Breaking the Temporal Resolution Limit by Superoscillating Optical Beats

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Band-limited functions can oscillate locally at an arbitrarily fast rate through an interference phenomenon known as superoscillations. Using an optical pulse with a superoscillatory envelope we experimentally break the temporal Fourier-transform focusing limit with a temporal feature that is approximately three times shorter than the duration of a transform-limited Gaussian pulse having a comparable bandwidth while maintaining 30% visibility. We experimentally demonstrate the ability of such signals to achieve temporal superresolution and show numerically in which cases such pulses can outperform transform-limited pulses.

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Introduction.—The ability to manipulate the waveform of optical pulses is essential for numerous applications [1] such as spectroscopy and coherent control [2,3], metrology [4], microscopy [5], and optical communications [6]. Any pulse shaping system is constrained by the available bandwidth, which sets a limit on the minimal possible pulse duration. This limit, described using a duration-bandwidth product, states that for a given bandwidth the minimal pulse duration is achieved when the spectral phase of the pulse is at most linear [7]. Such a pulse is termed Fourier-transform limited. However, band-limited signals can actually oscillate locally at an arbitrarily fast rate (concomitant with a decreased amplitude), thus breaking the Fourier-transform limit. This is achieved through an interference phenomenon now known as superoscillation, when the local oscillation rate exceeds the highest Fourier component of the signal [8]. Superoscillatory functions have been known since the early 1920s when attempts were made to produce antennas with extremely narrow radiation patterns [9]. Such functions were revived with the introduction of quantum weak measurements [10], which can yield values much larger than the largest eigenvalue of an observable. Berry and Popescu have introduced superoscillations into optics, predicting their use for spatial superresolution [11], which indeed was demonstrated experimentally in several works [12–16]. Superoscillatory diffraction-free beams [17,18] as well as a superoscillating pattern made of accelerating Airy beams were demonstrated [19]. Superoscillations were also used to realize supernarrow optical frequency conversion [20]. In the temporal regime, relatively little has been done with attempts to break the temporal Fourier limit. Two experiments done in 2006 and in 2005 used a quadratic spectral phase to break the Fourier limit by 20% [21] and 30% [22], respectively. A theoretical work discussed breaking the Fourier limit by superresolution pulse compression techniques [23] while experimentally, use of a compressive sensing technique was used to exceed the resolution limit of photodetectors [24]. In the radio-frequency regime temporal superoscillations were successfully demonstrated in 2011 [25] and in 2012 [26]. In addition it was suggested that optical temporal superoscillations can be used to overcome absorption in dielectric materials [27]. Here we experimentally break the temporal resolution limit in the optical domain in the sense of improving the visibility for the existence of distinct temporal features (without considering postprocessing and deconvolution). We first break the temporal Fourier-transform focusing limit of an ultrashort optical pulse by synthesizing a superoscillating pulse envelope. In particular, we achieve a temporal feature that is approximately three times narrower than a Fourier-limited Gaussian pulse having the same bandwidth, while maintaining a visibility (ratio between the amplitudes of the narrow feature and its adjacent fringes) of 30%. We then provide a temporal superresolution measurement by using this pulse in a nonlinear mixing process to resolve a temporal double-slit event (made of two consecutive test pulses) in a case where a transform-limited pulse with at least the same bandwidth fails to do the same. Further numerical simulations analyze in what cases the particular superoscillatory pulse we work with outperforms its transform-limited counterpart in resolving a temporal double slit, achieving temporal superresolution.

Theory.—We start with a known complex superoscillating function [11],

\[ f_{SO}(t) = \left[ \cos(\Omega_0 t) + ia \sin(\Omega_0 t) \right]^N, \]

\[ a > 1, \quad N \in \mathbb{N}^+ \]

whose highest Fourier component is \( N\Omega_0 \), while around \( t \approx 0 \) it superoscillates \( a \) times faster, at the rate of \( aN\Omega_0 \). We expand the real part of Eq. (1) into a cosine series,

\[ \text{Re}\{f_{SO(a,N)}(t)\} = \frac{1}{2^N} \sum_{k \in \text{Even}}^N \binom{N}{k} \sum_{i=0}^{N-k} \sum_{m=0}^k (-1)^i \binom{N-k}{i} \binom{k}{m} e^{i(2i+l+m-N)\Omega_0 t} = \sum_{n=0}^{\lfloor N/2 \rfloor} A_n \cos(q_n \Omega_0 t) \]

\[ q_n \equiv 2n + \mu_N; \quad \mu_N \equiv \text{mod} \,(N, 2), \]
to derive an exact set of modal real-valued amplitudes $A_{q_0}$ and frequencies creating a real-valued superoscillatory signal. Here mod is the modulo operation.

