The Dirty MIMO Multiple-Access Channel

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***Abstract*—In the scalar dirty multiple-access channel, in addition to Gaussian noise, two additive interference signals are present, each known non-causally to a single transmitter. It was shown by Philosof et al. that for strong interferences, an i.i.d. ensemble of codes does not achieve the capacity region. Rather, a structured-codes approach was presented, which was shown to be optimal in the limit of high signal-to-noise ratios (SNRs), where the sum-capacity is dictated by the minimal ("bottleneck") channel gain. In the present work, we consider the multiple-input multiple-output (MIMO) variant of this setting. In order to incorporate structured codes in this case, one can utilize matrix decompositions, which transform the channel into effective parallel scalar dirty multiple-access channels. This approach however suffers from a "bottleneck" effect for each effective scalar channel and therefore the achievable rates strongly depend on the chosen decomposition. It is shown that a recently proposed decomposition, where the diagonals of the effective channel matrices are equal up to a scaling factor, is optimal at high SNRs, under an equal rank assumption.**

I. INTRODUCTION

The dirty-paper channel, introduced by Costa [1], is given by

$$y = x + s + z, \tag{1}$$

where y is the channel output, x is the channel input subject to an average power constraint P, z is an additive white Gaussian noise (AWGN) of power 1, and s is an interference which is known non-causally to the transmitter but not to the receiver.

Costa [1] showed that the capacity of this channel, when the interference is i.i.d. and Gaussian, is equal to that of an interference-free channel $\frac{1}{2}\log(1+P)$, i.e., as if $s \equiv 0$. This result was subsequently extended to ergodic interference in [2] and to arbitrary interference in [3], where to achieve the latter, a structured lattice-based coding scheme was used.

This model serves as an information-theoretic basis for the study of interference cancellation techniques, and was applied to different network communication scenarios; see, e.g., [4].

Its multiple-input multiple-output (MIMO) variant as well as its extension to MIMO broadcast with private messages can be easily treated either directly or via scalar dirty-paper coding and an adequate orthogonal matrix decomposition, the most prominent being the singular-value decomposition (SVD) and the QR decomposition (QRD); see [5] and references therein.

Philosof et al. [6] extended the dirty-paper channel to the case of K multiple (distributed) transmitters, each transmitter knowing a different part of the interference:

$$y = \sum_{k=1}^{K} (x_k + s_k) + z, \tag{2}$$

where y and z are as before, x_k ($k = 1, \ldots, K$) is the input of transmitter k and is subject to an average power constraint P_k, and s_k is an *arbitrary* interference sequence which is known non-causally to transmitter k but not to the other transmitters nor to the receiver. The capacity region of this scenario, termed the *dirty multiple-access channel* (DMAC) in [6], was shown to be contained ("outer bound") in the region of all rate tuples $(R_1, \ldots, R_K)$ satisfying

$$\sum_{k=1}^{K} R_k \leq \frac{1}{2}\log\left(1 + \min_{k=1,\ldots,K} P_k\right), \tag{3}$$

and to contain ("achievable region") all rate tuples $(R_1, \ldots, R_K)$ satisfying[1]

$$\sum_{k=1}^{K} R_k \leq \frac{1}{2}\left[\log\left(\frac{1}{K} + \min_{k=1,\ldots,K} P_k\right)\right]^+, \tag{4}$$

where $[x]^+ \triangleq \max\{0, x\}$. These two regions coincide in the limit of high signal-to-noise ratios (SNRs) — $P_1, \ldots, P_K \gg 1$ — thus establishing the capacity region in this limit to be equal to the region of all rate tuples $(R_1, \ldots, R_K)$ satisfying

$$\sum_{k=1}^{K} R_k \leq \frac{1}{2}\log\left(\min_{k=1,\ldots,K} P_k\right). \tag{5}$$

That is, the sum-capacity suffers from a *bottleneck problem* and reduces to the minimum of the individual capacities in this limit. Interestingly, Costa's random binning technique does not achieve the rate region (4) or the high-SNR region (5), and structured lattice-based techniques need to be used [6].

