Algorithms for Optimal Control with Fixed-Rate Feedback

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***Abstract*— We consider a discrete-time linear quadratic Gaussian networked control setting where the (full information) observer and controller are separated by a fixed-rate noiseless channel. The minimal rate required to stabilize such a system has been well studied. However, for a given fixed rate, how to quantize the states so as to optimize performance is an open question of great theoretical and practical significance. We concentrate on minimizing the control cost for first-order scalar systems. To that end, we use the Lloyd–Max algorithm and leverage properties of logarithmically-concave functions to construct the optimal quantizer that greedily minimizes the cost at every time instant. By connecting the globally optimal scheme to the problem of scalar successive refinement, we argue that its gain over the proposed greedy algorithm is negligible. This is significant since the globally optimal scheme is often computationally intractable. All the results are proven for the more general case of disturbances with logarithmically-concave distributions.**

I. Introduction

The demand for new and improved control techniques over unreliable communication links is constantly growing, due to the rise of emerging opportunities in the Internet of Things realm, as well as due to new surprising applications in Biology and Neuroscience. One of the most widely studied such *networked control* setups is that of control over discretized packeted communication channels [1]–[5]. This setup can be further divided into two regimes: fixed-rate feedback — where exactly r bits can be noiselessly conveyed from the observer/encoder to the controller/decoder [6], [7], and variable-rate feedback — where r bits are available *on average* and the observer/encoder can decide how many bits to allocate at each time instant [8].

Although much effort has been put into determining the conditions for the stabilizability of such systems, less so has been done for determining the optimal attainable control costs — which are of great importance in practice — with several notable exceptions [9]–[11].

In this work, we construct algorithms for the fixed-rate feedback setting.

However, in contrast to the works of Minero et al. [12] and Yüksel [7], which concentrated on the conditions for system stabilizability, using *adaptive* uniform and logarithmic quantizers,[1] respectively, we attempt to optimize the control cost.

To that end, we concentrate on the class of disturbances that have logarithmically-concave (log-concave) probability density functions (PDFs) (the Gaussian PDF being an important special case), for which the Lloyd–Max algorithm [13, Ch. 6] is known to converge to the optimal quantizer [14]–[16]. Using Lloyd–Max quantization at every step, proposed previously by Nakahira [17] (albeit without any optimality claims), and proving that the resulting system state — which is composed of the scaled sums of quantization errors of the previous steps and the new disturbances — continues to have a log-concave PDF, leads to an optimal greedy algorithm.

To tackle the more challenging task of designing a globally optimal quantizer, we recast the problem as that of designing an optimal quantizer for the problem of *sequential coding of correlated sources* [18] (see also [19] and references therein).

An extreme case of this problem is provided by that of linear quadratic regulator (LQR) control, where the only disturbance is the initial state with log-concave PDF. This problem is equivalent, in turn, to that of *successive refinement* [20], which can be regarded as a special case of sequential coding of correlated sources. Surprisingly, for the latter, a computationally plausible variant of the Lloyd–Max algorithm exists [21] that is known to achieve globally optimal performance for log-concave functions [22].

Although greedy optimization is known to be suboptimal [23], simulations for the LQR case show that the gain of the globally optimal algorithm over the optimal greedy one is modest even at low rates (for which the gain is expected to be the largest). This, in turn, suggests that the optimal greedy algorithm will remain close in performance to the optimum for the more general case where the state is driven by i.i.d. log-concave disturbances, which includes linear quadratic Gaussian (LQG) control.

We conclude the paper by noting that in the limit of high rate, Bennett's approximated quantization law [13, Ch. 6.3] provides the (approximate) optimal quantizer, which is successively refinable, suggesting, in turn, that the optimal greedy algorithm is near optimal in this limit.

II. Problem Setup

We now formulate the control–communication setup, considered in this work. We use a discrete-time model spanning the time interval $[T] \triangleq \{1, 2, \cdots, T\}$. The plant is a discrete-time linear scalar stochastic system

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[1] It is impossible to stabilize an unstable system using fixed-rate static quantization if the distributions of the disturbances or the initial state have unbounded supports [4, Sec. III-A].

$$x_{t+1} = Ax_t + w_t + u_t, \qquad t+1 \in [T], \qquad (1)$$

where $x_t, w_t, u_t \in \mathbb{R}$ are the system state, disturbance and control action at time t, respectively. We consider two setups for the disturbance sequence $\{w_t\}$:

- **Independent and identically distributed (i.i.d.):** $\{w_t\}$ are i.i.d. according to a known log-concave PDF $f_w(w)$.
- **LQR:** w_0 is distributed according to a known log-concave PDF $f_w(w)$; $w_t = 0$ for all $t > 0$.

