Breaking The Temporal Resolution Limit By Super Oscillating Optical Beat

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by

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Abstract

The ability to create ultra short optical pulses or narrow fringes is important to achieve better temporal resolution in measurements and control of various processes as well as for various applications. It is generally assumed that the Fourier transform limit which gives the minimum pulse duration for a given spectrum also sets the resolution limit. However, band-limited functions can actually oscillate locally at an arbitrarily fast rate through an interference phenomenon known as superoscillations.

In this thesis we experimentally demonstrate the breaking of the temporal Fourier-transform resolution limit by generating a superoscillating optical beat (SOB) having a temporal feature which is approximately four times shorter than the duration of a transform-limited Gaussian pulse having the same bandwidth, and oscillating twice as fast as the fastest Fourier component of the signal while maintaining 25% visibility. Numerical simulations demonstrate the ability of such signals to achieve temporal super-resolution.
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1 Introduction

Ultrashort optical pulses are electromagnetic pulses that have time duration in the order of picoseconds or less, these pulses have wide spectral bandwidth which is inversely proportional to the pulses duration for the simple pulses. In this thesis, we discuss the construction of a Superoscillating Optical Beat (SOB) which is an optical pulse that have a temporal feature that is arbitrary shorter than the duration of a transform-limited Gaussian pulse having the same bandwidth (albeit at the price of decreased amplitude at the fast oscillation).

In order to construct the SOB, we first needed to build a pulse shaping system that can shape our original short pulse to our desire. When dealing with short pulses, electronic devices no longer have the time resolution to measure or control such pulses. So, to control the pulse in the time domain we first transform the pulse to the spectral domain. Then, we can control each spectral component amplitude and phase, and thus, control the pulse shape in the time domain. In the first chapter, we discuss the pulse shaper theory and describe the construction of the experimental setup.

In the second chapter we discuss our measurement system. A home-made Second Harmonic Generation Frequency Resolved Optical Gating (SHG FROG) was built. The SHG FROG deals with the resolution problem by measuring the pulse with the pulse itself. This is done by splitting the beam and generating a second harmonic with the two split beams. The second harmonic pulse spectrum is measured for different delays between the split pulses. Then, The original pulse amplitude and phase are reconstructed by an iterative algorithm.

In the last chapter we discuss the SOB, first, we explain the basic theorem of superoscillation, then, the theoretical derivation of the SOB is explained and the
experimental results are presented. Lastly, we show numerical simulations that demonstrate the ability of such signals to achieve temporal super-resolution.
2 Pulse Shaping

Pulse shaping systems are used to manipulate the waveform of optical pulses. Controlling an ultra short pulse (in the order of picoseconds or less) in the time domain is a very hard procedure, and there is no known routine to shape a pulse with good enough resolution. Alternately, if we look at a short pulse at the frequency domain, we can get a very good resolution by using a Spatial Light Modulator (SLM) in the Fourier plane of a 4f system, with the SLM we can control the phase and amplitude of each spectral component and thus get a desired temporal pulse [1]. A basic pulse shaper layout is presented in Fig. 2.1

In the last decades, Fourier-domain pulse shaping has become a widely common tool for many applications [2] such as spectroscopy and coherent control [3, 4], metrology [5], microscopy [6] and optical communications [7]. In this chapter we review the basic theorem of pulse shaping and describe our experimental setup.
2.1 Theory

As we can see in Fig. 2.1, the pulse shaper apparatus consist of a grating which disperse the pulse to it’s spectral components. Then, a lens collimate the beam and focuses each spectral component at what is called the Fourier plane. At that plane, a SLM is placed to control the amplitude and phase of each spectral component. Afterward, a second set of lens and grating recombine all the spectral components to a single collimated beam. If all elements are placed correctly, the original pulse is rebuild and the apparatus is called a “zero dispersion pulse compressor”[8].

At the Fourier plane, each spectral component $E_{in}(\omega)$ passes through an SLM pixel and a linear polarizer, and is thus multiplied by a response function $H(\omega)$:

$$E_{out}(\omega) = E_{in}(\omega)H(\omega)$$  \hspace{1cm} (2.1)

$H(\omega)$ has an absolute value smaller then one (only attenuation) and a phase range which can vary depending on $\omega$ and the SLM model (in some SLM models the only variable that can be controlled is the phase). So, in order to get a certain pulse $e_{out}(t)$, we can look at this pulse in the spectral domain $E_{out}(\omega) = \int_{-\infty}^{\infty} e_{out}(t)e^{-i\omega t}dt$. Then, we can modulate the SLM accordingly by changing Eq. 2.1 to:

$$H(\omega) = E_{out}(\omega)/E_{in}(\omega)$$  \hspace{1cm} (2.2)

Indeed, we can get a desired pulse with the main limitation being our original spectrum width and the resolution in the Fourier plane.

2.2 Experimental Setup

The pulse shaper experimental setup is presented in Figures 2.2 and 2.3. The beam, which is centered around 800nm and has about 60nm Full Width Half Maximum (FWHM), enter the system through a silver mirror and then hit a 1200 lines/mm holographic grating (custom made by Richardson Gratings), the grating separate the beam’s spectral components into different directions.
A 5 cm square mirror ("folding mirror") then changes the beams direction into a 35 cm focal length cylindrical mirror (8 cm long, custom made by CVI Laser Optics), the mirror then collimates the beams and focuses each of them after 35 cm into a thin line at the Fourier plane. We chose to focus the beams with a mirror and not with a lens in order to minimize dispersion induced chirp. A cylindrical mirror was chosen instead of parabolic one in order to minimize the peak intensity at the Fourier plane, high peak intensity would have damaged our SLM.

