\delta(h)$ with the average value. Nevertheless, analytical solutions are very convenient to demonstrate main features of physical processes and influence of governing parameters. Here we will give a new set of rigorous analytical solutions of the nonlinear equation in (9). Similar to static solution of equilibrium bottom profile in (18) in the form of the power law approximation, they characterize long-term average profiles in equilibrium or quasi-equilibrium state. These solutions are used to obtain the averaged relations between characteristics of the beach profile (bottom slope, erosion speed, etc.). The quasi-equilibrium regimes for which the sediment transport rate is nonzero at each point of the bottom profile are discussed in the following paragraph and some analytical solutions of (9) are presented.

Let us assume that the sediment fluxes at the breaker line and at the shoreline are equal and have the same sign, i.e., $Q_b = Q_s = Q$. In this case a steady solution of (9) can be found, which satisfies the boundary conditions in (10) and (11). This solution corresponds to an ordinary differential equation

$$\delta(h) \frac{dh}{dx} - K(h) = Q$$  \hspace{1cm} (20)

In the case $Q = 0$, the solution of (20) is the equilibrium bottom profile described earlier. If Kobayashi's expression in (15) for the function $K$ will be used ($K$ does not depend on $h$), the power law in (5) for the bottom profile is again the steady solution of (20), with $K$ replaced by $K + Q$. This solution presents the equilibrium bottom profile with nonzero net sediment transport rate, the coefficient $A_s$ in expression in (5) differs from one of the static profile (redefined now as $A_{sw}$) and depends on $Q$. In fact, using the measured bottom profile ($A_s$), we can find the cross-shore transport rate, which should support such an equilibrium bottom profile

$$Q = ne_s[A_s^{1/n} - A_{sw}^{1/n}]$$  \hspace{1cm} (21)

This new equilibrium bottom profile is one along which the cross-shore sediment flux is translated seaward (if $A_s > A_{sw}$) or shoreward (if $A_s < A_{sw}$) without changing form. Hence, the sediment flux $Q$ can induce the observed variations of the universal constants, in particular of the Dean's constant in (1). Such variations of the Dean's constant were studied for the Polish and Italian coasts by Pruszak (1993) and Pruszak and Aminti (1993). Small variations of constant and exponent in the power asymptotic [see (5)] for the Israeli beach are described briefly in the section on experimental data. In practical applications (21) can be used for estimation of the net cross-shore transport considering the temporal evolution of universal constants at the same morphologic conditions (sand fraction, its diameter, etc.). The nonzero sediment flux can be related, for instance, with the variability of the wind erosion (or deposition) of sediment materials at the beaches. Bird (1985) describes a few examples of such phenomenon, e.g., the sand flux to Morocco beaches from the Sahara Desert. Another source of sediment material is often used regular nourishment of the beach. For instance, this type of an operation is applied to the Caucasus Beach at the Black Sea, where the beach erosion was terminated by the sediment feeding carried out there for many years. Pruszak (1993) pointed out such factors, as occurrence and disappearance of sediment supply due to changing conditions of global dynamic shore equilibrium and the balance of sediment transport, for explanation of temporal variations of the Dean's constant. These processes are parametrized in the theoretical model by the sediment flux at the shoreline $Q_s(t)$ and lead to the variable coefficient $A(t)$ in the power law of an equilibrium or quasi-equilibrium bottom profile.

It is worth mentioning that if the expression in (19) is used for computation of parameters of the cross-shore transport $K$ and $\delta$, the solution of (20) describing the equilibrium bottom profile at the nonzero net transport will differ from the power law in (5). The equilibrium profile again can be obtained utilizing the integral expression in (18), as it was shown in Kit and Pelinovsky (1995). Because the function $K = enA_s^{1/n}$. $h^{(n-2)/n}$ at $n > 2/3$ tends to zero at $h \to 0$ and, hence, contribution of $Q$ becomes important, it can be readily seen from (20) that near the shoreline the Dean's shape of the bottom profile

$$h = \left(\frac{3Q}{2en}\right)^{2/n} x^{2/n}$$  \hspace{1cm} (22)

(with constant depending on the cross-shore transport) emerges in the vicinity of the shoreline. Therefore, the applicability of Dean's formula in (1) is significantly broader: it describes the equilibrium bottom profile with both zero and nonzero cross-shore transport.