A set of optical carrier modes with such amplitudes and frequencies would produce a temporal superoscillatory signal, which is quite a challenge as all the modes need to be phase locked as well as harmonics of a given fundamental frequency.

It is much easier to create a superoscillation in the envelope of a given pulse that is naturally made of phase-locked modes. An interference between two modes of slightly different frequencies produces a modulated envelope beating at the difference frequency of the two modes. Interference of several beat frequencies suited with the right amplitude ratios and phases would manifest a superoscillating beat (SOB)—an envelope with a temporal feature that breaks the temporal Fourier focusing limit.

For the following, we construct a SOB signal by first setting in Eq. (2) the parameters $N = 3$ ($M = 1$) and $\alpha = 2$, which gives

$$\text{Re}\{f_{\text{SO}(2,3)}(t)\} = -\frac{9}{4}\cos(\Omega_0 t) + \frac{13}{4}\cos(3\Omega_0 t). \quad (4)$$

This signal superoscillates at a rate of $6\Omega_0$. Next, the two cosine modes of Eq. (4) are mounted on a signal’s envelope according to a procedure outlined in Supplemental Material (Sec. I) [28], which maps each Fourier component to a beat constructed by two close-by-frequency components resulting in the following modes, amplitudes, and phases:

$q_k = \{-3,-1,+1,+3\}, \quad |A_{q_k}| = \{13/4,9/4,9/4,13/4\}, \quad \phi_{q_k} = \{0,\pi,\pi,0\}$. The beats spectral spacing $\Delta\omega$ is chosen such that $(2M + \mu_N)\Delta\omega$ fits within the available bandwidth. The frequency and time domain theoretical representations of this SOB signal are shown in Supplemental Material (Sec. I) [28] and in Fig. 1(a) below (in dashed lines).

Results.—Our experimental setup consists of a Ti:Sapphire femtosecond laser oscillator together with a home-built 4/f pulse shaper and a home-built frequency-resolved-optical gating (FROG) apparatus [31] used for pulse characterization (see the methods section in Supplemental Material [28]). Generally the SOB signal is prone to dispersion destructing the superoscillation after propagating a dispersion length of $(4\pi^2)/[GVD \times (N\Delta\omega/2)^2]$ where $GVD = \partial^2 k/\partial \omega^2 |_{\omega=\omega_0}$ and $N\Delta\omega/2$ is the bandwidth of the pulse. For the signals that we used, with bandwidth around 15 nm, the dispersion length in air (in Beta Barium Borate crystal) is in the order of one kilometer (centimeters), which is much longer than the optical path length we used (thickness of the crystal used in the FROG apparatus). Thus the distortions caused by dispersion could be ignored.

The first part of the experiment is dedicated to synthesizing a SOB pulse and comparing it to simple transform-limited pulses. Here the pulse shaper was first used to shape the original spectrum into a Gaussian and a rectangular shape, both with a flat spectral phase. The rectangle full width and the Gaussian full width at half maximum (FWHM) are 15 nm (7 THz).

In the time domain, the transform-limited Gaussian pulse has a FWHM of $140 \pm 5 \, \text{fs}$ while the transform-limited sinc pulse has the primary and secondary lobes FWHM of $169 \pm 5$ and $93 \pm 5 \, \text{fs}$ correspondingly. These numbers are within 97% (90%) of FWHM of ideal theoretical waveforms for the rectangular (Gaussian) pulse due to experimental imperfections in the waveform synthesis. The spectrogram (time-frequency FROG traces), retrieved spectrum, and retrieved temporal waveform of the Gaussian and sinc pulses are shown in Figs. 1(a) and 1(b), respectively.

Next, the pulse shaper is set to generate a single beat comprised of two modes set 15 nm apart [Fig. 1(c)]. This is a manifestation of the fastest possible single Fourier component within the given bandwidth and it is generally assumed that it gives the narrowest possible temporal features in the form of interference fringes. The fringes FWHM is $112 \pm 5$ fsec (which is off by 12% of the corresponding perfect waveform). This FWHM value is 80% of the Gaussian pulse’s FWHM and 66% of the sinc pulse’s central lobe FWHM. This is a very general and known result: the resolution available by a (spatial or temporal) double-slit interference is better than possible with a single-slit diffraction whose width is equivalent to the double-slit separation. This fact also came to fame with the introduction of Ramsey-fringes in atom interferometry [32].