At the Fourier plane, a one dimensional, dual-mask SLM (Jenoptik SLM-S640d) is placed to control each of the spectral components. Then, a 180 degree rotated system collimate the beam back together. The cylindrical mirrors are placed on an xy linear stage and each of the gratings is placed on a linear and a rotating stage. A polarizer then selects the polarization component that was shaped by the SLM and the desire pulse is created.

In order to align the system, we first started working with a 780 nm cw beam. To correctly place the elements, we calculated where the 780 nm cw beam is sup-
posed to hit the folding mirror and the cylindrical mirror. The assumption was, that 800nm is the central wavelength and thus, it needs to hit the middle of the folding mirror and the cylindrical mirror. Then, we placed the elements at their designated positions and made sure that all the spectral components can pass the entire pulse shaper without being physically blocked by the system elements.

Another use for the 780nm cw beam is to confirm that all the setup components are parallel to the optical table. To align the grating, we needed to make sure that both the reflected zero order and the first one are parallel to the optical table. The cylindrical mirror alignment was made by verifying that the reflected beam is parallel to the table no matter where the incoming beam hits the mirror (otherwise the cylindrical mirror is tilted horizontally).

In order to check if the pulse shaper is aligned properly, the most important things to ensure are the prevention of spatial and temporal chirp at the system output. To check the spatial chirp, we let the beam propagate a long distance (few meters) and project it on a screen, we then pass a thin object through the Fourier plane, if there is a spatial chirp, a shadow moving on the beam from side to side will be observed. To minimize the temporal chirp, we measure the output pulse
via autocorrelator and try to minimize the pulse duration by adjusting the linear stages. The most important distance parameter is the one between the two gratings [9].

The SLM contains two separate controllable pixeled liquid crystal arrays distanced 3mm apart from each other. In order to avoid a spatial offset between the two pixel arrays which decreases the spectral resolution, it is imperative to affirm that the SLM is perpendicular to the laser beam. After this is verified, each pixel has to be assigned a matching wavelength. This was done by a LabView program we have created for this purpose. The program sets $H = 1$ to a very small batch of close display pixels (allows full transfer) at a time, finds the wavelength that gives the maximum intensity of the spectrum and then matches between the center pixel and the corresponding wavelength. This calibration is done only one time after the initial alignment.

To generate the desired pulse we have designed a collection of Matlab and LabView scripts that control the SLM as explained in the theory section. $E_{in}(\omega)$ is taken from a FROG measurement of a pulse after the pulse shaper was set with $H(\omega) = 1$. 
3 Frequency Resolved Optical Gating

In order to measure an event in time, you must use a shorter one. But then, to measure the shorter event, you must use an even shorter one. And so on. So, now, how do you measure the shortest event ever created? [10]

In the last few decades the use of short pulses have increased rapidly and with it the necessity to know the exact shape of the pulse [11–15]. This challenge is addressed by letting a given pulse measure itself.

At first, the techniques that have used the pulse to measure itself have yielded blurry pictures of the pulse, this is similar to the use of a slow camera shutter that yields a blurry picture of a moving object.

In 1991 Daniel J. Kane and Rick Trebino introduced The Frequency Resolved Optical Gating (FROG) method which can measure the full time-dependent intensity and phase of ultrashort light pulses in a wide variety of circumstances. Other techniques are Spectral Phase Interferometry for Direct Electric-field Reconstruction (SPIDER)[16] and Multiphoton Intrapulse Interference Phase Scan (MIIPS)[17] which we won’t describe here.

3.1 Theory

The FROG is a correlation based technique that is capable of extracting both amplitude and phase information about the pulse [18].
\[ E(t) = \text{Re}[\sqrt{I(t)} \exp(i\omega_0 t - i\phi(t)) ] \] (3.1)

where \( I(t) \) and \( \omega(t) \) are the time-dependent intensity and phase of the pulse, and \( \omega_0 \) is a carrier frequency. The heart of the FROG is the measurement of the spectrogram:

\[ S(\omega, \tau) = \left| \int_{-\infty}^{\infty} E(t) g(t - \tau) e^{-i\omega t} dt \right|^2 \] (3.2)

Where \( g(t - \tau) \) is a gate pulse delayed by time \( \tau \) relative to the optical pulse \( E(t) \). In this work we discuss only the case where the gate pulse is a replica of the electrical pulse, this FROG version is called a Second Harmonic Generation (SHG) FROG. So, for SHG FROG we get the "SHG FROG trace" [19]:

\[ I_{\text{SHG FROG}}^{\text{SHG}}(\omega, \tau) = \left| \int_{-\infty}^{\infty} E(t) g(t - \tau) e^{-i\omega t} dt \right|^2 \] (3.3)

In fig.3.1 we can see a typical setup which gives the desired SHG FROG trace. Now, we need to find an algorithm to retrieve the original pulse \( E(t) \) from the measured \( I_{\text{SHG FROG}}^{\text{SHG}}(\omega, \tau) \). to do this we need to think of the problem as a "two-dimensional phase-retrieval problem."[20] We begin by marking \( E_{\text{sig}}(t, \tau) = E(t) g(t - \tau) \) and considering the Fourier transform of this function with respect to \( \tau \): \( \hat{E}_{\text{sig}}(t, \Omega) \), it is easy to see that if we know \( \hat{E}_{\text{sig}}(t, \Omega) \) we can simply find \( E(t) \) by looking at \( E(t) = \alpha \hat{E}_{\text{sig}}(t, \Omega = 0) \) where \( \alpha \) is a proportionality constant that is of little interest to us.