One of the possible applications of the dynamic equilibrium profile is a conceptual model of sediments bypassing a marine structure. At the updrift side of the structure, the shoreline migrates seaward after completion of structure construction and eventually reaches, on the average, a steady state. In the new steady state the sediment flux entering the region (e.g., from the south in the vicinity of marine structures in the Ashkelon area) should not cause further accretion of the sand at the shoreline and at the lower beach. It can be assumed that part of the sediments approaching the region in the vicinity of marine structure will bypass the structure driven by a long shore current, and the other part will be moved to the deeper water by an offshore flux $Q(s)$ where $s$ is the coordinate along the shore. In this case a variable $A_s(s)$ is expected, which should be greater than $A_s$, corresponding to a static equilibrium and the cross-shore flux can be expressed by an equation similar to (21)

$$Q(s) = ne_s[A(s)^{1/n} - A_{sw}^{1/n}]$$  \hspace{1cm} (23)

The measured and corresponding equilibrium bottom profiles in the vicinity of Katza harbor in the Ashkelon area for three different cross sections are shown in Fig. 5. The profiles appearing in the lower and upper part of the figure correspond to the bathymetric data measured in 1968 and 1986, before and 13 years after the completion of the Katza harbor construction. The construction of the cooling basin in 1986–1987 and hot water outlets in 1988 and 1991 modified strongly the local bathymetry at the selected cross sections. The computed values of $A(s)$ are presented in Table 1. Although the variations of $A(s)$, computed at different cross sections after the harbor was completed in 1973, are relatively small, the trend is ob-
\[ \frac{\delta dh}{dy} = K(h) + c h + P \]  
(24)
derived from (9) by integration; \( P \) is an arbitrary constant. Taking into account (14) and the boundary condition in (11), we can conclude that \( P \) coincides with the value of the sediment transport on the moving shoreline: \( P = q(h = 0) = cd + Q_s \). Thus

\[ \frac{\delta dh}{dy} = K(h) + c(h + d) + Q_s \]  
(25)
defines all possible theoretical forms of the "traveling" beaches. For simplicity we will consider again the Kobayashi's approximation in (15) for \( K \) (its independence of depth). In this case combining all constants \( (M = K + cd + Q_s) \) we can rewrite (24) as follows:

\[ \frac{\delta dh}{dy} = M + c h \]  
(26)

In the vicinity of the moving shoreline the last term in (26) can be ignored; the corresponding equation has the same functional form as for static conditions in (20) and, therefore, its solution is again expressed via the power law in (5) (constant \( A_n \) depends on \( M \)). It is convenient to use the "observed" value of the constant \( A_n \) and to express \( M \) [this relation is obtained from (26) when the last term is ignored]

\[ M = n e A_n^{[n]} \]  
(27)

Therefore, the solution of (26) and the shape of the bottom profile depend on two measured parameters: a speed of the beach displacement (translation) \( c \) and the constant \( A_n \), which can be determined in the vicinity of the moving shoreline. In particular, the solution of (26) for \( n = 2/3 \) has the following form:

\[ x + ct = \frac{2e}{c} (\sqrt{h} - \sqrt{2eA_n^{[2/3]} c \arctan \sqrt{3ch/2eA_n^{[2/3]}}} \]  
(28)

In the vicinity of shoreline this solution is, of course, approximated by the Dean's asymptotic \( h \sim x^{2/3} \) (can be obtained readily from the Taylor expansion) while far enough from the coast the solution yields the form \( h \sim x^2 \). Profile in (28) is shown in Fig. 6(a) for \( A = 0.13 \text{ m}^2/\text{sec} \) and for three values of the erosion speed: 0.2, 1, and 5 m/d. At lower speed the bottom profile coincides practically with the Dean equilibrium profile in (1), while speed increases the bottom profile becomes steeper and erosion processes will result in retreating of the shoreline into the landward direction. Such solutions should be considered together with boundary conditions in (9) and (10). If a plane berm of the height \( d \) above still water will be assumed, the sediment flux at the shoreline caused by erosion can be expressed as

\[ q_s = c \cdot d + Q_s \]  
(29)

From (25) and (29) \( Q_s \) can be written in the following form:

\[ Q_s = M - K - cd \]  
(30)

Similarly, we can find the sediment transport \( Q_o \) at the breaker line and, for the traveling beach in the form of (28), it is equal

\[ Q_o = Q_s + c(h_o + d) \]  
(31)

and is greater than the sediment flux at the shoreline because of the additional transport caused by the erosion of the beach. It follows from (24) that the sediment transport in each point of the bottom profile is proportional to water depth, and its gradient is constant and equal to the translation speed \( c \). The relation between sediment transport at the shoreline and at the