The MIMO counterpart of the problem is given by

$$\boldsymbol{y} = \sum_{k=1}^{K} (\mathbf{H}_k \boldsymbol{x}_k + \boldsymbol{s}_k) + \boldsymbol{z}. \tag{6}$$

For simplicity, we assume for now that all vectors are of equal length N.[2] We further assume, without loss of generality, that the square channel matrices all have unit determinant, since any other value can be absorbed in P_k. The AWGN vector $\boldsymbol{z}$ has i.i.d. unit-variance elements, while the interference vectors $\{\boldsymbol{s}_k\}$ are arbitrary as in the scalar case. The transmitters are subject to average power constraints $\{P_k\}$.

[1] In addition to (4), other inner bounds which are tighter in certain cases are derived in [6].

[2] We will depart from this assumption later.

In the high-SNR limit (where all powers satisfy $P_k \gg 1$), the individual capacity of the k-th user is given by:[3]

$$\frac{N_r}{2} \log \left(\frac{P_k}{N_r} \right).$$

Thus, similarly to the scalar case (5), one can expect the high-SNR capacity region to be given by

$$\sum_{k=1}^{K} R_k \leq \frac{N_r}{2} \log \left(\frac{\min\limits_{k=1,\ldots,K} P_k}{N_r} \right). \tag{7}$$

However, in contrast to the single-user setting (1), the extension of the scalar DMAC to the MIMO case is not straightforward. As structure is required even in the scalar case (2), one cannot use a vector random codebook. To overcome this, we suggest to employ N_r parallel scalar schemes, each using the lattice coding technique of [6]. This is in the spirit of the capacity-achieving SVD [7] or QRD [8] based schemes, that were proposed for MIMO communications (motivated by implementation considerations). The total rate is split between multiple scalar codebooks, each one enjoying a channel gain according to the respective diagonal value of the equivalent channel matrix obtained by the channel decomposition.

Unfortunately, for the MIMO DMAC problem, neither the SVD nor the QRD is suitable, i.e., their corresponding achievable rates cannot approach (7). Applying the SVD is not possible in the MIMO DMAC setting as joint diagonalization with the same orthogonal matrix on one side does not exist in general. Applying the QRD to each of the orthogonal matrices, in contrast, is possible as it requires an orthogonal operation only at the transmitter.[4] However, the resulting matrices will have non-equal diagonals in general, corresponding to non-equal SNRs. Specifically, denoting the i-th diagonal element of the k-th matrix by $t_{k;i}$, the resulting high-SNR sum-rate would be limited to

$$\sum_{k=1}^{K} R_k \leq \sum_{i=1}^{N_r} \frac{1}{2} \log \left(\frac{\min\limits_{k=1,\ldots,K} (P_k t_{k;i}^2)}{N_r} \right) \tag{8}$$

in this case. As this represents a *per-element* bottleneck, the rate is in general much lower than (7).

In this work we make use of a recently proposed joint orthogonal triangularization [9] to remedy the problem, i.e., to transform the per-element bottleneck (8) into a global one as in (7). Specifically, the decomposition allows to transform two matrices (with equals determinants) into triangular ones with *equal diagonals*, using the same orthogonal matrix on the left — corresponding to a common operation carried at the receiver — and different orthogonal matrices on the right — corresponding to different operations applied by each of the transmitters. The equal diagonals property implies that the minimum in (8) is not active and hence the per-element bottleneck problem, incurred in the QRD-based scheme, is replaced by the more favourable vector bottleneck (7).

The paper is organized as follows. We start by introducing the channel model in Section II. We then present the ingredients we use: the matrix decomposition is presented in Section III, and a structured coding scheme for the single-user "dirty" MIMO channel is presented in Section IV. Our main result, the high-SNR capacity of the two-user MIMO DMAC (6) is given in Section V, using a structured scheme. We extend this result to the K-user case in Section VI. We discuss the usefulness of this scheme for physical-layer MIMO network coding and conclude the paper in Section VII.