With this substitution we can now rewrite Eq.3.3 as:

\[ I_{\text{FROG}}^{\text{SHG}}(\omega, \tau) = \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{E}_{\text{sig}}(t, \Omega) e^{-i\omega t - i\Omega \tau} dt d\Omega \right|^2 \] (3.4)

Here, we can see that the SHG FROG trace \( I_{\text{FROG}}^{\text{SHG}}(\omega, \tau) \) is the squared magnitude of the 2D Fourier transform of \( \alpha \hat{E}_{\text{sig}}(t, \Omega = 0) \), thus, the only thing missing to find \( E(t) \) is the phase of \( \alpha \hat{E}_{\text{sig}}(t, \Omega = 0) \). This problem was solved in the field of image processing and is called the 2D phase-retrieval problem [21]. In our system we used the generalized projections method [19].
3.2 Experimental Setup

In Fig.3.2 the experimental setup we built is presented, the schematics of this setup is described in Fig.3.1. The pulse enters the system through a silver mirror and then splits to two replicas in a low Group Delay Dispersion (GDD) ultrafast beam-splitter (UFBS5050 by Thorlabs). In order to keep both of the pulses equal, the reflected beam goes through an infrasil window to balance the GDD (UDP10 by Thorlabs).

The beam that goes through the beam splitter goes to a delay stage (MFA-PPD by Newport) that is controlled by a computer and sets the $\tau$ parameter in Eq.3.3. Then, the beams are focused by an off axis parabolic mirror (MPD149-P01 by Thorlabs) on an SHG crystal (BBO-600H by Eksma optics). Thus, a second harmonic pulse $E(t)E(t-\tau)$ is created and measured by a spectrometer at different $\tau$.

In order to correctly convert the delay stage movement into the time delay $\tau$,
the stage motion has to be parallel to the input and output rays. For high precision, this can be checked by inserting the input beam into a lens that is placed on the stage. If the alignment isn’t good, the beam spot would move on the lens with the stage movement. And so, if we look far away from the lens, we will see a significant movement of the beam for different stage displacements. A similar procedure can be done for the output beam with the lens placed in front of the output beam spot.

Another alignment step worth noting is the beams’ focusing, we first needed to verify that the beams arrive parallel to the off axis parabolic mirror, then, we checked that the beams meet after 4” at a 90 degree angle. If the beams focus on the same spot on the SHG crystal and the optic path of the two beams is about the same, a second harmonic will be generated.

From Eq.3.3 we conclude that for an ideal SHG FROG the SHG trace is symmetric in $\tau$. This fact can help verify that the FROG alignment is good. For example, if we remove the compensating infrasil window, the beam that passes through the beam splitter gains more GDD than the reflected one, and so, the SHG trace is no longer symmetric. In Fig.3.3 we can see that an SHG trace taken...
with a compensating infrasil window is highly symmetric (Fig.3.3.a) and a trace measurement taken without a compensating window has a small tilt (Fig.3.3.b). For a normal beam splitter (and not a low GDD one), the tilt would have been bigger because of the higher GDD gained by the passing beam.

Figure 3.3: FROG traces with (a) and without (b) a compensating window. The dotted line is placed on the zero delay and the dot dashed line represent the trace angle

A Labview program was written to perform the trace measurement. A Matlab script by Adam Wyatt which was modified by us, takes the raw data and produces the reconstructed field envelope shape and phase profile. Fig.3.3 shows a final result of our FROG system for a Gaussian pulse.
Figure 3.4: A typical measurement of our FROG that include the original measured SHG FROG trace, the constructed trace after 1252 iteration with 0.755% error, and the constructed intensity (blue, normalize to the $[0, 2\pi]$ range) and phase (green) both in time and in wavelength.
A main constraint for any pulse shaping system is the original pulse bandwidth which sets a limit on the minimal possible pulse length. This limit is usually described using a duration-bandwidth product, stating that for a given bandwidth the minimal pulse duration would be achieved when the spectral phase of the pulse is at most linear [22]. Such a pulse is termed Fourier-transform-limited.

However, band-limited signals can actually oscillate locally at an arbitrarily fast rate (albeit at the price of decreased amplitude at the fast oscillation), thus breaking the Fourier-transform limit resolution-wise. This is achieved through an interference phenomenon now known as superoscillation [23]. Superoscillatory functions are known since the early 1920 when attempts were made to produce antennas with extremely narrow radiation patterns (see ref.[24] for an historical account).

The concept of functions that oscillate arbitrarily fast has revived with the introduction of quantum weak measurements [25], where a measurement can yield values much larger than the largest eigenvalue of an observable. Berry and Popescu have introduced this concept of superoscillatory functions into optics, predicting their use for spatial super-resolution [26]. Following this prediction, super-resolution with superoscillatory spatial optical waveforms was demonstrated experimentally to break the optical diffraction limit in several works [27–30]. Superoscillatory diffraction-free beams were predicted [31], and later demonstrated experimentally [32]. Lately, a superoscillatory waveform in momentum space of a nonlinear photonic crystal was shown experimentally to yield super-narrow frequency conversion [33].

In the temporal regime, relatively little has been done with attempts to break
the temporal resolution limit. Two experiments done in 2006 and in 2005 demonstrated the use of quadratic spectral phase to break the Fourier resolution limit by 20% [34] and 30% [35] respectively. A theoretical work discussed breaking the Fourier limit by super-resolution pulse compression techniques [36]. In the radio-frequency regime temporal superoscillations were successfully demonstrated in 2011 [37]. In addition it was suggested that optical temporal superoscillations can be used to overcome absorption in dielectric materials, thus achieving optical super-transmission [38].

