# Event-triggered stochastic control via constrained quantization

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**Abstract**

We consider a discrete-time linear quadratic Gaussian networked control setting where the (full information) observer and controller are separated by a fixed-rate noiseless channel. We study the event-triggered control setup in which the encoder may choose to either transmit a packet or remain silent. We recast this problem into that of fixed-rate quantization with an extra symbol that corresponds to the silence event. This way, controlling the average transmission rate is possible by constraining the minimal probability of the silence symbol. We supplement our theoretical framework with numerical simulations.

## Introduction

The demand for new and improved control techniques over unreliable communication links is constantly growing, due to the rise of emerging opportunities in the Internet of Things and Cyber-physical systems. The cyber part of the latter is relies, in turn, on *networked control*, in which control is carried over discretized packeted communication channels [1–5].

Unfortunately, the rapid rise in the number of users sharing the same physical media for communications often creates network congestion problems and calls for a demand for reduced-communication approaches.

To reduce the number of physical packets transmitted over the network, communication techniques that convey information by relying on silence and timing have been proposed both for transmitting data [6, 7] and for stabilizing control system [8–13].

However, although much effort has been put into determining the conditions for the stabilizability of such systems, less so has been done for determining the optimal attainable control costs.

In this work, we consider the event-triggered control problem, in which the encoder can either transmit packet of a fixed rate $R$ (which can be zero; this corresponds to transmitting empty packets, i.e., packet baring no content that only signal a transmission) or remain silent. We develop a quantization framework for this setting by adding a quantization cell that corresponds to "silence", i.e., $2^R + 1$ cells in total. By requiring the probability of this cell to be above a minimal value, we are able to control the average transmission rate of the scheme (which is equal to the sum of the probabilities of the remaining cells). Clearly the additional "silent cell" allows to convey extra information such that the effective rate exceeds the physical transmission rate.

We concentrate on the case of discrete-time linear plants with disturbances that have logarithmically-concave (log-concave) probability density functions (PDFs) (with the Gaussian PDF being an important special case). Such PDFs are unimodal—a property that allows to concentrate on quantizers with contiguous quantization cells. Furthermore, it has been recently proved that for disturbances having a log-concave PDF, the resulting control states are guaranteed to have log-concave PDFs at every step [14] (this does not hold for general unimodal PDFs [15]) and that therefore applying the Lloyd–Max algorithm [16, Ch. 6.3] w.r.t. this PDF, at every step, is (greedily) optimal in for time-triggered control (when the encoder cannot remain silent).

## Problem Setup

We now formulate the control–communication setup, considered in this work. We use a discrete-time model spanning the time interval $[1:T] \triangleq \{1, 2, \cdots, T\}$ for $T \in \mathbb{N}$, where $[i:j] \triangleq \{i, i+1, \dots, j\}$ for $i, j \in \mathbb{Z}$, such that $i \leq j$. The plant is a discrete-time linear scalar stochastic system

$$X_{t+1} = aX_t + W_t + U_t, \qquad t \in [0:T-1], \tag{1}$$

where $X_t, W_t, U_t \in \mathbb{R}$ are the system state, disturbance and control action at time $t$, respectively. $\{W_t\}$ $\{W_t\}$ are independent and identically distributed (i.i.d.) according to a known log-concave PDF $f_W(w)$ with variance $\sigma_W^2$, and assume, w.l.o.g., that it has zero mean.

**Definition 1** (Log-concave function; see [17])**.** A function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is said to be log-concave if $\log \circ f$ is concave. We use the extended definition that allows $f(x)$ to assign zero values: $\log f(x) \in \mathbb{R} \cup \{-\infty\}$.

*Remark* 1. The Gaussian PDF is a log-concave function.

We assume the observer has perfect access to $x_t$ at time $t$. However, in contrast to traditional control, the observer is not colocated with the controller and may communicate with it instead via a noiseless channel of *data rate* $R$. We shall consider two settings.