II. Problem Statement

The K-user MIMO DMAC is given by:

$$\boldsymbol{y} = \sum_{k=1}^{K} (\mathbf{H}_k \boldsymbol{x}_k + \boldsymbol{s}_k) + \boldsymbol{z}, \tag{9}$$

where $\boldsymbol{y}$ is the channel output vector of length N_r,[5] $\boldsymbol{x}_k$ ($k = 1, \ldots, K$) is the input vector of transmitter k of length $N_t^{(k)}$ and is subject to an average power constraint P_k defined formally in the sequal, $\boldsymbol{z}$ is an AWGN vector with an identity covariance matrix, and $\boldsymbol{s}_k$ is an interference vector of length N_r which is known non-causally to transmitter k but not to the other transmitters nor to the receiver. The interference vector signals $\{\boldsymbol{s}_k\}$ are assumed to be arbitrary sequences. We consider a closed-loop scenario, meaning that the $N_r \times N_t^{(k)}$ channel matrix $\mathbf{H}_k$ is known everywhere and that it satisfies the following properties.

Definition 1 (Proper)**.** A matrix $\mathbf{H}$ of dimensions $N_r \times N_t$ is said to be proper if it has no fewer columns than rows, i.e., $N_r \leq N_t$, is full rank (namely of rank N_r) and satisfies[6]

$$\det \left(\mathbf{H}\mathbf{H}^T \right) = 1. \tag{10}$$

Transmission is carried out in blocks of length n. The input signal transmitted by transmitter k is given by $\boldsymbol{x}_k^n = f_k (w_k, \boldsymbol{s}_k^n)$, where we denote by a^n blocks of a at time instants $1, 2, \ldots, n$, i.e., $a^n = a[1], \ldots, a[n]$, w_k is the conveyed message by this user which is chosen uniformly from $\left\{1, \ldots, \lceil 2^{nR_k} \rceil \right\}$, R_k is its transmission rate, and f_k is the encoding function. The input signal $\boldsymbol{x}_k$ is subject to an average power constraint

$$\frac{1}{n} \sum_{t=1}^{n} \boldsymbol{x}_k^2[t] \leq P_k.$$

The receiver reconstructs $\hat{w}_1, \ldots \hat{w}_K$ from the channel output, using a decoding function g: $(\hat{w}_1, \ldots, \hat{w}_K) = g(\boldsymbol{y}^n)$. A rate tuple $(R_1, \ldots, R_K)$ is said to be achievable if for any $\epsilon > 0$, however small, there exist n, f and g, such that the error probability is bounded from the above by ϵ, i.e., $\Pr(\hat{w}_1 \neq w_1, \ldots, \hat{w}_K \neq w_K) \leq \epsilon$. The capacity region is defined as the closure of all achievable rate tuples.

[3]The optimal covariance matrix in the limit of high SNR is white; see Lemma 1 in the sequel.

[4]More precisely, RQ decompositions need to be applied to the channel matrices in this case.

[5]All vectors in this paper are assumed column vectors.

[6]There is no loss in generality in the communication settings we consider, since a scalar coefficient can always be absorbed in the power constraint.

III. Background: Orthogonal Matrix Triangularization

In this section we briefly recall some important matrix decompositions that will be used in the sequel. In Section III-A we recall the generalized triangular decomposition (GTD) and some of its important special cases. Joint orthogonal triangularizations of two matrices are discussed in Section III-B.

A. Single Matrix Triangularization

Let $\mathbf{A}$ be a proper matrix of dimensions $M \times N$. A generalized triangular decomposition (GTD) of $\mathbf{A}$ is given by:

$$\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{V}^T, \tag{11}$$

where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices of dimensions $M \times M$ and $N \times N$, respectively, and $\mathbf{T}$ is a generalized lower-triangular matrix. Namely, it has the following structure:

$$\mathbf{T} = {\scriptstyle M}\left\{ \overbrace{\begin{bmatrix} T_{11} & 0 & 0 & \cdots & 0 & \cdots & 0 \\ T_{21} & T_{22} & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots & \ddots & \vdots \\ T_{M,1} & T_{M,2} & \cdots & T_{MM} & 0 & \cdots & 0 \end{bmatrix}}^{N} \right. .$$

The diagonal entries of $\mathbf{T}$ always have a unit product. Necessary and sufficient conditions for the existence of a GTD for a prescribed diagonal $\{T_{ii}\}$ are known, along with explicit constructions of such a decomposition [10], [11].