Finally we synthesize the SOB signal given above for which we set the following: mode amplitudes are $|A_{\{-3,-1,+1,+3\}}| = \{13/4,9/4,9/4,13/4\} \times A_0$ ($A_0$ is a common amplitude factor); center frequencies of the modes are $\nu_{\{-3,-1,1,3\}} = \omega_{\{-3,-1,1,3\}} / (2\pi) = \{370.9,373.2,375.6,377.9\}$ THz; the frequency difference between adjacent modes is $\Delta\nu = 2.33$ THz ($\Delta\lambda \approx 5$ nm) and the mode phases are $\phi_{\{-3,-1,1,3\}} = \{0,\pi,\pi,0\}$.

The modes possess an approximate Gaussian form whose width $\Delta\nu$ is inversely proportional to an overall 0.8 ps pulse duration. The SOB spectrogram, frequency, and temporal waveforms are shown in Fig. 1(d) (continuous lines). It is evident that around time 0 a superoscillating feature emerges, with a FWHM of $48 \pm 4 \, \text{fs}$ (the half-maximum value was taken between the peak maximum and its adjacent minima). The SOB FWHM is approximately twice as narrow as the fringes of the corresponding single beat (double-slit) pattern, three times narrower compared with the transform-limited Gaussian pulse and 3.5 times narrower than the central lobe of the transform-limited sinc pulse.

Note that although both the SOB and single beat signals have spectral content extending beyond the designated spectral width of 15 nm (due to the finite bandwidth of each mode), it does not make the oscillating features within the envelope narrower, and so it is irrelevant to our result. This excess spectral content only limits the overall duration of the entire signal.
The ideal SOB signal is shown in a dashed line in Fig. 1(d). The agreement between the ideal and synthesized signal is quite good, especially for the superoscillating feature. The \(87/\sqrt{C_6^5}\) temporal full width delimiting the synthesized superoscillation corresponds to a 5.75 THz local frequency. This local frequency differs by 18% from the theoretical value of \(a \times N \times \Delta \nu = 2\frac{7}{4}\) THz, which is twice the corresponding single beat frequency (which by itself corresponds to the fastest Fourier component in the SOB spectrum). The visibility of this SOB is 30%.

Because superoscillations are an interference phenomena they rely on keeping the correct amplitude and phases of their constituting modes. Still, there is some resilience to changes (see Supplemental Material Sec. II [28]). We note that a theoretical treatment for the sensitivity of superoscillating signals to amplitude and phase changes was made in Ref. [27].

For SOB signals, the increase in local frequency comes at the expense of increased side lobes resulting in decreased visibility. For spatial superresolution the existence of large side lobes sets fundamental limits on the resolving power of the optical system [33]. However, this limitation becomes irrelevant when the narrow feature interacts with an isolated small enough object, especially for superoscillating functions for which the side lobes are relatively distant from the superoscillation (such functions were used for superresolution microscopy [13,14]).

For the function we chose to work with, it is possible to continuously tune the SOB waveform between better temporal focusing to better visibility of the superoscillating feature. Most simply this is done by changing the \(a\) parameter in Eq. (1) that sets the ratio between the superoscillating frequency to the highest frequency in the spectrum of the signal (see Supplemental Material Sec. II [28]).

Practically, when considering pulse shaping, to get a narrower SOB feature, better resolution and control is needed in the spectral domain to allow for synthesizing the required waveform. In addition, in this work, we have chosen to work with a specific family of superoscillatory functions described with Eq. (1). Alternatively it is possible to work with other superoscillatory functions that optimize the duration and amplitude of the superoscillating features [34].