We further denote the variance of f_w by W and assume, w.l.o.g., that it has zero mean.

Definition 1 (Log-concave function; see [24])**.** A function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is said to be log-concave if $\log \circ f$ is concave:

$$\log f\big(\lambda x + (1-\lambda)y\big) \geq \lambda \log f(x) + (1-\lambda)\log f(y),$$

for all $\lambda \in [0,1]$ and $x, y \in \mathbb{R}$; we use the extended definition that allows $f(x)$ to assign zero values: $\log f(x) \in \mathbb{R} \cup \{-\infty\}$.

Remark 1. The Gaussian PDF is a log-concave function.

We assume the observer has perfect access to x_t at time t. However, in contrast to classical control settings, it is not co-located with the controller and communicates with it instead via a noiseless channel of *data rate* r. That is, at each time t, the observer, which also takes the role of the encoder $\mathcal{E}_t$, can perfectly convey a message (or "index") of r bits, $\ell_t \in \{0, \ldots, 2^r - 1\}$, of the past states, to the controller:

$$\ell_t = \mathcal{E}_t(x^t),$$

where we denote $a^t \triangleq (a_1, a_2, \ldots, a_t)$ and use the convention that $a^t = \emptyset$ for $t \leq 0$. We further set $\ell_0 = \ell_T = 0$.

The controller at time t, which also takes the role of the decoder $\mathcal{D}_t$, recovers the observed codeword ℓ^t and uses it to generate the control action

$$u_t = \mathcal{D}_t\left(\ell^t\right).$$

Our goal is to minimize the following average-stage linear quadratic (LQ) cost upon reaching the time horizon $T \in \mathbb{N}$:

$$\bar{J}_T \triangleq \frac{1}{T}\mathbb{E}\left[Q_T x_T^2 + \sum_{t=1}^{T-1}\left(Q_t x_t^2 + R_t u_t^2\right)\right] = \frac{1}{T}\sum_{t=1}^{T} J_t\,, \quad (2)$$

where $\{J_t\}$ are the instantaneous costs

$$J_T \triangleq \mathbb{E}\left[Q_T x_T^2\right]; \; J_t \triangleq \mathbb{E}\left[Q_t x_t^2 + R_t u_t^2\right], \; t \in [T-1]. \quad (3)$$

The weights $\{Q_t\}$ and $\{R_t\}$ penalize the state deviation and actuation effort, respectively.

III. Greedy Optimal Control

In this section we consider the i.i.d. disturbance setting. We recall the Lloyd–Max algorithm and its optimality guarantees in Sec. III-A, which are subsequently used in Sec. III-B to constructed a greedy optimal control policy.

A. Quantizer Design

Definition 2 (Scalar quantizer)**.** A scalar quantizer $\mathcal{Q}$ of rate r is described by an encoder $\mathcal{E}_\mathcal{Q} : \mathbb{R} \to \{0, \ldots, 2^r - 1\}$ and a decoder $\mathcal{D}_\mathcal{Q} : \{0, \ldots, 2^r - 1\} \to \{c[0], \ldots, c[2^r - 1]\} \subset \mathbb{R}$. We define the quantization operation $\mathcal{Q} : \mathbb{R} \to \{c[0], \ldots, c[2^r - 1]\}$ as the composition of the encoding and decoding operations: $\mathcal{Q} = \mathcal{D}_\mathcal{Q} \circ \mathcal{E}_\mathcal{Q}$.[2] The *reproduction points* $c \triangleq \{c[0], \ldots, c[2^r - 1]\}$ are assumed to be ordered, w.l.o.g.:[3]

$$c[0] < c[1] < \cdots < c[2^r - 1].$$

We denote by $\mathcal{I}[\ell]$ the collection of all points that are mapped to index ℓ (equivalently to the reproduction point $c[\ell]$):

$$\mathcal{I}[\ell] \triangleq \{x | x \in \mathbb{R}, \mathcal{E}_\mathcal{Q} = \ell\} = \{x | x \in \mathbb{R}, \mathcal{Q} = c[\ell]\}.$$

We shall concentrate on the class of *regular quantizers*.