Three important special cases of the GTD include the SVD — in which the matrix $\mathbf{T}$ is diagonal, the QRD — in which $\mathbf{V}$ is equal to the identity matrix, and the geometric mean decomposition (GMD) — in which the matrix $\mathbf{T}$ has a constant diagonal which is equal to the geometric mean of its singular values of $\mathbf{A}$.[7]

B. Joint Matrix Triangularization

Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two proper matrices of dimensions $M_1 \times N$ and $M_2 \times N$, respectively. A joint triangularization of these two matrices is given by:

$$\mathbf{A}_1 = \mathbf{U}_1\mathbf{T}_1\mathbf{V}^T, \tag{12a}$$

$$\mathbf{A}_2 = \mathbf{U}_2\mathbf{T}_2\mathbf{V}^T, \tag{12b}$$

where $\mathbf{U}_1$, $\mathbf{U}_2$ and $\mathbf{V}$ are orthogonal matrices of dimensions $M_1 \times M_1$, $M_2 \times M_2$ and $N \times N$, respectively, and $\mathbf{T}_1$ and $\mathbf{T}_2$ are generalized lower-triangular matrices.

It turns out that the existence of such a decomposition depends on the diagonal *ratios* $\{T_{1;ii}/T_{2;ii}\}$. Necessary and sufficient conditions were given in [9]. Specifically, it was shown that there always exists a decomposition with unit ratios, i.e.,

$$T_{1;ii} = T_{2;ii}\,, \qquad\qquad i = 1, \dots, N\,.$$

Such a decomposition is coined the joint equi-diagonal decomposition (JET).[8] Technically, the existence of JET is an extension of the existence of the (single-matrix) GMD.

Unfortunately, JET of more than two matrices is not possible in general [12]. Nonetheless, in Section VI we present a way to overcome this obstacle.

[7]See [12] for a geometrical interpretation of these decompositions.

[8]See [12] for a geometrical interpretation of the JET.

IV. Background: Single-User MIMO Dirty-Paper Channel

In this section we review the (single-user) MIMO dirty-paper channel, corresponding to setting $K = 1$ in (9):

$$\boldsymbol{y} = \mathbf{H}\boldsymbol{x} + \boldsymbol{s} + \boldsymbol{z}. \tag{13}$$

We drop the user index of $\boldsymbol{x}$, $\boldsymbol{s}$ and $\mathbf{H}$ in this case.

For an i.i.d. Gaussian interference vector, a straightforward extension of Costa's random binning scheme achieves the capacity of this channel,

$$\mathrm{C}(\mathbf{H}, \mathbf{K}) \triangleq \max_{\mathbf{K}:\, \mathrm{trace}(\mathbf{K}) \le P} \frac{1}{2} \log \left| \mathbf{I} + \mathbf{H}\mathbf{K}\mathbf{H}^T \right|, \tag{14}$$

which is, as in the scalar case, equal to the interference-free capacity. In the high-SNR limit, we have the following.

Lemma 1 (See [13])**.** *The capacity of the single-user MIMO dirty-paper channel* (13) *satisfies* $\lim_{P \to \infty} [C - R_{\mathrm{HSNR}}] = 0$, *where*

$$R_{\mathrm{HSNR}} \triangleq \frac{N_r}{2} \log \frac{P}{N_r}\,. \tag{15}$$

Moreover, this rate is achieved by the input covariance matrix

$$\mathbf{K} = \frac{P}{N_r} \mathbf{I}_{N_r}\,.$$

The Costa-style scheme for the MIMO dirty-paper channel suffers from two major drawbacks. First, it requires vector codebooks of dimension N_t, which depend on the specific channel $\mathbf{H}$. And second, it does not admit an arbitrary interference. Both of these can be solved by using the orthogonal matrix decompositions of Section III to reduce the coding task to that of coding for the *scalar* dirty-paper channel (1). For each scalar channel, the interference consists of two parts: a linear combination of the elements of the "physical interference" $\boldsymbol{s}$ and a linear combination of the off-diagonal elements of the triangular matrix which also serve as "self interference". When using the lattice-based scheme of [3], the capacity (14) is achieved for arbitrary interference sequences.