Despite the obvious limitations mentioned above of superoscillatory wave functions, several recent works already proved experimentally that in microscopy, for some specific cases, such waveforms can outperform transform-limited beams, achieving superresolution [13,14,16]. In the second part of our experiment we use a straightforward temporal analogy to a spatial imaging process to demonstrate temporal superresolution using the SOB signal. A spatial imaging system is described through the convolution of a point-spread function and the object to be imaged. With a superoscillating point-spread function superresolution is achieved [14,16]. In our temporal case—the physical signal to be used in a generic measurement is an optical polarization vector proportional to the mixing of the SOB signal with a temporal event (test) signal \(g(t)\): \(P \propto f_{\text{SOB}}(t)g(t - \tau)\). Here \(\tau\) is the relative delay between the two real-valued signals. If we further assume that the overall interaction length is short, then a slow intensity detector would measure the cross-correlation signal \(S(\tau) = \int |f_{\text{SOB}}(t)g(t - \tau)|^2 dt\). In the experiment we...
used two different signal pulses: a transform-limited Gaussian pulse and a SOB signal having the same energy. The Gaussian FWHM bandwidth is equal to the SOB complete bandwidth, which is 15 nm (7 THz), clearly giving an advantage to the Gaussian pulse ability to resolve temporal features [these are the same signals depicted in Figs. 1(a) and 1(d)]. We used several temporal double-slit test signals comprised of two Gaussian pulses separated by $t_{\text{sep}} = 29, 43, 57, 71\ \text{fs}$ and having FWHM of 14 fs each.

In the experiment we synthesized the signal and the double-slit pulses with a delay of 971 fs between them, and sent them to our FROG apparatus, this time being used as an autocorrelator. We isolated from the full autocorrelation trace only the part that is the cross-correlation between the SOB/Gaussian signal and the double slit. The trace for the 200 fs around time 0 of the cross-correlation is shown in Fig. 2 for the different cases. Global or local minima at time 0 of the cross-correlation attest to the existence of two separated test pulses. Thus, the Gaussian pulse is able to resolve the existence of the double slit only for $t_{\text{sep}} = 57, 71\ \text{fs}$, while the SOB pulse is also able to resolve the existence of the double slit for $t_{\text{sep}} = 43\ \text{fs}$, proving superresolution. For $t_{\text{sep}} = 29\ \text{fs}$ both the Gaussian and the SOB pulse fail to resolve the double slit as there is a maximum in time 0 of the cross-correlation.

These results agree well with simulations we did (see Supplemental Material Sec. III [28]), which further analyze in which cases our SOB signal would outperform a transform-limited signal (this time with exactly the same bandwidth) in terms of resolving a temporal double slit. The conclusion is that whenever the double-slit separation is shorter than half the period associated with the diffraction limited signal, there exists a SOB signal that would outperform it. In the experiment, half the period associated with the 15 nm bandwidth of the Gaussian pulse is exactly $t_{\text{sep}} = 71\ \text{fs}$.

Two important notes are in order. First, were we to use a SOB signal that superoscillates faster (that is, has a larger $a$ parameter) we could resolve the double slit for the shortest $t_{\text{sep}}$ value (see Supplemental Material Sec. III [28]). Second, we note that if we were using a superoscillating signal with the superoscillation being more isolated from its adjacent side lobes (needing many more beat modes than the few we were able to use with our setup, and a different functional form [13,14]), then the whole structure of the temporal double slit could be reconstructed in the cross-correlation.

![FIG. 2. Resolving a temporal double slit with a SOB signal and a transform-limited Gaussian pulse having full bandwidth/FWHM bandwidth respectively of 15 nm (7 THz).](image-url)
not just the existence of a gap between the two pulses (this is akin to imaging of the double slit in the equivalent spatial case [14]). Regardless of this point, this part of the experiment constitutes a proof of concept for the ability of superoscillating signals to provide superresolution measurements in the time domain.

To conclude we have applied the concept of superoscillations to the temporal domain of ultrashort optical pulses. We experimentally demonstrated a superoscillating optical beat having a temporal fringe that is three times narrower than a Gaussian pulse whose FWHM equals its full bandwidth, breaking the temporal Fourier-transform limit given with transform-limited Gaussian pulses by 67% while maintaining visibility of 30%. Importantly, using this superoscillating signal, we experimentally demonstrated temporal superresolution detection of a temporal double slit. The superresolution achieved is in the sense of the Rayleigh and Abbe limit, without any improvement the observability of distinct temporal features in the time domain.

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The superresolution achieved is in the spirit of the Rayleigh and Abbe limit, without any improvement the observability of distinct temporal features in the time domain. Although it should not be expected that superoscillatory signals could gain superresolution for any general signal (e.g., for cases where the signal has a significant interaction with the side lobes of the superoscillatory function), we expect that our results are important for some of the cases requiring high temporal resolution, such as measuring a short duration isolated event or to induce dynamics on fast time scales. Thus our results can be relevant for many applications relying on ultrashort optical pulses such as spectroscopy, nonlinear optics, and metrology.

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