*Time-triggered control.* Here, the encoder *always* sends a packet of rate $R$. That is, at each $t$, the observer, which also takes the role of the encoder $\mathcal{E}_t$, can perfectly convey a message (or "index") of $R$ bits, $\ell_t \in \left[0 : 2^R - 1\right]$, of the past states, to the controller:

$$\ell_t = \mathcal{E}_t(X^t), \tag{2}$$

where we denote $a^t \triangleq (a_1, a_2, \dots, a_t)$ and use the convention that $a^t = \emptyset$ for $t \leq 0$. We further set $\ell_0 = \ell_T = 0$.

The controller at time $t$, which also takes the role of the decoder $\mathcal{D}_t$, recovers the observed codeword $\ell_t$ and uses it to generate the control action

$$U_t = \mathcal{D}_t\left(\ell^t\right).$$

*Event-triggered control.* The encoder may avoid sending a packet to reduce the average transmission rate. We shall view this event as an extra possible index value.

Our goal is to minimize the following average-stage linear quadratic (LQ) cost upon reaching the time horizon $T \in \mathbb{N}$:

$$\bar{J}_T \triangleq \frac{1}{T}\mathbb{E}\left[q_T X_T^2 + \sum_{t=1}^{T-1}\left(q_t X_t^2 + r_t U_t^2\right)\right] = \frac{1}{T}\sum_{t=1}^{T} J_t\,,$$

where $\{q_t\}$ and $\{r_t\}$ are non-negative weights, and $\{J_t\}$ are the instantaneous costs

$$J_t \triangleq \mathbb{E}\left[q_t X_t^2 + r_t U_t^2\right], \qquad t \in [1:T-1], \tag{3a}$$

$$J_T \triangleq \mathbb{E}\left[q_T X_T^2\right]. \tag{3b}$$

## Optimal Greedy Control

In this section we consider the time-triggered setting. We first recall the Lloyd–Max algorithm and its optimality guarantees and then use them to construct a greedy optimal control policy.

*Quantizer Design*

**Definition 2** (Scalar quantizer)**.** A scalar quantizer $\mathcal{Q}$ of rate $R$ is described by an encoder $\mathcal{E}_\mathcal{Q} : \mathbb{R} \to \left[0 : 2^R - 1\right]$ and a decoder $\mathcal{D}_\mathcal{Q} : \left[0 : 2^R - 1\right] \to c \triangleq \{c[0], \ldots, c[2^R - 1]\} \subset \mathbb{R}$. We define the quantization operation $\mathcal{Q} : \mathbb{R} \to c$ as the composition of the encoding and decoding operations: $\mathcal{Q} = \mathcal{D}_\mathcal{Q} \circ \mathcal{E}_\mathcal{Q}$.[1] The *reproduction points* $\{c[0], \ldots, c[2^R - 1]\}$ are assumed to be ordered, w.l.o.g.:[2] $c[0] < c[1] < \cdots < c[2^R - 1]$. We denote by $\mathcal{I}[\ell]$ the collection of all points that are mapped to index $\ell$:

$$\mathcal{I}[\ell] \triangleq \{x | x \in \mathbb{R}, \mathcal{E}_\mathcal{Q} = \ell\} = \{x | x \in \mathbb{R}, \mathcal{Q} = c[\ell]\}.$$

We shall concentrate on the class of *regular quantizers*.

**Definition 3** (Regular quantizer)**.** A scalar quantizer is *regular* if every cell $\mathcal{I}[\ell], \ell \in \left[0 : 2^R - 1\right]$, is a contiguous interval that contains its reproduction point $c[\ell]$:

$$c[\ell] \in \mathcal{I}[\ell] = [p[\ell], p[\ell + 1]), \qquad \ell \in \left[0 : 2^R - 1\right],$$

where $p \triangleq \left\{p[0], \ldots, p[2^R]\right\}$ is the set of *partition levels*—the boundaries of the cells. Hence, a regular scalar quantizer can be represented by the input partition-level set $p$ and the reproduction-point set $c \triangleq \left\{c[0], \ldots, c[2^R - 1]\right\}$. We further take $p[0]$ and $p[2^R]$ to be the left-most and right-most values of the support of the source's PDF.