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**FIG. 5.** Temporal and Spatial Variability of Cross-Shore Bottom Profiles at Various Distances from Marine Structure (Distances from Structure: \( \delta, 450 \text{ m}; \Delta, 780 \text{ m}; \bigcirc, 1,050 \text{ m} \))

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**TABLE 1.** Variations of Dean’s Constant \( A \) at Different Cross Sections

<table>
<thead>
<tr>
<th>Time:distance</th>
<th>( A(s) )</th>
<th>Distance of Shoreline from Reference Position (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structure (1)</td>
<td>( 450 \text{ m} )</td>
<td>( 750 \text{ m} )</td>
</tr>
<tr>
<td>June 1968</td>
<td>( 0.118 )</td>
<td>( 0.120 )</td>
</tr>
<tr>
<td>November 1976</td>
<td>( 0.130 )</td>
<td>( 0.124 )</td>
</tr>
<tr>
<td>November 1977</td>
<td>( 0.130 )</td>
<td>( 0.129 )</td>
</tr>
<tr>
<td>June 1980</td>
<td>( 0.130 )</td>
<td>( 0.125 )</td>
</tr>
<tr>
<td>June 1986</td>
<td>( 0.133 )</td>
<td>( 0.129 )</td>
</tr>
</tbody>
</table>

vicious, indicating the increase of \( A(s) \) with decrease of the cross-section distance from the main breakwater of Kataza harbor. The slight variations of background values of \( A(s) \), computed for the same cross sections in 1968, apparently are related to the measurements uncertainties and to the choice of specific locations.

Following the same approach, decrease of \( A(s) \) should be expected at the downdrift side (to the north of Kataza harbor) of the structure. However, at the shoreline a rocky beach is exposed as a result of strong erosion because of the lack of sediments caused by decreased bypassing. It is obvious that such a bottom profile (including a rocky section and, therefore, not attending equilibrium) supports an onshore sediment flux although the evaluation of its value is more complicated.

The case when the beach profiles migrate with a constant speed is considered in the following. These beach profiles can be represented mathematically as a function of a "running" coordinate \( y = x + ct \), where \( c \) is an unknown velocity of displacement (\( c > 0 \) corresponds to the erosion of the beach, and \( c < 0 \) corresponds to its accretion). Such solutions can be obtained from an ordinary differential equation (provided the mean sea level \( \eta \) does not change in time)
breaking line or the sediment transport gradient determines the speed of steadily traveling beach.

Other solutions of (26) also can be analyzed. In particular, the analytical solution describing the accreted beach profile for $n = 2/3$ is

$$\frac{|c|x - |c|t}{e \sqrt{h_i}} = 2\sqrt{h_i} + \ln \frac{1 - \sqrt{h_i}}{1 + \sqrt{h_i}},$$

(32)

where $h_i = 2eA^{3/2}/|c|$ (the absolute value of the speed is used). It is shown in Fig. 6(b) for $A = 0.13$ m$^{3/2}$ and for three values of the speed of migration: 0.1, 1, and 4 m/d. Near the shoreline (32) coincides with the Dean’s profile (it can be shown using the Taylor’s series). Far from the shoreline the bottom depth tends to the constant value $h_i$ (underwater terrace). The moving bottom profile in this case is more gentle in slope than the equilibrium profile and it corresponds to accretion of the beach. Obviously, the correct boundary conditions should be provided.

It is necessary to mention that the stationary traveling bottom profile is an oversimplification for description of real beach transformation during storms (mainly because of specific boundary sediment fluxes and geometry of the problem). Such solutions at least can be realized as an intermediate asymptotic profile in the process of beach deformation (it is known from the theory of nonlinear diffusion equations that the intermediate asymptotic solutions are stable for a wide class of disturbances). Applications of the obtained solutions for the description of long-term beach evolution will be presented in the following section.