Scheme (Single-user zero-forcing MIMO dirty-paper coding).

Offline:

- Apply any orthogonal matrix triangularization (11) to the channel matrix: $\mathbf{H} = \mathbf{U}\mathbf{T}\mathbf{V}^T$.
- Denote the vector of the diagonal entries of $\mathbf{T}$ by $\boldsymbol{t} \triangleq \mathrm{diag}(\mathbf{T})$.
- Construct N_r good unit-power scalar dirty-paper codes with respect to SNRs $\{t_i^2 P/N_r\}$.

Transmitter: At each time instant:

- Generates $\{\tilde{x}_i\}$ in a successive manner from first ($i = 1$) to last ($i = N_r$), where $\tilde{x}_i$ is the corresponding entry of the codeword of over sub-channel i, the interference over this sub-channel is equal to

$$\sum_{\ell=1}^{i-1} T_{i,\ell}\tilde{x}_\ell + \sum_{\ell=1}^{N_r} V_{i,\ell} s_\ell,$$

and $T_{i,\ell}$ is the (i, ℓ) entry of $\mathbf{T}$.

- Forms $\tilde{\boldsymbol{x}}$ with its first N_r entries being $\{\tilde{x}_i\}$ followed by $(N_t - N_r)$ zeros.
- Transmits $\boldsymbol{x}$ which is formed as folows: $\boldsymbol{x} = \mathbf{V}\tilde{\boldsymbol{x}}$.

Receiver:

- At each time instant forms $\tilde{\boldsymbol{y}} = \mathbf{U}^T \boldsymbol{y}$.
- Decodes the codebooks using dirty-paper decoders, where $\tilde{x}_i$ is decoded from $\tilde{y}_i$.

As is well known, the zero-forcing dirty-paper coding scheme approaches capacity for proper channel matrices in the limit of high SNR. This is formally stated as folows.

Corollary 1. *For any proper channel matrix* $\mathbf{H}$*, the zero-forcing MIMO dirty-paper coding scheme achieves* R_{HSNR} (15)*. Thus, it approaches the capacity of the MIMO dirty-paper channel* (13) *in the limit* $P \to \infty$*.*

Proof: The zero-forcing MIMO dirty-paper coding scheme achieves a rate of

$$\begin{aligned} R_{\text{ZF}} &= \sum_{i=1}^{N_r} \frac{1}{2} \log \left(1 + \frac{P}{N_r} t_i^2 \right) \\ &\geq N_r \cdot \frac{1}{2} \log \left(\frac{P}{N_r} \right) + \frac{1}{2} \log \left(\prod_{i=1}^{N_r} t_i^2 \right) \\ &= \frac{N_r}{2} \log \left(\frac{P}{N_r} \right), \end{aligned}$$

where the last equality follows from (10). ■

Remark 1. A minimum mean square error (MMSE) variant of the scheme achieves capacity for any SNR and any channel matrices (not necessarily proper); see, e.g., [5]. Unfortunately, extending the MMSE variant of the scheme to the DMAC setting is not straightforward, and therefore we shall concentrate on the zero-forcing variant of the scheme.

V. Two-User MIMO DMAC

We now derive outer and inner bounds on the capacity region of the two-user MIMO DMAC. We show that the two coincide for proper channel matrices in the limit of high SNRs.

The outer bound of [6] for the scalar case (3) is easily extended to the MIMO setting (9): Both users convey a common message with one interference nullified and the variances of the other taken to infinity. This upper bounds the sum-capacity by the individual capacity of each user and is stated formally next; a detailed proof is available in [14].

Proposition 1 (Two-user sum-capacity outer bound)**.** *The sum-capacity of the two-user MIMO DMAC* (9) *is bounded from above by the minimum of the individual capacities:*

$$R_1 + R_2 \leq \frac{1}{2} \log \min_{k=1,2} \max_{\mathbf{K}_k \colon \operatorname{trace}(\mathbf{K}_k) \leq P_k} \left| \mathbf{I} + \mathbf{H}_k \mathbf{K}_k \mathbf{H}_k^T \right| \quad (16)$$

We next introduce an inner bound that approaches the upper bound (16) in the limit of high SNRs.