The cost we wish to minimize is the mean squared error distortion between the source $W$ with a given PDF $f_W$ and its quantization $\mathcal{Q}(W)$:

$$D \triangleq \mathbb{E}\left[\{W - \mathcal{Q}(W)\}^2\right] = \sum_{\ell=0}^{2^R-1} \int_{p[\ell]}^{p[\ell+1]} (w - c[\ell])^2 f_W(w) dw. \tag{4}$$

Denote by $D^*$ the minimal achievable $D$; the *optimal quantizer* achieves $D^*$.

*Remark* 2. We shall concentrate on log-concave PDFs $f_W$, which are therefore continuous [17]. Hence, the inclusion or exclusion of the boundary points in each cell does not affect the distortion of the quantizer, meaning that the boundary points can be broken systematically.(semi-i)nfinite support, then the leftmost and/or rightmost intervals of the quantizer are open ($p[0]$ and/or $p[2^R]$ may take infinite values).

[1] The encoder and decoder that give rise to the same parameter are unique up to a permutation of the labeling of the index $\ell$.

[2] If some inequalities are not strict, then the quantizer can be reduced to a lower-rate quantizer.

The optimal quantizer satisfies the following necessary conditions [16, Ch. 6.2].

**Proposition 1** (Centroid condition)**.** *For a fixed partition-level set* $p$ *(fixed encoder), the reproduction-point set* $c$ *(decoder) that minimizes the distortion* $D$ (4) *is*

$$c[\ell] = \mathbb{E}\left[w \,\middle|\, p[\ell] < w \leq p[\ell+1]\right], \qquad \ell \in \left[0 : 2^R - 1\right]. \tag{5}$$

**Proposition 2** (Nearest neighbor condition)**.** *For a fixed reproduction-point set* $c$ *(fixed decoder), the partition-level set* $p$ *(encoder) that minimize the distortion* $D$ (4) *is*

$$p[\ell] = \frac{c[\ell-1] + c[\ell]}{2}, \qquad \ell \in \left[1 : 2^R - 1\right], \tag{6}$$

*where the leftmost/rightmost boundary points* $p[0]/p[2^R]$ *are equal to the smallest/largest values of the support of* $f_W$*.*

The optimal quantizer must simultaneously satisfy both (5) and (6); iterating between these two necessary conditions gives rise to the Lloyd–Max algorithm.

**Algorithm 1** (Lloyd–Max quantization)**.**
***Initial step.*** Pick an initial partition-level set $p$.
***Iterative step.*** Repeat the two steps

1. Fix $p$ and set $c$ as in (5),
2. Fix $c$ and set $p$ as in (6),

interchangeably, until the decrease in the distortion $D$ per iteration goes below a desired accuracy threshold.

Props. 1 and 2 suggest that the distortion at every iteration decreases; since the distortion is bounded from below by zero, the Lloyd–Max algorithm is guaranteed to converge to a local optimum.

Unfortunately, multiple local optima may exist in general, rendering the algorithm sensitive to the initial choice $p$.

Nonetheless, sufficient conditions for the existence of a unique global optimum were established in [18–20]. These guarantee that the algorithm converges to the global optimum for any initial choice of $p$. An important class of PDFs that satisfy these conditions is that of log-concave PDFs.

**Theorem 1** ([18–20])**.** *Let the PDF* $f_W$ *be log-concave. Then, Alg. 1 converges to a unique solution that minimizes the mean squared error distortion* (4)*.*

*Controller Design*

We now describe the optimal greedy control policy. To that end, we make use of the following lemma that extends the control–estimation separation principle to networked control.