**COMMENTS**

Usually, mathematical models of sediment transport in the surf zone are applied for analysis of dune erosion or beach nourishment evolution during one storm and provide reasonable results. Meanwhile, cross-shore sediment transport at large time scales (months or years) remains a challenging problem that has received less attention. The long-term dynamics of the cross-shore beach profile evolution has been studied in terms of the quasi-equilibrium Dean’s profile in (1) with time-dependent parameters [some other empirical profile evolution models are discussed briefly by Roelvink and Broker (1993)]. In particular, Pruszak (1993) has analyzed 1964–1991 data for the Polish Baltic Sea shore and has shown that Dean’s constant varies periodically with amplitude 0.022 m$^{3/2}$ (mean value 0.075 m$^{3/2}$) and the period of approximately 25–30 years (each series of measurements consists of 15–20 cross-shore profiles with 70–100 points in each profile). Such variations of Dean’s constant are linked by Pruszak (1993) and Pruszak and Aminti (1993) to the long-term cyclic changes of some hydro- and morphodynamic parameters (long-term sea level changes, migration of coastal forms, occurrence and disappearance of sediment supply due to changing conditions of global dynamic shore equilibrium, and the balance of sediment transport). Israeli data, described in the section on experimental data, have shown a moderate time and spatial variation of Dean’s parameter in the range of approximately 5–10%.

Anomalous high speed of translation of the Carmel beach in California has been reported by Bascom (1964). The beach there migrates in a seaward direction approximately 90 m for four months during the summer (the speed of accretion is around 20 m/month) and in opposite direction in winter time. The large amount of data related to the long-term migration of the beaches in different countries are summarized by Bird (1985). He found that 70% of all beaches erode with speed greater than 10 cm/yr. For example, the sand beach on Rod Island (USA) propagates with the mean velocity 0.7 m/yr for the period of 1939–1975. Some Polish beaches at the Baltic Sea erode with the speed of 5 m/yr.

Both phenomena described in the foregoing paragraph (the long-term variation of Dean’s constant for the equilibrium bottom profile and the beach migration) can be interrelated in the mathematical model of the cross-shore transport based on analytical treatments presented in the foregoing section. All solutions for the equilibrium and quasi-equilibrium beaches can be presented as solutions of the balance equation for the sediment transport

$$q = \delta \frac{\partial h}{\partial x} - K = Q + c \cdot h$$

(33)

In particular, the “classical” equilibrium bottom profile (Dean’s profile) for the zero sediment transport at each point of the bottom profile is a solution of (33) at $c = 0$ and $Q = 0$. The dynamic equilibrium profile with nonzero sediment transport translated seaward or shoreward is a solution of (33) too at $c = 0$, and $Q$ is the sediment transport at the shore or breaking lines (they should be equal for equilibrium). Uniformly migrating beach is also a solution of (33), when $Q$ coincides with parameter $P = cd + Q_i$, [see (24) and (25)], where $Q_i$ is the sediment flux from the beach and $d$ is the berm height. Here we will consider for simplicity the case $n = 2/3$ and will use the expression in (14) together with (16) for the cross-shore sediment transport. It is convenient to rewrite (33) using these formulas as

$$Q + c \cdot h = \frac{2e}{3} \left( \frac{\partial h^{3/2}}{\partial x} - A^{3/2} \right)$$

(34)
where \( A \) = equilibrium value of Dean's constant in (2) as it was introduced by Dean (it is a function of granulometry and geomorphology only). We have shown in the foregoing section that in the vicinity of the shoreline (both for steady and for traveling waves), when \( h \rightarrow 0 \), the power law (in our case the Dean's profile \( A^h \)) is valid with different values of Dean's constant \( A \). Thus, the following expression for \( Q \) from (34) can be derived at \( h \rightarrow 0 \):
\[
Q = \frac{2e}{3} (A_{5}^{h} - A_{i}^{h})
\]
(35)

Therefore, considering, at first, steady beaches, the observed variability of Dean's parameter could be explained by the sediment flux from the beach to the sea (\( A \) increases) or in opposite direction (\( A \) decreases). The necessary sediment transport rate can be estimated from (35); e.g., 5–10% variation of the Dean's parameter for Israeli beaches can be provided by sediment flux \( Q = 2.4 \times 10^{-3} \) m/yr per s. Variations of Dean's constant for Polish beaches described by Pruszak (1993) can be caused by the sediment flux ranging from \( 10^{-4} \) m/yr per s in seaward direction to 0.7 \( 10^{-4} \) m/yr per s in landward direction.