Theorem 1. *For the two-user MIMO DMAC* (9) *with any proper channel matrices* $\mathbf{H}_1$ *and* $\mathbf{H}_2$*, the region of all non-negative rate pairs* (R_1, R_2) *satisfying*

$$R_1 + R_2 \leq \frac{N_r}{2} \left[\log \left(\frac{\min\{P_1, P_2\}}{N_r} \right) \right]^+ \quad (17)$$

is achievable.

We give a constructive proof by adapting the single-user MIMO dirty-paper coding scheme of Section IV to the two-user DMAC. To this end, we replace the GTD of Section III-A with the JET of Section III-B. Applying the JET to the channel matrices translates the two-user MIMO DMAC (9) into parallel SISO DMACs with equal channel gains (corresponding to equal diagonals). Over the resulting SISO DMACs, good SISO DMAC codes as in [6] are used to attain (17).

Proof of Theorem 1: The proposed scheme achieves any rate pair (R_1, R_2) whose sum-rate is bounded from below by

$$\begin{aligned} R_1 + R_2 &= \sum_{i=1}^{N_r} (r_{1;i} + r_{2;i}) && \text{(18a)} \\ &\geq \sum_{i=1}^{N_r} \frac{1}{2} \left[\log \left(\frac{1}{2} + t_i^2 \cdot \frac{\min\{P_1, P_2\}}{N_r} \right) \right]^+ && \text{(18b)} \\ &\geq \sum_{i=1}^{N_r} \frac{1}{2} \log \left(t_i^2 \cdot \frac{\min\{P_1, P_2\}}{N_r} \right) && \text{(18c)} \\ &= \frac{N_r}{2} \log \left(\frac{\min\{P_1, P_2\}}{N_r} \right), && \text{(18d)} \end{aligned}$$

where $r_{k;i}$ is the achievable rate of transmitter k ($k = 1, 2$) over sub-channel i ($i = 1, \ldots, N_r$), (18b) follows from (4), and (18d) holds true due to (10). ■

By comparing Proposition 1 with Theorem 1 in the limit of high SNR, the following corollary follows.

Corollary 2. *The capacity region of the two-user MIMO DMAC* (9) *with any proper channel matrices* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *is given by* $\mathcal{C}_{\text{HSNR}} + o(1)$*, where* $\mathcal{C}_{\text{HSNR}}$ *is given by all rate pairs satisfying:*

$$R_1 + R_2 \leq \frac{N_r}{2} \log \left(\frac{\min\{P_1, P_2\}}{N_r} \right)$$

and $o(1)$ *vanishes as* $\min\{P_1, P_2\} \to \infty$*.*

Remark 2. At any finite SNR, the scheme can achieve rates outside $\mathcal{C}_{\text{HSNR}}$. Specifically, inequality (18c) is strict, unless the achievable sum-rate is zero. However, in that case the calculation depends upon the exact diagonal values $\{t_i\}$; we do not pursue this direction.

VI. K-User MIMO DMAC

In this section we extend the results of Section V to MIMO DMACs with $K > 2$ users. The outer bound is a straightforward extension of the two-user case of Proposition 1.

Proposition 2 (K-user sum-capacity outer bound)**.** *The sum-capacity of the* K*-user MIMO DMAC* (9) *is bounded from the above by the minimum of the individual capacities:*

$$\sum_{k=1}^{K} R_k \leq \frac{1}{2} \log \min_{k=1,\ldots,K} \max_{\mathbf{K}_k \colon \operatorname{trace}(\mathbf{K}_k) \leq P_k} \left| \mathbf{I} + \mathbf{H}_k \mathbf{K}_k \mathbf{H}_k^T \right|.$$

For an inner bound, we would have liked to use JET of $K > 2$ matrices. As such a decomposition does not exist in general, we present a "workaround", following [12].