**Lemma 1** ([21], [22])**.** *The optimal controller is* $U_t = -k_t\hat{X}_t$*, where* $\hat{X}_t \triangleq \mathbb{E}\left[X_t|\ell^t\right]$ *is the MMSE estimate of* $X_t$*, and* $K_t$ *is the optimal LQR control gain, given by [23]:*

$$k_t = \frac{s_{t+1}}{s_{t+1}+r_t}a, \qquad\qquad s_t = q_t + \frac{s_{t+1}r_t}{s_{t+1}+r_t}a^2,$$

*with* $s_T = q_T$ *and* $s_{T+1} = k_T = 0$*. Moreover, this controller achieves a cost of*

$$\bar{J}_T = \frac{1}{T}\sum_{t=1}^{T}\left(s_t\sigma_{W_t}^2 + g_t\mathbb{E}\left[(X_t-\hat{X}_t)^2\right]\right),$$

*with* $g_t = s_{t+1}a^2 - s_t + q_t$*.*

*Remark* 3. Lem. 1 holds true for any memoryless channel, with $\hat{X}_t = \mathbb{E}\left[X_t|\ell^t\right]$, where $\ell_t$ is the channel output at time $t$.

The optimal greedy algorithm minimizes the estimation distortion $\mathbb{E}\left[(X_t-\hat{X}_t)^2\right]$ at time $t$, without regard to its effect on future distortions. To that end, at time $t$, the encoder and the decoder calculate the PDF of $x_t$ conditioned on $\ell^{t-1}$, $f_{X_t|\ell^{t-1}}$ via sequential Bayesian filtering [24], and apply the Lloyd–Max quantizer to this PDF. We refer to $f_{X_t|\ell^{t-1}}$ and to $f_{X_t|\ell^t}$ as the *prior* and posterior PDFs, respectively.

**Algorithm 2** (Optimal greedy control)**.**

***Initialization.*** Both the encoder and the decoder set

1. $\{s_t, k_t | t \in [1:T]\}$ as in Lem. 1, for the given $T$, $\{q_t\}$, $\{r_t\}$ and $a$.
2. $\ell_0 = X_0 = U_0 = 0$.
3. The prior PDF: $f_{X_1|\ell_0}(x_1|0) \equiv f_W(x_1)$.

***Observer/Encoder.*** At time $t \in [1:T-1]$:

1. Observes the current state $x_t$.
2. Runs the Lloyd–Max algorithm (Alg. 1) with respect to the prior PDF $f_{X_t|\ell^{t-1}}$ to obtain the quantizer $\mathcal{Q}_t(x_t)$ of rate $R$; denote its partition and reproduction sets by $p_t$ and $c_t$, respectively, and the cell corresponding to $p_t[l]$—by $\mathcal{I}_t[l]$.
3. Quantizes the system state $x_t$ [recall Def. 2]: $l_t = \mathcal{E}_{\mathcal{Q}_t}(x_t) =: \mathcal{E}_t(x^t), \hat{x}_t = \mathcal{Q}_t(x_t) = \mathcal{D}_{\mathcal{Q}_t}(l_t)$, where $\mathcal{E}_t(x^t)$ is the overall action of the observer/encoder at time $t$ (2).
4. Transmits the quantization index $l_t$.
5. Calculates the posterior PDF $f_{X_t|\ell^t}(x_t|l^t)$:[3]

$$f_{X_t|\ell^t}(x_t|l^t) = \begin{cases} f_{X_t|\ell^{t-1}}(x_t|l^{t-1})/\gamma, & x_t \in \mathcal{I}_t[l_t] \\ 0 & \text{otherwise} \end{cases} \quad ; \quad \gamma \triangleq \int_{p_t[l_t]}^{p_t[l_t+1]} f_{X_t|\ell^{t-1}}(\alpha|l^{t-1})d\alpha.$$

[3]We use here the regularity assumption.