Definition 3 (Regular quantizer)**.** A scalar quantizer is *regular* if every cell $\mathcal{I}[\ell]$ ($\ell = 0, \ldots, 2^r - 1$) is a contiguous interval that contains its reproduction point $c[\ell]$:

$$c[\ell] \in \mathcal{I}[\ell] = [p[\ell], p[\ell+1])\,, \qquad \ell = 0, \ldots, 2^r - 1,$$

where $p \triangleq \{p[0], \ldots, p[2^r]\}$ is the set of *partition levels* — the cells boundaries. Hence, a regular scalar quantizer can be represented by the input partition-level set p and the reproduction-point set $c \triangleq \{c[0], \ldots, c[2^r - 1]\}$.

Cost: The cost we wish to minimize is the mean squared error distortion between a source w with a given PDF f_w and its quantization $\mathcal{Q}(w)$:

$$D \triangleq \mathbb{E}\left[(w - \mathcal{Q}(w))^2\right] \quad (4a)$$

$$= \sum_{\ell=0}^{2^r-1} \int_{p[\ell]}^{p[\ell+1]} (w - c[\ell])^2 f_w(w) dw. \quad (4b)$$

Denote by D^* the minimal achievable distortion D; the *optimal quantizer* is the one that achieves D^*.

Remark 2. We shall concentrate on log-concave PDFs f_w, which are therefore continuous [24]. Hence, the inclusion or exclusion of the boundary points in each cell does not affect the distortion of the quantizer meaning that the boundary points can be broken systematically.

Remark 3. If the input PDF has an infinite/semi-infinite support, then the leftmost and/or rightmost intervals of the quantizer are open ($p[0]$ and/or $p[2^r]$ take infinite values).

The optimal quantizer satisfies the following necessary conditions [13, Ch. 6.2].

Proposition 1 (Centroid condition)**.** *For a fixed partition-level set p (fixed encoder), the reproduction-point set c (decoder) that minimizes the distortion D* (4) *is*

$$c[\ell] = \mathbb{E}\left[w \,\middle|\, p[\ell] < w \leq p[\ell+1]\right], \; \ell = 0, \ldots, 2^r - 1. \quad (5)$$

Proposition 2 (Nearest neighbor condition)**.** *For a fixed reproduction-point set c (fixed decoder), the partition-level set p (encoder) that minimize the distortion D* (4) *is*

$$p[\ell] = \frac{c[\ell-1] + c[\ell]}{2}, \qquad \ell = 1, 2, \ldots, 2^r - 1, \quad (6)$$

where the leftmost/rightmost boundary points $p[0]/p[2^r]$ are equal to the smallest/largest values in the support of f_w.

[2] The encoder and decoder that give rise to the same parameter are unique up to a permutation of the labeling of the index ℓ.

[3] If some inequalities are not strict, then the quantizer can be reduced to another quantizer with lower rate.

The optimal quantizer must simultaneously satisfy both (5) and (6); iterating between these two necessary conditions gives rise to the Lloyd–Max algorithm.

Algorithm 1 (Lloyd–Max)**.**

Initial step. Pick an initial partition-level set p.

Iterative step. Repeat the two steps

1) Fix p and set c as in (5),
2) Fix c and set p as in (6),

interchangeably, until the decrease in the distortion D per iteration goes below a desired threshold.

Props. 1 and 2 suggest that the distortion at every iteration decreases; since the distortion is bounded from below by zero, the Lloyd–Max algorithm is guaranteed to converge to a local optimum.

Unfortunately, multiple local optima may exist in general, rendering the algorithm sensitive to the initial choice p.

Nonetheless, sufficient conditions for the existence of a unique global optimum were established in [14]–[16]. These guarantee that the algorithm converges to the global optimum for any initial choice of p. An important class of PDFs that satisfy these conditions is that of log-concave PDFs.

Theorem 1 ([14]–[16])**.** *Let the PDF f_w be log-concave. Then, the Lloyd–Max algorithm converges to a unique solution that minimizes the mean squared error distortion* (4).

B. Controller Design

We now describe the optimal greedy control policy. To that end, we make use of the following lemma that extends the control–estimation separation principle to networked control.