If the source of the sediment materials is absent or not sufficient, a regime of the steady dynamic equilibrium is unrealizable and the beach should migrate. Here we will consider only simple scenario of such a migration when the beach travels with a constant speed. The speed of beach migration can be estimated from (34). When there is no source of sediment material (zero sediment flux) from (34) and (29) the speed of the beach propagation can be derived
\[
c(\log d) = \frac{2e}{3} \left( \frac{sh^{5}}{3x} - A_{i}^{5} \right)
\]
where \( d \) = berm height above the still water. In view of the fact that the bottom profile near the moving shoreline follows Dean's equation with variable parameter \( A \), the migration speed via Dean's constant can be estimated
\[
c = \frac{2e}{3d} (A_{5}^{h} - A_{i}^{h})
\]
(37)

This relation yields \( c \approx 0.6–1.2 \) m/d for the Israeli coast for the berm height \( d = 2 \) m and observed 5–10% variation of Dean's constant. Taking into account that strong storms occur a few days per year (it follows from the wave statistics that approximately 2% of waves exceed 3 m significant height), we conclude that the seasonal horizontal displacement of the beach is approximately 4–8 m. These values are in agreement with observed 5–10 m seasonal beach displacements. The annual speed of erosion 0.3 m/yr for Israeli beaches cited in the literature (Bird 1985) explains the very low variability of Dean's constant. Our analysis of Pruszak's data shows that the migration speed varies in the range of 3–7 m/d for Polish beaches and it does not contradict the observed annual speed of erosion (Bird 1985).

CONCLUDING REMARKS

The bottom profiles measured at the Israeli coast during 27 years (1968–1995) have been used to examine the concept of the equilibrium profile. It was shown that the asymptotic power laws (in particular, Dean's profile) describe satisfactorily the shape of the averaged bottom profile as well as local bottom profiles with no bars. Bar development has been studied also. The top position of the bar is described satisfactorily by empirical function in (6); the speed of bar displacement reached 40 m/month during the stormy winter of 1991–1992.

Theoretical model of beach dynamics in the surf zone based on the hypothesis of Kriebel and Dean (1985) and Kobayashi (1987) has been applied to demonstrate new solutions describing the dynamical regimes of the evolution of the quasi-equilibrium bottom profiles. It has been shown that two types of equilibrium profiles can occur in the suggested theoretical model: an equilibrium profile corresponding to a balance of sediment transport at any location (with zero net transport) and one with nonzero net sediment transport, corresponding to a situation when the nearshore zone acts as a "carrier" of sediments between the coast and the sea. The equilibrium profiles are the same, although the physical mechanisms are different. This fact can explain the frequent use of power asymptotic law for description of bottom profiles and the observed temporal and spatial variations of Dean's parameter.

A new class of solutions, describing uniform migration of bottom profiles, was obtained. These solutions describe both erosion and accretion of beaches depending on boundary conditions and they can be realized as intermediate asymptotic solutions in the course of beach transformation under the action of waves. It is confirmed that the slopes of eroded beaches are steeper and the slopes of accreted beaches are milder than the slope of the equilibrium bottom profile.

Long-term beach dynamics has been studied within the framework of the present theory. The relation between the variable Dean's parameter and the speed of the beach migration has been established. The estimate of the seasonal speed of the beach migration obtained from the theoretical model is comparable with field data from several sites.

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APPENDIX I. REFERENCES


**APPENDIX II. NOTATION**

The following symbols are used in this paper:

- $A$ = empirical (Dean’s) constant;
- $A_w$ = empirical constant for generalized bottom power profile;
- $A_{bw}$ = empirical constant for generalized equilibrium bottom power profile;
- $B$ = empirical coefficient;
- $c$ = speed of beach displacement, positive in landward direction;
- $d$ = particle diameter or berm height;
- $d_{16}$, $d_{50}$, $d_{94}$ = grain size exceeding by weight 16%, 50%, and 84% of particles, respectively;
- $e$ = empirical constant;
- $e_w$ = empirical constant for generalized power profile;
- $g$ = gravity acceleration;
- $h$ = local depth;
- $h_b$ = local depth of equilibrium bottom profile;
- $h_i$ = initial water depth;
- $h_t$ = depth of water terrace;
- $M$ = constant;
- $n$ = power exponent;
- $n_p$ = porosity of sediments;
- $P$ = arbitrary constant;
- $q$, $Q$ = specific depth-integrated sediment transport rate;
- $Q_{bw}$ = cross-shore sediment flux at breaker line;
- $Q_t$ = cross-shore sediment flux at shoreline;
- $s$ = distance from structure along the shore;
- $t$ = time;
- $w$ = fall velocity;
- $x$ = distance from shoreline;
- $x_b$ = breaker location;
- $x_s$ = shoreline location;
- $x_0$ = empirical coefficient;
- $x_b$ = bed level;
- $\delta$ = empirical function of water depth;
- $\eta$ = wave setup;
- $\mu$ = empirical coefficient; and
- $\sigma$ = geometric standard deviation of particles.