6. Determines the next prior PDF using (1), $u_t = -k_t \hat{x}_t$:

$$f_{X_{t+1}|\ell^t}\left(x_{t+1}|l^t\right) = \frac{1}{|a|} f_{X_t|\ell^t}\left(\frac{x_{t+1}-u_t}{a}\middle| l^t\right) * f_W\left(x_{t+1}\right),$$

where '$*$' denotes the convolution operation, and the two convolved terms correspond to the PDFs of the quantization error $a(X_t - \hat{X}_t)$ and the disturbance $W_t$.

***Controller/Decoder.*** At time $t \in [1 : T-1]$:

1. Runs the Lloyd–Max algorithm (Alg. 1) w.r.t. the prior PDF $f_{X_t|\ell^{t-1}}$ as in Step 2 of the observer/encoder protocol.
2. Receives the index $l_t$.
3. Reconstructs the quantized value: $\hat{x}_t = \mathcal{D}_{\mathcal{Q}_t}(l_t)$.
4. Generates the control actuation $u_t = -k_t \hat{x}_t := \mathcal{D}_t(\hat{x}^t)$.
5. Calculates $f_{X_t|\ell^t}$ and $f_{X_{t+1}|\ell^t}$ as in Steps 5 and 6 of the observer/encoder protocol.

**Theorem 2.** *Let $f_W$ be log-concave. Then, Alg. 2 is the optimal greedy control policy.*

This theorem was proved in [14], using Thm. 1 and the following result.

**Assertion 1** ([14])**.** *The prior PDFs $\{f_{X_t|\ell^t} : t \in [1 : T]\}$ are log-concave if regular quantizers are used at every time step.*

## Event-triggered Control

In this section, we adopt the greedy algorithm to the event-triggered control setting.

We concentrate on the cases of packets of zero (empty packets) and single bit rates, as in these regimes the advantage of the algorithm is most pronounced and the exposition of the algorithm is the simplest; extension to higher rates is straightforward.

The two [one] cells corresponding to the single-bit [empty] packet along with the silence symbol form a three-level [two-level] algorithm. We add a constraint $\delta$ on the minimal probability of the silent symbol; clearly, the average transmission rate is equal to $\bar{R} \triangleq \mathbb{E}[R] \equiv 1-\delta$ for $R = 1$. To optimize performance, the silence symbol needs to be assigned to the cell with the maximal probability:

$$\max_{\ell} \int_{p[\ell]}^{p[\ell+1]} f_W(w) dw \geq \delta, \tag{7}$$

where $\ell \in \{0,1,2\}$ [$\ell \in \{0,1\}$]. The cell-index $\ell$ that achieves the maximum in (7) corresponds to the *silent cell*; we denote this index by $\ell^*$.

Hence, the standard Lloyd–Max quantizer of Alg. 1 in each time step should be replaced by the following algorithm, which first checks whether standard three-level [two-level] Lloyd–Max quantization satisfies the constraint (7) and, if not, runs the algorithm with the constraint (7) imposed on a different cell each time, and chooses the one that achieves minimal average distortion. With the constraint imposed on a particular cell, the algorithm iterates between two steps: choosing the optimum $c$ for a fixed $p$ and choosing the optimum $p$ for a fixed $c$. The first step is the same as the standard Lloyd-Max quantizer. For the second step, the Karush–Khun–Tucker (KKT) conditions are employed [25, Ch. 5]. We start with the simple case of $R = 0$.

**Algorithm 3** (Minimal cell-probability constrained quantization for $R = 0$)**.** Apply Alg. 1. If the constraint (7) is satisfied for the resulting quantizer, use this quantization law. Else increase move the (only) boundary to increase the probability of the larger-probability cell until it satisfies (7).

**Algorithm 4** (Minimal cell-probability constrained quantization for $R = 1$)**.** *Unconstrained algorithm.* Apply Alg. 1. If the constraint (7) is satisfied for the resulting quantizer, use this quantization law. Else, set $p[0]$ and $p[3]$ to the smallest and largest values of the support of $f_W$, and run the following.