In this chapter we experimentally demonstrate the breaking of the temporal resolution limit of an ultra-short optical pulse by synthesizing the envelope function of the pulse waveform in the form of a Superoscillatory Optical Beat (SOB). In particular, we achieve a temporal feature which is approximately four times narrower than the corresponding width of a Fourier limited Gaussian pulse having the same bandwidth, while maintaining a visibility (the ratio between the amplitude of the narrow feature to the amplitude of its adjacent fringes) of 25%. Numerical simulations further demonstrate the ability of SOB signals to achieve super-resolution in the time domain.

4.1 Superoscillation

Berry and Popescu [26] has investigated the following family of complex superoscillating functions:

\[ f_{SO}(t) = \left[ \cos(\Omega_0 t) + i\alpha \sin(\Omega_0 t) \right]^N \quad a > 1, \quad N \in \mathbb{N}^+ \tag{4.1} \]

and showed that while the highest Fourier transform component of Eq. 4.1 is \( N\Omega_0 \), around \( t \approx 0 \) the function superoscillates \( a \) times faster, at the rate of \( aN\Omega_0 \):

\[ f_{SO}(t \ll \pi/\Omega_0) \approx \exp \left[ N\log(1 + i\alpha\Omega_0 t) \right] \approx \exp(iaN\Omega_0 t) \tag{4.2} \]

It is possible to expand the real part of Eq. 4.1 into the following binomial expansion and Fourier cosine series:
and derive an exact set of modal amplitudes and frequencies for a real-valued superoscillatory signal. Here \( \text{mod} \) is the modulo operation, the \( A_{q_n} \) amplitudes are real valued.

### 4.2 SOB Theory

In theory, a set of optical carrier modes bearing the above amplitude ratios and carrier frequencies can produce a superoscillatory signal in time where the interfered waveform would superoscillate. In practice it is quite a challenge due to the requirement of using mutually coherent and phase locked optical sources which are harmonics of a given fundamental frequency.

It is much easier, however, to create a superoscillation in the envelope of a given pulse as naturally all the modes creating the envelope are phase locked. An interference between two modes of slightly different frequencies produces a periodic modulation of the signal’s envelope in time. Such a modulation is known as a "Beat" where the beat frequency is the difference frequency between the frequencies of the two modes. Interference of several beat frequencies suited with the right amplitude ratios and phases would manifest a SOB - an envelope with a temporal feature which breaks the temporal Fourier focusing limit.

The SOB signal is the sum of \( M + 1 \) beats, where those who have a beat frequency different than zero are composed of two modes having the same amplitude
and phase:

\[ f_{SOB}(t) = \left( 1 - \mu_N \right) B_0 \cos(\omega_0 t + \phi_0) + \]
\[ + \sum_{m=-M}^{M} \left( 1 - \delta_{m,0} \cdot \mu_N \right) B_m \cos(\omega_m t + \phi_m) \]
\[ B_m = B_{-m}, \quad \phi_m = -\phi_{-m}, \quad \omega_0 = \omega_c \]
\[ = 2 \left( 1 - \mu_N \right) \cdot B_0 \cos(\omega_0 t + \phi_0) + \]
\[ + 2 \left( 1 - \mu_N \right) \cdot B_0 \cos \left( \frac{\omega_m + \omega_m}{2} \right) t \cdot \cos \left( \frac{\omega_m - \omega_m}{2} \right) t + \phi_m \]

Here \( \delta_{i,j} \) is the Kronecker delta function and \( \omega_c \) is a carrier frequency within the bandwidth of the pulse. In addition the \( B_m \) amplitudes are positive valued. The beats are chosen to have the same mean frequency, while their beat frequencies are integer multiplication of an arbitrary fundamental beat frequency:

\[ \forall m: \quad \frac{\omega_m + \omega_m}{2} = \omega_c \quad ; \quad \frac{\omega_m - \omega_m}{2} = \left( \frac{2m - \mu_N}{2} \right) \Delta \omega \]

These reduce Eq. 4.5 to:

\[ f_{SOB}(t) = 2 \cos(\omega_c t + \phi_0) \cdot \]
\[ \cdot \left( 1 - \mu_N \right) \cdot B_0 + \sum_{m=1}^{M} B_m \cos \left( \left( \frac{2m - \mu_N}{2} \right) \Delta \omega \cdot t + \phi_m \right) \]

Here \( \cos(\omega_c t + \phi_0) \) is the common carrier signal of the beats that oscillates at the frequencies \( \left( \frac{2m - \mu_N}{2} \right) \Delta \omega \). Provided that the beats’ amplitudes and phases are determined by Eq. 4.3 i.e. \( B_m \equiv |A_{q_m - \mu_N}| \quad ; \quad \phi_m = \frac{\pi}{2} \cdot (1 - \text{sgn}(A_{q_m - \mu_N})) \) (where \( \text{sgn} \) indicates the sign function), the envelope of \( f_{SOB}(t) \) would be superoscillating. While the highest beat frequency of the envelope is bounded by \( \left( \frac{2M + \mu_N}{2} \right) \Delta \omega \), the envelope would locally superoscillate at a higher beat frequency of \( a \left( \frac{2M + \mu_N}{2} \right) \Delta \omega \).