We process jointly N channel uses and consider them as one time-extended channel use. The corresponding channel is

$$\mathcal{y} = \sum_{k=1}^{K} (\mathcal{H}_k \mathcal{x}_k + \mathcal{s}_k) + \mathcal{z},$$

where $\mathcal{y}$, $\mathcal{x}_k$, $\mathcal{s}_k$, $\mathcal{z}$ are the time-extended vectors composed of N "physical" (concatenated) output, input, interference and noise vectors, respectively. The corresponding time-extended matrix $\mathcal{H}_k$ is a block-diagonal matrix whose N blocks are all equal to $\mathbf{H}_k$:

$$\mathcal{H}_k = \mathbf{I}_N \otimes \mathbf{H}_k \, , \tag{19}$$

where $\otimes$ is the Kronecker product operation. As the following result shows, for such block-diagonal matrices we can achieve equal diagonals, up to edge effects that can be made arbitrarily small by taking a sufficient number of time extensions N.

Theorem 2 (K-JET with edge effects [12])**.** *Let $\mathbf{H}_1, \ldots, \mathbf{H}_K$ be K proper matrices of dimensions $N_r \times N_t^{(1)}, \ldots, N_r \times N_t^{(K)}$, resp., and construct their time-extended matrices with N blocks, $\mathcal{H}_1, \ldots, \mathcal{H}_K$, resp., according to* (19)*. Denote $\tilde{N} \triangleq N - N_r^{K-2} + 1$. Then, there exist matrices $\mathcal{U}, \mathcal{V}_1, \ldots, \mathcal{V}_K$ with orthonormal columns of dimensions $N_r N \times N_r \tilde{N}$, $N_t^{(1)} N \times N_r \tilde{N}, \ldots, N_t^{(K)} N \times N_r \tilde{N}$, resp., such that*

$$\mathcal{T}_k = \mathcal{U}^T \mathcal{H}_k \mathcal{V}_k^T \, , \qquad\qquad k = 1, \ldots, K \, ,$$

where $\mathcal{T}_1, \ldots, \mathcal{T}_K$ are lower-triangular matrices of dimensions $N_r \tilde{N} \times N_r \tilde{N}$ with equal diagonals of unit product.

Since $\lim_{N \to \infty} \tilde{N}/N = 1$, we have the following.

Theorem 3. *For the K-user MIMO DMAC* (9)*, the region of all non-negative rate tuples $(R_1, \ldots, R_K)$ satisfying*

$$\sum_{k=1}^{K} R_k \leq \frac{N_r}{2} \log \left(\frac{\min_k P_k}{N_r} \right)$$

is achievable.

Proof: Fix some large enough N. Construct the channel matrices $\mathcal{H}_1, \ldots, \mathcal{H}_K$ as in (19), and set $\mathcal{U}$, $\mathcal{V}_1, \ldots, \mathcal{V}_K$ and $\mathcal{T}_1, \ldots, \mathcal{T}_K$ according to Theorem 2. Now, over nN consecutive channel uses, apply the natural extension of the scheme of Section V to K users, replacing $\{U_k\}, V, \{T_k\}$ with the obtained matrices. As in the proof of Theorem 1, we can attains any rate approaching

$$\sum_{k=1}^{K} R_k \leq \frac{N_r \tilde{N}}{2} \log \frac{\min\limits_{k=1,\ldots,K} P_k}{N_r} \, .$$

As we used the channel nN times, we need to divide these rates by N; Using $\lim\limits_{N \to \infty} \tilde{N}/N = 1$ completes the proof. ■

By comparing Proposition 2 with Theorem 3 in the limit of high SNR, we can extend Corollary 2 as follows.

Corollary 3. *The capacity region of the K-user MIMO DMAC* (9) *with any proper channel matrices $\mathbf{H}_1, \ldots, \mathbf{H}_K$ is given by $\mathcal{C}_{\mathrm{HSNR}} + o(1)$, where $\mathcal{C}_{\mathrm{HSNR}}$ is given by all rate pairs satisfying:*

$$\sum_{k=1}^{K} R_k \leq \frac{N_r}{2} \log \left(\frac{\min\limits_{k=1,\ldots,K} P_k}{N_r} \right)$$

and $o(1)$ vanishes as $\min\limits_{k=1,\ldots,K} P_K \to \infty$.