0) $\ell^* = 0$.

   (a) Set $p[1]$ such that $\int_{p[0]}^{p[1]} f_W(w)dw = \delta$.

   (b) Compute $c[0]$ as in (5).

   (c) Run Alg. 1 for the remaining two cells (with $p[0], p[1], c[0]$ remain fixed), to determine $p[2]$ and $c[2]$.

   (d) Denote the resulting overall quantizer and distortion by $\mathcal{Q}_0$ and $D_0$, respectively.

1. $\ell^* = 1$.
   ***Initial step.*** Pick an initial partition-level set $p$.
   ***Iterative step.*** Repeat the following steps

   (a) Fix $p$ and set $c$ as in (5),

   (b) Fix $c$ and set $p$ as in (6),

   (c) If $p$ does not satisfy the constraint (7), set $p$, in accordance with the KKT conditions, as the solution of

$$\begin{cases} \delta = \displaystyle\int_{p[1]}^{p[2]} f_W(w)dw & \text{(8a)} \\ p[2] = \dfrac{c[0] - c[1]}{c[2] - c[1]} p[1] + \dfrac{c[2]^2 - c[0]^2}{2(c[2] - c[1])} & \text{(8b)} \end{cases}$$

   (d) If no solution to (8) exists, replace (8b) with the choice that gives the smaller distortion out of $p[1] = p[0]$ and $p[2] = p[3]$,

   until the decrease in the distortion $D$ per iteration is below a desired accuracy threshold. Denote the resulting quantizer and distortion by $\mathcal{Q}_1$ and $D_1$, respectively.

2. $\ell^* = 2$.

(a) Set $p[2]$ such that $\int_{p[2]}^{p[3]} f_W(w)dw = \delta$.

(b) Compute $c[2]$ as in (5).

(c) Run Alg. 1 for the remaining two cells (with $p[2], p[3], c[2]$ remain fixed), to determine $p[1]$ and $c[1]$.

(d) Denote the resulting overall quantizer and distortion by $\mathcal{Q}_2$ and $D_2$, respectively.

3. Set the quantizer to $\mathcal{Q}_i$, where $i^* = \arg\min_{i=0,1,2} D_i$.

Replacing the Lloyd–Max quantizer of Alg. 1 with the constrained variant of Algs. 3 or 4 gives rise to the following event-triggered variant of Alg. 2.

**Algorithm 5** (Greedy event-triggered control)**.**
***Initialization.*** Both the encoder and the decoder

1. Run steps 1–3 of the initialization of Alg. 2.
2. Set $\delta = 1 - \bar{R}$.[4]

***Observer/Encoder.*** At time $t \in [1 : T - 1]$:

1. Observes $x_t$.
2. Runs Alg. 4 with respect to the prior PDF $f_{X_t|\ell^{t-1}}$ and the maximal probability constraint $\delta$ to obtain the quantizer $\mathcal{Q}_t$; denote its partition and reproduction sets by $p_t$ and $c_t$, respectively, the index of the silent cell—by $\ell_t^*$, and the cell corresponding to $p_t[\ell]$—by $\mathcal{I}_t[\ell]$.
3. Quantizes the system state $x_t$ as in Step 3 of the observer/encoder protocol of Alg. 2.
4. If $l_t \neq l_t^*$, transmits the index $l_t$; otherwise, remains silent.
5. Calculates the posterior PDF $f_{X_t|\ell^t}$ and the next prior PDF $f_{X_{t+1}|\ell^t}$ as in Steps 5 and 6 of the observer/encoder protocol of Alg. 2, respectively.