In practice, the modes constituting the SOB would have some spectral width, inducing a finite envelope width \( \sigma_t \) for the temporal SOB signal while essentially not modifying the superoscillating frequency. In this case the SOB signal and its
spectrum are given by:

\[ f_{SOB}(t) = 2\exp\left(-\frac{t^2}{2\sigma^2}\right)\cos(\omega_c t + \phi_0) \cdot \left((1 - \mu_N) |A_{q_0}| + \sum_{m=1-\mu_N} M |A_{qm}| \cos \left(\frac{2m + \mu_N}{2} \cdot \Delta \omega \cdot t + \phi_{qm}\right)\right) \]  

\( F_{SOB}(\omega > 0) = \frac{\sqrt{2\pi\sigma^2}}{2} \cdot \sum_{k=\pm\delta\mu_N} M |A_{q_k}| \exp \left(-\frac{1}{2}\sigma^2 \left(\omega - \left[\omega_c + \frac{2k + \mu_N}{2}\right] \Delta \omega\right)\right)^2 + i\phi_{q_k} \)  

We construct a SOB signal by first setting in Eq. 4.1 the parameter \( N = 3 (M = 1) \). This gives us:

\[ \text{Re}\{f_{SO(a,3)}(t)\} = \frac{3(1 - a^2)}{8} \cos(\Omega_0 t) + \frac{1 + 3a^2}{8} \cos(3\Omega_0 t) \] (4.10)

To which we set the arbitrary value of \( a = 2 \):

\[ \text{Re}\{f_{SO(2,3)}(t)\} = -\frac{9}{8} \cos(\Omega_0 t) + \frac{13}{8} \cos(3\Omega_0 t) \] (4.11)

The Fourier representation for positive frequencies of this signal is depicted in Fig. 4.1.(a) while its time domain form is given in Fig. 4.1.(b). Together with this function we also depict a cosine oscillating at the highest Fourier component of the signal \( 3\Omega_0 \), and a cosine oscillating at the superoscillation frequency \( 6\Omega_0 \). It is apparent that the signal superoscillates around time zero.

Next, the two cosine modes of Eq. 4.11 are mounted on a signal’s envelope according to a procedure outlined in Eq.4.4 to Eq.4.9 which results in the following modes’ amplitudes and phases: \( |A_{q_k}| = \{\frac{13}{8}, \frac{9}{8}, \frac{9}{8}, \frac{13}{8}\} \), \( \phi_{q_k} = \{0, \pi, \pi, 0\} \), \( q_k = \{-3, -1, +1, +3\} \). \( \Delta \omega \) and \( \omega_c \) are chosen such that \( (2M + \mu_N)\Delta \omega \) fits within the available bandwidth and \( \omega_c \) is the designated carrier frequency. Thus the SOB has been generated. Its frequency domain and its time domain representations are
Figure 4.1: Construction of a superoscillatory optical beat. A real valued superoscillatory (SO) function is first defined through its Fourier modes which are harmonic multiples of a fundamental frequency. These modes are then reflected around a central carrier frequency to generate the superoscillating-optical-beat (SOB): a superposition of beat frequencies with a superoscillatory envelope function. (a) The positive frequency components of the SO signal (b) Temporal waveform of the SO signal (thick continuous purple line) together with its fastest Fourier component (dashed red line) and the Fourier component corresponding to the superoscillation (dot-dashed blue line). (c) The positive frequency components of the SOB signal. (d) The temporal waveform of the SOB (continuous blue line) together with a trace of the superoscillating envelope (thick continuous purple line) and the pulse finite envelope (dashed red line) which is due to the finite width of the constituting Fourier modes.
shown in Fig. 4.1.(c) and Fig. 4.1.(d) respectively.

4.3 Experimental Results

Our experimental setup consists of a Ti:Sapphire femtosecond laser oscillator together with a 4f pulse shaper and a FROG apparatus used for pulse characterization (see Fig. 4.2). All FHWM measurements were done using a 2nd order polynomial fit over the retrieved waveforms. Indicated uncertainties in experimentally retrieved values are based on the temporal and spectral resolution of our FROG apparatus.

We note that the SOB signal is prone to dispersion and the superoscillating feature would be destroyed after propagating a dispersion length $4\pi^2 GV D \cdot (N\Delta \omega/2)^2$ where $GV D = \frac{\partial^2 k}{\partial \omega^2} |_{\omega = \omega_0}$ (Group Velocity Dispersion) and $N\Delta \omega/2$ is the bandwidth of our pulses in $rad/sec$. For the signals that we used, the bandwidth is around 8 nm and the dispersion length in air is in the order of one kilometer (which is much longer than the optical path length we used) and in the BBO crystal used in the FROG apparatus it is in the order of centimeters (much longer than the thickness of the crystal). Thus the distortions caused by dispersion could be ignored.

The experiment went as follows: first, the pulse shaper was used to shape the original spectrum into a Gaussian and a rectangular shape, both with a flat spectral phase and with the same full width of 15 nm (7 THz). In this case the Gaussian has a full-width at half maximum (FWHM) of 7.9 nm (3.7 THz).

In the time domain, the transform limited Gaussian pulse feature has a FWHM of $245 \pm 9$ fs with good agreement to $239$ fs in theory, while the transform limited Sinc pulse has the primary and secondary lobes FWHM of $169 \pm 5$ fs and $93 \pm 5$ fs correspondingly (172 fs and 95 fs in theory). Our FWHM measurements are within 97% of FWHM of ideal theoretical waveforms due to experimental imperfections in the waveform synthesis. The FROG traces and the retrieved spectrum and temporal waveform of the Gaussian and Sinc pulses are shown in Fig. 4.3.(a) and Fig. 4.3. (b) respectively.