Lemma 1 ([25], [9])**.** *The optimal controller is given by*

$$u_t = -K_t \hat{x}_t, \qquad \hat{x}_t \triangleq \mathbb{E}\left[x_t \middle| \ell^t\right],$$

where K_t is the optimal LQR control gain, given by [26]:[4]

$$K_t = \frac{L_{t+1}}{L_{t+1}+R_t}A, \qquad L_t = Q_t + \frac{L_{t+1}R_t}{L_{t+1}+R_t}A^2, \quad (7)$$

with $L_{T+1}=0$. Moreover, this controller achieves the cost

$$\bar{J}_T = \frac{1}{T}\sum_{t=1}^{T}\Big(L_t W + G_t \mathbb{E}\left[(x_t-\hat{x}_t)^2\right]\Big), \qquad (8)$$

with $G_t = L_{t+1}A^2 - L_t + Q_t$.

Remark 4. Lem. 1 holds true for any memoryless channel, with $\hat{x}_t = \mathbb{E}\left[x_t|\ell^t\right]$, where ℓ_t is the channel output at time t.

The optimal greedy algorithm minimizes the estimation distortion $\mathbb{E}\left[(x_t-\hat{x}_t)^2\right]$ at time t, without regard to its effect on future distortions. To that end, at time t, the transmitter and the receiver calculate the the PDF of x_t conditioned on ℓ^{t-1}, $f_{x_t|\ell^{t-1}}$, and apply the Lloyd–Max quantizer to this PDF. We refer to $f_{x_t|\ell^{t-1}}$ and to $f_{x_t|\ell^t}$ as the *prior* and posterior PDFs, respectively.

Algorithm 2 (Optimal greedy control)**.**

Initialization. Both the encoder and the decoder set

1) $\{L_t, K_t | t\in[T]\}$ as in Lem. 1, for the given T, $\{Q_t\}$, $\{R_t\}$ and A.
2) $\ell_0 = x_0 = u_0 = 0$.
3) The prior PDF: $f_{x_1|\ell_0}(x_1|0) \equiv f_{w_0}(x)$.

Observer/Encoder. At time $t\in[T]$:

1) Observes x_t.
2) Runs the Lloyd–Max algorithm (Alg. 1) with respect to the prior PDF $f(x_t|\ell^{t-1})$ to obtain the quantizer $\mathcal{Q}_t(x_t)$ of rate r; denote its partition and reproduction sets by p_t and c_t, respectively.
3) Quantizes the system state x_t [recall Def. 2]:
$$\ell_t = \mathcal{E}_{\mathcal{Q}_t}(x_t), \qquad \hat{x}_t = \mathcal{Q}_t(x_t) = \mathcal{D}_{\mathcal{Q}_t}(\ell_t).$$
4) Transmits the quantization index ℓ_t.
5) Calculates the posterior PDF $f(x_t|\ell^t)$:[5]
$$f(x_t|\ell^t) = \begin{cases} f_{x_t|\ell^{t-1}}(x_t|\ell^{t-1})/\gamma, & x_t \in [p_t[\ell_t], p_t[\ell_t+1]) \\ 0 & \text{otherwise} \end{cases}$$
where $\gamma \triangleq \int_{p_t[\ell_t]}^{p_t[\ell_t+1]} f_{x_t|\ell^{t-1}}(\alpha|\ell^{t-1})d\alpha$.
6) Determines the next prior PDF using (1), $u_t = -K_t\hat{x}_t$:
$$f_{x_{t+1}|\ell^t}\left(x_{t+1}|\ell^t\right) = \frac{1}{|A|} f_{x_t|\ell^t}\left(\frac{x_{t+1}-u_t}{A}\middle|\ell^t\right) * f_w\left(x_{t+1}\right),$$
where '$*$' denotes the convolution operation, and the two convolved terms correspond to the PDFs of the quantization error $A(x_t-\hat{x}_t)$ and the disturbance w_t.

Controller/Decoder. At time $t\in[T]$:

1) Runs the Lloyd–Max algorithm (Alg. 1) w.r.t. the prior PDF $f(x_t|\ell^{t-1})$ as in Step 2 of the observer/transmitter protocol.
2) Receives the index ℓ_t.
3) Reconstructs the quantized value: $\hat{x}_t = \mathcal{D}_{\mathcal{Q}_t}(\ell_t)$.
4) Generates the control actuation $u_t = -K_t\hat{x}_t$.
5) Calculates $f(x_t|\ell^t)$ and $f(x_{t+1}|\ell^t)$ as in Steps 5 and 6 of the observer/transmitter protocol.