VII. Discussion: General Channel Matrices

The proposed scheme can be readily applied to the MIMO two-way channel, which can be seen as containing a MIMO DMAC; see [14]–[16].

Furthermore, in this paper we restricted attention to full rank channel matrices having more columns than rows. In this case, the column spaces of both matrices are equal. Indeed, the scheme and inner bound of Sections V and VI can be extended to work for the general case as well; this requires, however, introducing an output projection at the receiver, which transforms the channel matrices to effective proper ones. Since all interferences need to be canceled out for the recovery of the transmitted messages, it seems that such a scheme would be optimal in the limit of large transmit powers $P_1, \ldots, P_K \to \infty$. Unfortunately, the upper bound of Proposition 1, which is equal to the maximal individual capacity, is not tight in the non-proper matrix case, and calls for further research.

References

[1] M. H. M. Costa, "Writing on dirty paper," *IEEE Trans. Inf. Theory*, vol. 29, no. 3, pp. 439–441, May 1983.

[2] A. S. Cohen and A. Lapidoth, "The Gaussian watermarking game," *IEEE Trans. Inf. Theory*, vol. 48, no. 6, pp. 1639–1667, Jun. 2002.

[3] U. Erez, S. Shamai, and R. Zamir, "Capacity and lattice strategies for canceling known interference," *IEEE Trans. Inf. Theory*, vol. 51, no. 11, pp. 3820–3833, Nov. 2005.

[4] B. Nazer and R. Zamir, *Lattice coding for signals and networks*. Cambridge: Cambridge University Press, 2014, ch. Gaussian Networks.

[5] Y. Jiang, W. Hager, and J. Li, "Uniform channel decomposition for MIMO communications," *IEEE Trans. Sig. Proc.*, vol. 53, no. 11, pp. 4283–4294, Nov. 2005.

[6] T. Philosof, R. Zamir, U. Erez, and A. Khisti, "Lattice strategies for the dirty multiple access channel," *IEEE Trans. Inf. Theory*, vol. 57, no. 8, pp. 5006–5035, Aug. 2011.

[7] E. Telatar, "Capacity of the multiple antenna Gaussian channel," *Europ. Trans. Telecommun.*, vol. 10, no. 6, pp. 585–595, Nov. 1999.

[8] G. Foschini, "Layered space–time architecture for wireless communication in a fading environment when using multi-element antennas," *Bell Sys. Tech. Jour.*, vol. 1, no. 2, pp. 41–59, 1996.

[9] A. Khina, Y. Kochman, and U. Erez, "Joint unitary triangularization for MIMO networks," *IEEE Trans. Sig. Proc.*, vol. 60, no. 1, pp. 326–336, Jan. 2012.

[10] Y. Jiang, W. Hager, and J. Li, "The generalized triangular decompostion," *Math. of Comput.*, vol. 77, no. 262, pp. 1037–1056, 2008.

[11] P. Kosowski and A. Smoktunowicz, "On constructing unit triangular matrices with prescribed singular values," *Computing*, vol. 64, no. 3, pp. 279–285, 2000.

[12] A. Khina, I. Livni, A. Hitron, and U. Erez, "Joint unitary triangularization for Gaussian multi-user MIMO networks," *IEEE Trans. Inf. Theory*, vol. 61, no. 5, pp. 2662–2692, May 2015.

[13] E. Martinian, "Waterfilling gains at most O(1/SNR) at high SNR," 2004. [Online]. Available: web.mit.edu/~emin/www.old/research/wfill.pdf

[14] A. Khina, U. Erez, and Y. Kochman, "The dirty MIMO multiple-access channel," *IEEE Trans. Inf. Theory*, submitted, Oct. 2015. [Online]. Available: arXiv:1510.08018

[15] A. Khina, Y. Kochman, and U. Erez, "Physical-layer MIMO relaying," in *Proc. IEEE Int. Symp. Info. Theory (ISIT)*, St. Petersburg, Russia, July/Aug. 2011, pp. 2437–2441.

[16] ——, "Improved rates and coding for the MIMO two-way relay channel," in *Proc. IEEE Int. Symp. Info. Theory and Its Apps. (ISITA)*, Melbourne, Vic., Australia, Oct. 2014, pp. 658–662.