Next, the pulse shaper is set to generate a single beat comprised of two modes set 15 nm apart (Fig. 4.3.(c)). This is a manifestation of the fastest possible single
Figure 4.2: Experimental setup. The pulses emitted by an ultra-fast laser oscillator are shaped in a 4f Fourier domain pulse shaper. The shaped pulses amplitude and phase are retrieved through a measurement in a FROG apparatus. M=Mirror, CM=Cylindrical Mirror, G=Grating, BS=beam Splitter, PM=Off-axis Parabolic Mirror, SHGC=Second-Harmonic-Generation Crystal, B=Beam Blocker.
Figure 4.3: Transform-limited signals and a super-oscillating beat. Experimental measurements of SHG FROG time-frequency spectrograms (left column) and the retrieved signals in the frequency (middle column) and time (right column) domains for four different signals sharing the same bandwidth (which is delimited by purple dot-dashed lines): (a) transform-limited Gaussian pulse (b) transform limited Sinc pulse (c) highest frequency beat signal (d) super-oscillating optical beat (SOB). The superoscillation is circled with a dot-dashed line.
Fourier component within the given bandwidth and it is generally assumed that it gives the narrowest possible temporal features in the form of interference fringes. The fringes FWHM is \(112 \pm 5\) fs (which is off by \(12\%\) of the corresponding perfect waveform). This FWHM value is \(46\%\) of the Gaussian pulse’s FWHM and \(66\%\) of the Sinc pulse’s central lobe FWHM. This is a very general and known result: the resolution available by a double-slit interference (whether in space or in time) is better than possible with a single slit diffraction whose width is equivalent to the double-slit separation. This fact also came to fame with the introduction of Ramsey-fringes in atom interferometry [39].

Finally we synthesize with the pulse shaper the SOB signal according to the recipe and example given above. For this we set \(\alpha = 2\) and \(N = 3\) as in Eq. 4.11 and then according to Eq. 4.9, the required temporal beat shape was constructed using the following modes’ amplitudes and phases: \(|A_{\{-3,-1,+1,+,+3\}}| = \left\{ \frac{13}{8}, \frac{9}{8}, \frac{9}{8}, \frac{13}{8} \right\} \cdot A_0; \ \nu_{\{-3,-1,+1,+,+3\}} = \frac{\omega_{\{-3,-1,+1,+,+3\}}}{2\pi} = \{370.89, 373.22, 375.56, 377.89\} THz; \ \Delta \nu = 2.334 \ THz \ (\Delta \lambda \approx 5 \ nm) ; \ \phi_{\{-3,-1,+1,+,+3\}} = \{0, \pi, \pi, 0\}.\) Here \(|A_{\{-3,-1,+1,+,+3\}}|\) are the amplitudes of the modes with \(A_0\) as a common amplitude factor, \(\nu_{\{-3,-1,+1,+,+3\}}\) are the center frequencies of the modes. \(\Delta \nu\) is the frequency difference between adjacent modes. \(\phi_{\{-3,-1,+1,+,+3\}}\) are the corresponding required phases for each mode.

The modes possess an approximate Gaussian form whose width \(\Delta \nu\) is inversely proportional to an overall 0.8ps pulse duration. The SOB spectrogram, frequency and temporal waveforms are shown in (Fig. 4.3.(d)). It is evident that around time zero a superoscillating feature emerges, with a FWHM of \(58 \pm 5\) fs. The SOB FWHM is approximately twice as narrow as the fringes of the corresponding single beat (double-slit) pattern, four times narrower compared with the transform limited Gaussian pulse and 2.9 times narrower than the central lobe of the transform-limited Sinc pulse.

Note that both the SOB and single beat signals have modes that, due to their finite spectral width, have spectral content extending beyond the designated spectral width of 15 nm. Essentially, for these signals, this spectral broadening does not contribute to narrower features, it just limits the overall extent of the signals in the time domain. Conversely if we were to take a Gaussian whose spectral full width extends beyond 15 nm it would have resulted in a narrower pulse. For ex-
ample, a Gaussian spectrum whose FWHM (and not its full width) is the same 15 nm would give a pulse whose temporal width is 125 fs. This is still wider by 2.15 times than the SOB superoscillation.

The difference between the theoretical ideal SOB signal and the one that was synthesized can be seen in Fig. 4.4(a) where the agreement is quite good. The differences, including the reduced visibility of the super-oscillation, are mostly due to the difference between the required spectral phase function to the one applied by the pulse-shaper. The $87 \pm 5$ fs temporal full-width delimiting the synthesized superoscillation corresponds to a $5.75 \text{ THz}$ local frequency. This local frequency differs by 18\% from the theoretical value of $a \cdot N \cdot \Delta \nu / 2 = 7 \text{ THz}$, which is twice the corresponding single beat frequency (which by itself corresponds to the fastest Fourier component in the SOB spectrum). The visibility of this SOB is 25\%.

Because superoscillations are an interference phenomena they rely on keeping the correct amplitude and phases of their constituting modes. Still, there is some resilience to changes. For example, we have decreased the lowest beat’s phase by 20\% and measured the resulting temporal shape (see Fig. 4.4(b)). In this case the measured FWHM is $72 \pm 5$ fs which is wider by approximately 20\% than the full width of the unmodified SOB signal. We note that a theoretical treatment for the sensitivity of superoscillating signals to amplitude and phase changes was made in Ref.[38].