Theorem 2. *Let f_w be a log-concave PDF. Then, Alg. 2 is the optimal greedy control policy.*

The following is an immediate consequence of the log-concavity of the Gaussian PDF.

Corollary 1. *Let f_w be a Gaussian PDF. Then, Alg. 2 is the optimal greedy control policy.*

Recall that the Lloyd–Max Algorithm converges to the global minimum for log-concave PDFs. Consequently, in order to prove the greedy optimality of Alg. 2, we need to show that all the prior PDFs $\{f(x_t|\ell^t) : t\in[T]\}$ are log-concave. We provide a formal proof of this result in [27], which relies on the following log-concavity properties.

[4]We set $L_T = Q_T$ and $K_T = 0$ for $R_T = 0$.

[5]This expression follows from the regularity assumption.

Assertion 1 ([24])**.** *Let $f(x)$ and $g(x)$ be log-concave functions. Then, the following are also log-concave functions:*

- ***Affinity:*** *$cf(ax+b)$ for any constants $a,b,c\in\mathbb{R}$.*
- ***Truncation:*** $\begin{cases} f(x) & x\in I \\ 0 & \text{otherwise}\end{cases}$*, for any interval I, possibly (semi-)infinite.*
- ***Convolution:*** *$f(x)*g(x)$.*

IV. Globally Optimal LQR Control

In this section, we study the LQR control setting, namely, the case where w_0 has a log-concave PDF f_w and $w_t=0$ for all $t\in[T-1]$. Clearly, this is equivalent to the case of a random initial condition x_0 and $w_t\equiv 0$ for all t, and is therefore referred to as LQR control.

We construct a globally optimal control policy in Sec. IV-B by connecting the problem to that of scalar successive refinement [21], [22], which is formulated and reviewed in Sec. IV-A. The resulting quantizers are commonly referred to as *multi-resolution scalar quantizers* (MRSQs).

A. Successive Refinement

A T-step MRSQ successively quantizes a single source sample $w\in\mathbb{R}$ with PDF f_w using a series of rate r quantizers $\mathcal{Q}^T$: At stage $t\in[T]$, r bits are available for the re-quantization of the source w, and are encoded into an index $\ell\in\{0,\ldots,2^r-1\}$. ℓ_t, along with all previous indices ℓ^{t-1}, is then used for the construction of a refined description $\hat{w}_t=\mathcal{Q}_t(w)$.

Definition 4 (MRSQ)**.** A T-step MRSQ of rate r is described by a series of T encoders $(\mathcal{E}_{\mathcal{Q}_1},\ldots,\mathcal{E}_{\mathcal{Q}_T})$ and a series of T decoders $(\mathcal{D}_{\mathcal{Q}_1},\ldots,\mathcal{D}_{\mathcal{Q}_T})$, with $\mathcal{E}_{\mathcal{Q}_t}:\mathbb{R}\to\{0,\ldots,2^r-1\}$ and $\mathcal{D}_{\mathcal{Q}_t}:\{0,\ldots,2^r-1\}^t\to\{c_t[0],\ldots,c_t[2^{tr}-1]\}$ serving as the encoder and decoder at time t, respectively. We define the quantization operation $\mathcal{Q}_t:\mathbb{R}\to\{c_t[0],\ldots,c_t[2^{tr}-1]\}$, at time t, as the composition of all the encodings until time t and the decoding at time t: $\mathcal{Q}_t=\mathcal{D}_{\mathcal{Q}_t}\circ(\mathcal{E}_{\mathcal{Q}_1},\ldots,\mathcal{E}_{\mathcal{Q}_t})$.

This definition means that, although the overall effective rate of the quantizer at time t is tr, only the last r bits, corresponding to ℓ_t, are determined during time step t. At the decoder, these bits are appended to the previously determined and received $(t-1)r$ bits (corresponding to ℓ^{t-1}), for the construction of a description of w at time t, $\hat{w}_t=\mathcal{Q}_t(w)$.