***Controller/Decoder.*** At time $t \in [1 : T - 1]$:

1. Runs Alg. 4 [Alg. 3] w.r.t. the prior PDF $f_{X_t|\ell^{t-1}}$ as in Step 2 of the observer/encoder protocol.
2. Receives the index $l_t$: in case of silence, recovers $l_t = l_t^*$.
3. Reconstructs the quantized value: $\hat{x}_t = \mathcal{D}_{\mathcal{Q}_t}(l_t)$.
4. Generates the control actuation $u_t = -k_t \hat{x}_t$.
5. Calculates the posterior PDF $f_{X_t|\ell^t}$ and the next prior PDF $f_{X_{t+1}|\ell^t}$ as in Steps 5 and 6 of the observer/encoder protocol of Alg. 2, respectively.

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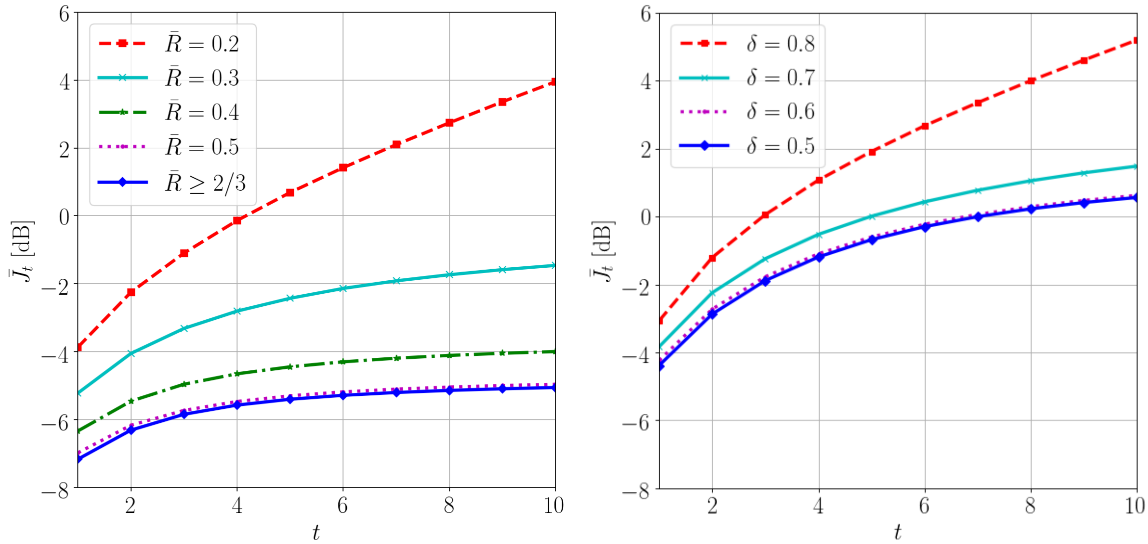
[4]Recall that we assume $\bar{R} \in (0, 1]$.




(a) $R = 1$

(b) $R = 0$

Figure 1: Average-stage LQ cost $\bar{J}$ versus time $t$ for $a = 1.5, q_t \equiv 1, r_t \equiv 0$ and an i.i.d. standard Gaussian disturbance sequence in (1), under event-triggered control with different (minimal) silence probabilities $\delta$, for $R = 0$ and $R = 1$.

## Numerical Calculations

We compare in Fig. 1 the performance of Alg. 2 of rate $R = 1$ with the event-triggered algorithm (Algs. 3 and 4) for various transmission rates $\bar{R}$, for the LQG setup with $a = 1.5$, $q_t \equiv 1$, $r_t \equiv 0$, and i.i.d. standard Gaussian disturbance ($\sigma_W^2 = 1$).

Note that for $\delta \leq 0.5$ in Fig. 1b and rate $\bar{R} = 1$ in Fig. 1a correspond to two- and three-level 2, respectively, since the constraint 7 is trivially satisfied in these cases.

## Future Work

(Unconstrained) Lloyd–Max quantization (Alg. 1) is guaranteed to converge to the global optimum for log-concave PDFs. It would be interesting to prove a similar result for its constrained variant—Alg. 4. We verified this numerically for Gaussian, exponential and Laplace PDFs and conjecture that it holds true for all log-concave PDFs.

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