A more acute modification in the modes - such as equalizing their phases would completely ruin the superoscillation (see Fig. 4.4(c)). This last case is illuminating: it is very well known that a signal is transform limited when its spectral phase is linear as this condition minimizes the overall root-mean-square width of the pulse [1]. Thus a linear spectral phase minimizes a global feature of the pulse - its overall width, as is the case seen here in Fig. 4.4(c) (as well as in the examples shown in 4.3(a)-(c)). In contrast, when a super-oscillating function is constructed - the spectral phase is no longer linear - thus the overall width of the pulse is not minimized. What we gain, however, is a fringe (or several fringes) which is (are) narrower than the fringes of a transform limited pulse. In view of this, for super-oscillatory functions, the optimization, instead of being global is a local optimization: narrowing a given fringe while keeping the magnitude of its surrounding side-lobes as low as possible.
Figure 4.4: Phase modifications of a SOB signal. Experimentally retrieved waveforms (solid lines) vs. optimal theoretical waveforms (dashed lines) in the frequency domain (Left) and the time domain (right) for three different instances of a SOB signal: (a) original SOB signal (b) modified by lowering the lowest beat phase by 20% (c) modified to have a flattened spectral phase.
For SOB signals, the increase in local frequency comes at the expense of increased side lobes resulting in decreased visibility. For spatial super-resolution the existence of large side lobes sets fundamental limits on the resolving power of the optical system [40]. However, this limitation becomes irrelevant when the narrow feature interacts with an isolated small enough object, or when the side lobes can be cut by the application of a nonlinear filter. In microscopy, this means the use of a pupil close to the scanned object. The pinhole projects the superoscillation into a real Fourier component. Thus the super resolving spot becomes evanescent but without the side-lobes. We would gain resolution (compared to illuminating the pinhole with a plane wave) when the superoscillating spot is smaller than the diameter of the pinhole, while the pinhole still cuts the side lobes. Similarly for the time domain - the effects of the side-lobes can be mitigated when interacting with isolated short events, or when an additional time gating could be used. Another interesting possibility would be the use of pre-processing for repeated applications of the SOB waveform, while changing one of its parameters, for isolating the effect of the superoscillating feature (see e.g. Ref.[41]).

Regarding the above mentioned trade off it is theoretically possible to continuously tune the SOB waveform between better temporal focusing to better visibility of the superoscillating feature. Most simply this is done by changing $a$ in Eq. 4.2 which sets the ratio between the superoscillating frequency to the highest frequency in the spectrum of the signal. This is shown in Fig. 4.5(a) where keeping the same number of modes (with $N=3$) and their spectral width while continuously changing $a$ between 1 to 6 results in gradually increased temporal focusing and decreased visibility. We realized experimentally three instances of the SOB with different values of $a = \{1.63, 2, 2.5\}$. The SOB with $a = 2$ was shown already in Fig. 4.3 and Fig. 4.4 where the FWHM of the superoscillation was $58 \pm 5$ fs and the visibility was 25%. Fig. 4.5(b) shows a SOB signal with $a = 1.63$ where the superoscillation FWHM and visibility are $78 \pm 5$ fs and 41% respectively, while Fig. 4.5(c) shows a SOB signal with $a = 2.5$ where the superoscillation FWHM and visibility are $45 \pm 4$ fs and 16.8% respectively.

Practically, when considering pulse shaping, to get a narrower SOB feature, better resolution and control is needed in the spectral domain to allow for synthesizing the required waveform. In addition, in this work, we have chosen to
Figure 4.5: Tuning the SOB signal between better resolution to better visibility. (a) Numerical modification of the $a$ parameter in the Frequency domain (left column) and in the Temporal domain (right column). The increase in $a$ results in better temporal resolution of the SOB signal at the expense of visibility. Notice that the super-oscillating portion of the waveform is around time zero. The dashed white lines indicates the $a$ values for which waveforms where experimentally retrieved (solid lines) and compared with optimal theoretical waveforms (dashed lines) as shown in the frequency domain (Left) and the time domain (right) for: (b) $a = 1.63$ (c) $a = 2.5$. 
work with a specific family of superoscillatory functions described with Eq. 4.1. Alternatively it is possible to work with other superoscillatory functions which optimize the duration and amplitude of the superoscillating features [42].

4.4 Temporal Super-Resolution With SOB Signals

Despite the obvious limitations mentioned above of superoscillatory wave-functions, several recent works already proved experimentally that in microscopy such waveforms can outperform transform limited beams, achieving super-resolution [28, 29]. In the last part of this work we bring the results of numerical simulations showing this is also the case for the temporal counterpart under consideration here. The analogy with the spatial case is quite straightforward.

A spatial imaging system is described through the convolution of a point-spread-function and the object to be imaged. With a superoscillating point-spread-function super-resolution is achieved [29]. In our case - the physical signal to be used in a generic measurement would be an optical polarization vector proportional to the mixing of the SOB signal with a temporal event signal \( g(t) \): \( P \propto f_{SOB}(t)g(t - \tau) \). Here \( \tau \) is the relative delay between the two real-valued signals. If we further assume that the overall interaction length is short, then a slow intensity detector would measure the cross-correlation signal \( S(\tau) = \int [f_{SOB}(t)g(t - \tau)]^2 dt \).

We wish to analyze the detection of a temporal double-peak modeled as two separate Gaussian pulses: \( g(t) = \left[ \exp \left( -\frac{(t-t_{sep}/2)^2}{2\sigma^2} \right) + \exp \left( -\frac{(t+t_{sep}/2)^2}{2\sigma^2} \right) \right] \cdot \cos(\omega_g t) \) with a carrier frequency \( \omega_g \). For the following we fix \( \sigma = 0.15 \cdot t_{sep} \). In the simulations we set the carrier frequencies of both the SOB signal and \( g(t) \) to zero to factor out the fast oscillations associated with the carrier frequency of the polarization (formally this is equivalent to the application of a low-pass filter to the cross-correlation).

We numerically calculated the cross-correlation for various values of \( t_{sep} \) and for various SOB signals by modifying the \( a \) parameter for two values of the \( N \) parameter: \( N = 3, 4 \). The SOB signals are normalized by their energy. In Fig. 4.6(a) we see two examples of SOB signals with \( N = 3, 4 \) and with \( a = 3.4, 3.25 \) (correspondingly) superimposed with a temporal double peak signal \( g(t) \) with some
small separation $t_{sep}$. In Fig. 4.6(b) the cross-correlations are given separately for $N = 3, 4$ for a specific value of $t_{sep} = 0.32 \left[ \frac{2\pi}{Na\Delta\omega} \right]$ while $a$ is modified. The cross-correlations are shown only around time-delay-zero where the superoscillating feature is interacting with the double-pulse. The curved white lines delimits the range $\tau \in \frac{1}{4}[-T_{SOB}, T_{SOB}]$ (where $T_{SOB} = \frac{4\pi}{aN\Delta\omega}$) which reflects the temporal delays in which the double pulse interactions with the lobes outside the superoscillatory feature is minimal. The delay between the two straight vertical white lines is equal to $t_{sep}$.