Definition 5 (Regular MRSQ)**.** A T-step MRSQ is *regular* if the quantizer at each step $t\in[T]$ is regular and the partitions of subsequent stages are nested, as follows. For each time $t\in\{2,\ldots,T\}$:

$$p_t\left[\ell\cdot 2^r\right]=p_{t-1}[\ell];\qquad \ell=0,\ldots,(t-1)r-1,\quad (9)$$

where p_t is the partition-level set of the quantizer at time t.

Remark 5. The relation in (9) implies that given p_T, the partitions of all the previous stages can be deduced.

Remark 6. Counterexamples for both discrete and continuous PDFs have been devised, for which regular MRSQs are strictly suboptimal [28], [29]. However, none such are known for the case of log-concave input PDFs. Furthermore, by using Bennett's law [13, Ch. 6.3], regular MRSQs have been proved to be optimal in the limit of high rate [22, Sec. VII].

Our goal here is to design an MRSQ that minimizes the weighted time-average squared quantization error $\bar{D}$ of an input w with a given PDF $f_w(w)$ and positive weights $\{\tilde{G}_t\}$:

$$\bar{D}=\sum_{t=1}^{T}\tilde{G}_t\mathbb{E}\left[\{w-\hat{w}_t\}^2\right].\qquad (10)$$

We next present a *Generalized Lloyd–Max Algorithm* due to Brunk and Farvardin [21] for constructing MRSQs, which is in turn an adaptation of an algorithm for scalar multiple descriptions by Vaishampayan [30]. Similarly to the standard Lloyd–Max algorithm (Alg. 1), the generalized variant iterates between structuring the reproduction point sets c^T given the partition p_T (recall Rem. 5), and vice versa.

Furthermore, the centroid condition of Prop. 1 remains unaltered, as it does not have any direct effect on other stages, and is calculated separately for each stage. The partition of earlier stages, on the other hand, has a direct effect on the boundaries of newer stages, due to the nesting property (9). Consequently, the nearest neighbor condition of Prop. 2 is replaced by a weighted variant [21], [30].

Proposition 3 (Weighted nearest neighbor)**.** *The optimal partition p_T for a given sequence of reproduction-point sets c^T is determined by the weighted nearest neighbor condition:*

$$p_T[\ell]=\max_{0\le i\le 2^{Tr}-1:\alpha_i<\alpha_\ell}\frac{\beta_\ell-\beta_i}{2(\alpha_\ell-\alpha_i)}\qquad (11a)$$

$$p_T[\ell+1]=\min_{0\le i\le 2^{Tr}-1:\alpha_i>\alpha_\ell}\frac{\beta_\ell-\beta_i}{2(\alpha_\ell-\alpha_i)}\qquad (11b)$$

for $0\le\ell\le 2^{Tr}-1$, where

$$a_t[\ell]\triangleq\left\lceil(\ell+1)2^{(T-t)r}\right\rceil-1,$$

$$\alpha_\ell\triangleq\sum_{t=1}^{T}\tilde{G}_tc_t[a_t[\ell]],\qquad \beta_\ell\triangleq\sum_{t=1}^{T}\tilde{G}_tc_t^2[a_t[\ell]].$$

Remark 7. α_ℓ and β_ℓ can be viewed as weighted centroid and squared centroid, respectively. In these terms, the partition points in (11a) and (11b) reduce to the midpoints of adjacent centroids of the standard Lloyd–Max algorithm (6).

Similarly to the optimal one-stage quantizer of Sec. III-A, the optimal MRSQ has to satisfy both the centroid condition of Prop. 1 and the weighted nearest neighbor condition of Prop. 3, simultaneously. Iterating between these conditions gives rise to the Generalized Lloyd–Max algorithm.

Algorithm 3 (Generalized Lloyd–Max)**.**
Initial step. Pick an initial partition p_T.
Iterative step. Repeat the two steps

1) Fix p_T and evaluate c^T as in (5),
2) Fix c^T and evaluate p_T as in (11),

interchangeably, until the decrease in the weighted distortion $\bar{D}$ is below a desired accuracy threshold.

As in the standard Lloyd–Max algorithm, Alg. 3 may converge to different local minima for different initializations

p_T. And similarly, sufficient conditions can be derived for the existence of a unique local — and thus also global — minimum [22]. Log-concave PDFs satisfy these conditions, suggesting that Alg. 3 is globally optimal for such PDFs.

Theorem 3 ([22])**.** *Let the PDF $f(w)$ be log-concave and $\{\tilde{G}_t\}$ a positive weight sequence. Then, Alg. 3 converges to a unique solution that minimizes the weighted mean square error distortion* (10) *with weights $\{\tilde{G}_t\}$.*

B. Controller Design

The following is the counterpart of Lem. 1 for LQR control, the proof of which can be found in [25], [9].

Lemma 2. *Consider the LQR setting: w_0 has a PDF f_w, and $w_t \equiv 0$ for $t \in [T-1]$. Then, the optimal controller for the robust control problem* (2) *is the same as in Lem. 1 with* (8) *replaced by $J_T = L_1 W + \sum_{t=1}^{T} G_t \mathbb{E}\left[(x_t - \hat{x}_t)^2\right]$.*

In order to construct a globally optimal control policy, we need to find a quantizer that minimizes

$$\sum_{t=0}^{T} G_t \mathbb{E}\left[(x_t - \hat{x}_t)^2\right]. \tag{12}$$

The following simple result connects this problem with that of designing an MRSQ that minimizes (10).

Lemma 3. *Let $\hat{w}_t$ be the estimate of the source sample w_0 at time $t \in \{1, \dots, T\}$, produced by the MRSQ that minimizes* (10) *with weights*

$$\tilde{G}_t = A^{2(t-1)} G_t \,. \tag{13}$$

Then, the quantizer $\hat{x}_t$ that minimizes (12) *is given by*

$$\hat{x}_t = A\hat{x}_{t-1} + u_{t-1} + A^{t-1}\left(\hat{w}_t - \hat{w}_{t-1}\right), \tag{14}$$

with $u_t = -K_t \hat{x}_t$ and K_t given in (7)*.*

We are now ready to present the globally optimal control policy for the LQR problem.

Algorithm 4 (Globally optimal LQR control)**.**

Initialization. Both the encoder and the decoder:

1) Construct $\{L_t, K_t, G_t | t \in [T]\}$ as in Lem. 1 for the given T, $\{Q_t\}$, $\{R_t\}$ and A.
2) Set $\tilde{G}_t = A^{2(t-1)} G_t$ as in (13).
3) Construct the T-step MRSQ sequence $(\mathcal{Q}_1, \dots, \mathcal{Q}_T)$ using Alg. 3 for the source w_0 and weights $\tilde{G}_t$.

Observer/Encoder. Observes w_0. At time $t \in [T]$:

1) Generates the quantizer index: $\ell_t = \mathcal{E}_{\mathcal{Q}_t}(w_0)$.
2) Transmits ℓ_t.

Controller/Decoder. At each time $t \in [T]$:

1) Receives ℓ_t.
2) Generates the description: $\hat{w}_t = \mathcal{D}_{\mathcal{Q}_t}(\ell^t) = \mathcal{Q}_t(w_0)$.
3) Generates $\hat{x}_t$ as in (14).
4) Generates the control actuation $u_t = -K_t \hat{x}_t$.

Combining Lemmata 2 and 3, and Thm. 3 leads to the global optimality of Algorithm 4.

Theorem 4. *Let f_w be a log-concave PDF. Then, Alg. 4 achieves the minimum possible average-stage LQ cost* (2)*.*

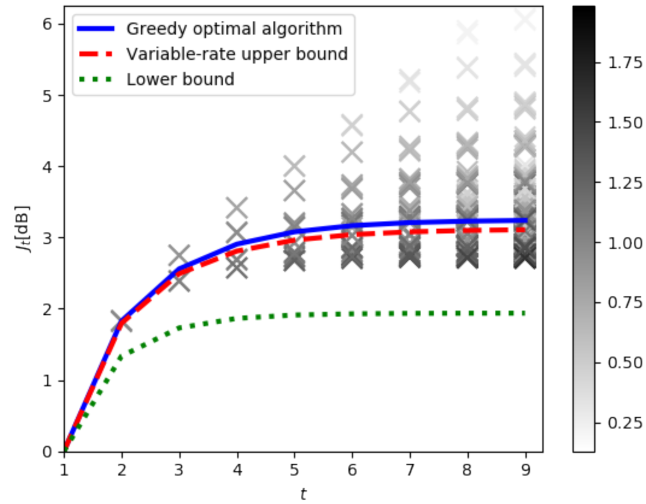

Fig. 1. Instantaneous costs J_t of the fixed-rate optimal greedy algorithm, variable-rate upper bound J_t^{UB} and J_t^{LB}, as a function of the time t for $A = 1.2$, $W = 1$, $r = 1$, $Q_t \equiv 1$, $R_t \equiv 0$. The 'x' marks correspond to instantaneous costs J_t for 2^{tr} contiguous intervals at time t for the fixed-rate setting, where the gray-scale intensity of the 'x' marks for each interval indicates the relative magnitude of probability of falling into that interval at time t.