The two-pulse structure is resolved when a minima occurs at time-zero of the cross-correlation (for a single pulse we would get a maxima at this location). However the resolving power is really a matter of visibility - how well is this feature observable. We calculate the visibility of the central feature of the cross-correlation (not to be confused with the visibility of the superoscillating feature) for different values of $a$, where the cross-correlation visibility is $\left| \frac{\max(S) - \min(S)}{\max(S) + \min(S)} \right|$. The maxima and minima are calculated for the range $\tau \in \frac{1}{4}[-T_{SOB}, T_{SOB}]$. The $a$ value where the visibility is maximal is denoted with a horizontal straight white line. The greatest visibility is achieved for $a$ for which the two-pulse separation matches the distance between the closest zeros of the superoscillation feature. This condition is approximately given by: $T_{SOB}/2 = t_{sep}$. This happens for both $N = 3, 4$. Furthermore - when we repeat the calculation of the visibility for different values of $t_{sep}$ we still see this identical behavior. This is shown in Fig. 4.6(c) depicting the visibility as a function of both $t_{sep}$ and $a$. The maximal visibility approximately matches the line $t_{sep} = 0.5 \cdot T_{SOB}$ (shown with black dots).

The explanation for the fact that there is an optimal value of $a$ for resolving the double-pulse is simple: it is the result of trade-off between higher local frequency associated with higher values of $a$ and lower visibility due to lower ratio of the amplitude of the superoscillations compared to its adjacent side-lobes.

In any case, the conclusion is obvious: if the double-pulse separation is shorter than half the period associated with diffraction limited signals, than a superoscillating signal would be better for detecting or resolving it (compared with a transform-limited pulse for which $a = 1$), achieving super-resolution in the time-domain.

We would like to add that the temporal resolving power is a function of the
Figure 4.6: Temporal super-resolution with SOB signals (a) SOB signals (blue line) with $N = 3, a = 3.4$ (left) $N = 4, a = 3.25$ (right) superimposed with a temporal double peak signal $g(t)$ (red) with some small separation. (b) Cross-correlation function of the SOB signal together with a double-peak signal with a specific separation $t_{sep} = 0.32 \cdot \frac{2\pi}{N\Delta\omega}$, given separately for $N = 3$ (left) $N = 4$ (right) as a function of time delay and the $a$ parameter. The separation of the two vertical straight lines is $t_{sep}$. The horizontal white line marks the $a$ value for which the visibility of the cross-correlation is maximal in the range $\tau \in \left[ -\frac{1}{2}T_{SOB}, T_{SOB} \right]$. This range is delimited between the two curved white lines. (c) Visibility as a function of $t_{sep}$ and $\frac{1}{2}T_{SOB} = \frac{1}{a}$ (in the units used in the graph) over the range $\tau \in \left[ -\frac{1}{2}T_{SOB}, T_{SOB} \right]$. The maximal visibility approximately matches the line $t_{sep} = 0.5 \cdot T_{SOB}$ (shown with black dots).
signal to be resolved. In analogy with imaging - regular microscopes are defined usually by their Modulation Transfer Function (MTF) which shows the visibility when imaging a specific Fourier component. The MTF is irrelevant for microscopes based on super-oscillations, as the power of the later lies in their ability to resolve signals made of a limited number of oscillations (not a Fourier component). If the number of oscillations extends too much into the side-lobes - they would not be resolved. In our case this reflects the cases where $t_{sep}$ is fixed and $a$ is increased too much.

As we have seen, the performance of the SOB signals would outperform transform-limited signals for cases where the signal to be resolved does not extend into the side-lobes of the SOB signal. This is the temporal counterpart to super-resolution imaging demonstrated experimentally with super-oscillating microscopes [28, 29].
5 conclusion

In this thesis, we applied the concept of superoscillations to the temporal domain of ultra short optical pulses. We experimentally demonstrated a superoscillating optical beat, having a temporal fringe which is four times narrower than a Gaussian pulse having the same bandwidth, breaking the temporal Fourier-transform limit given with transform limited Gaussian pulses by 75% while maintaining visibility of 25%.

As demonstrated numerically, this sub-Fourier focusing could be used for temporal super-resolution and so have important consequences in applications relying on ultra-short pulses such as spectroscopy, nonlinear optics and metrology.

Our results were submitted for publication[43].
References


היכולת לייצר פולסים אופטיים קצרים או שולייםทรงית היא חשובה להג OMITא של תהליכים שונים, גם במדידות ובחROKE של תהליכים שונים, וגם ליישומים שונים. יש להניח כי הגדרה קיצונית של גבול המריר של פוריה הידועה כסופר אוסצילציות (SOB),SUPEROSCILLATING OPTICAL BEAT (SOB), עבורה המגאומטריה של פולס יוצר קצרצר ותלוי במספר אופוניםfects ובעלת גלגל של פוריה מרחיב שלם. בסיס של תיאור המריר של פוריה הוא באמצעות תיאור hSUM ה TEMPORALゾ LOSION ו TEMPORALゾ LOSION של פוריה בצלVEN ל TEMPORALゾ LOSION של פוריה בצלVEN ל TEMPORALゾ LOSION של פוריה בצלVEN ל TEMPORALゾ LOSION של פוריה בצלVEN ל TEMPORALゾ LOSION של פוריה בצלVEN L

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