V. Numerical Calculations

A. Greedy LQG Control

We now evaluate the instantaneous costs (3) of Alg. 2 for a standard Gaussian i.i.d. disturbance sequence $\{w_t\}$, $Q_t \equiv 1$, $R_t \equiv 0$, and $A = 1.2$. These costs are depicted in Fig. 1 along with $\mathbb{E}\left[(\hat{x}_t - x_t)^2 | \ell^t\right]$ for all admissible ℓ^t sequences. We compare them to the following upper and lower bounds, also depicted in Fig. 1, which are valid for the less restrictive case of variable-rate feedback [13, Ch. 9.9], where the *expected rate* at time t is limited by r.

Proposition 4 ([10], [11], [19], [31])**.** *Consider the setting of an variable rate subject to an expected-rate constraint r, i.i.d. Gaussian disturbances of variance W, $Q_t \equiv 1$ and $R_t \equiv 0$. Then, the instantaneous cost J_t is bounded as $J_t^{LB} \le J_t \le J_t^{UB}$, with $J_0^{UB} = J_0^{LB} = 0$ and*

$$J_{t+1}^{LB} = A^2 J_t^{LB} 2^{-2r} + W,$$
$$J_{t+1}^{UB} = \frac{2\pi e}{12} A^2 J_t^{UB} 2^{-2r} + W.$$

B. LQR Control

We now compare the performance of the optimal greedy and globally optimal algorithms for LQR control with $A = 1.5$, $Q_t \equiv 1$, $R_t \equiv 0$, standard Gaussian disturbance w_0, $w_t = 0$ for $t \ge 1$, and time horizon $T = 9$. The accumulated costs (2) for $t = 1, \cdots, 9$ are tabulated in Table I.

VI. Discussion

As is seen in Sec. V-B, even in the extreme case of low-rate $r = 1$, the improvement of the globally optimal algorithm is negligible compared to the achievable results using the optimal greedy algorithm (fractions of a percent) — a fact previously noticed in [23]. For high rates, the gap is even smaller, as Bennett's law [13, Ch 6.3] suggests that scalar quantizers become successively refinable in this limit.

t	Greedy	Optimal
1	1.0000	1.0000
2	1.8176	1.8177
3	2.4125	2.4126
4	2.8099	2.8064
5	3.0614	3.0514
6	3.2156	3.2001
7	3.3079	3.2877
8	3.3624	3.3381
9	3.3941	3.3688

TABLE I

OPTIMALITY GAP OF THE GREEDY ALGORITHM FOR TIME HORIZON $T = 9$ AND $A = 1.5$.

This seems to extend to the more general case of i.i.d. disturbances (LQG case included), as adding a noise can only reduce the gap between the two quantizers, suggesting that the optimal greedy algorithm is essentially optimal for all practical purposes.

An interesting avenue would be to explore an even lower-complexity algorithm. A noteworthy attempt was made by Yüksel who considered a low-complexity uniform adaptive quantizer. Unfortunately, such quantizers, being inherently symmetric, cannot stabilize any unstable system using one-bit quantization rate, as no zooming in/out is possible this way. This is in stark contrast to the Lloyd–Max based algorithms that can become non-symmetric even for a rate of one bit, via repeated convolution of same size PDF tails; this is evident from Sec. V.

The algorithms in this work can be extended to the case of a time-varying rate budget — including the important special case of transmission over packet-erasure channels [12], [19], [31] — where exactly r_t bits are available at time t. For this case, the greedy algorithm does not need to know the whole rate-budget sequence at the start of transmission, whereas the globally optimal algorithm needs to take into account the exact nature of the statistics of the rate sequence (which might not be available), complicating further its derivation and implementation. Similar ideas seem to extend also to the case of delayed packet arrivals, and are left for future research.

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