REMATCH AND FORWARD: JOINT SOURCE/CHANNEL CODING FOR COMMUNICATIONS

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ABSTRACT

The Gaussian parallel relay network, introduced by Schein and Gallager, consists of a concatenation of a Gaussian additive broadcast channel from a single encoder to a layer of relays followed by a Gaussian multiple-access channel from the relays to the final destination (decoder), where all noises are independent. The original setup, with white channels of equal bandwidth, was recently extended by the authors to colored noises, by applying a coding scheme coined "Rematch and Forward" (R&F). In this work we broaden the scope of this approach in a few respects: We show that the R&F strategy may employ a variety of joint source/channel coding schemes over the broadcast section of the network, and present examples of such schemes. We make the connection to a predictive view of colored channels. We show how the R&F approach can be employed, and combined with other approaches, to a more general class of networks, which we call "Layered Networks". Finally we discuss a direction for improvement of the scheme by applying joint source/channel processing to the broadcast section as well.

1. INTRODUCTION

The emerging field of communication over networks witnesses the collapse of the traditional distinction between channel, source and joint source/channel problems. Specifically, consider relay-type problems, in which a message source node wishes to pass information to a destination node, while other nodes act as relays, whose sole purpose is to help in this data transfer. Though the end-to-end problem is a channel problem, the techniques used to solve it are diverse. Consider the best known relaying techniques (see e.g. [1]), where each one is known to be optimal under different conditions of network topology and signal-to-noise ratios:

1. A channel coding approach: *Decode and Forward* (D&F), where a relay decodes the message, and then re-encodes it.
2. A source and channel coding approach: *Compress and Forward* (C&F), where a relay treats its input as a source, compresses it, and then uses a channel code to forward it.

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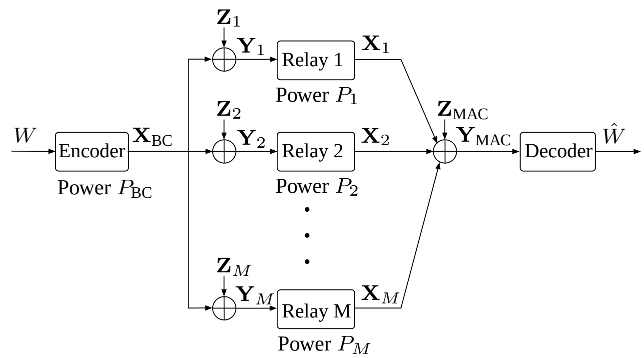

Figure 1: The Parallel Relay Network

3. A joint source/channel coding (JSCC) approach: *Amplify and Forward* (A&F), where a relay simply forwards its input, only applying power adjustment.

The last is indeed a JSCC approach, since it does not opt to decode the input, thus it treats it as a source, and then the analog treatment of this source, reminiscent of analog transmission in Gaussian point-to-point communications [3], relies upon matching between the statistics of that "source" and of the channel which initiates at the relay.

In this work we concentrate on a simple test-case: The Gaussian parallel relay network, first introduced by Schein and Gallager [10]. In this network, all the relays are ordered in a parallel manner; The message source node is connected to the relays by a Gaussian broadcast channel (BC), while the relays are connected to the destination node by a Gaussian multiple access channel (MAC). In the original setting, all noises are white and the channels all have the same bandwidth (BW). In the limit of high signal to noise ratio (SNR) in the MAC section, as well as in the limit of many relays [2], the A&F approach is optimal. In a recent work [4] we extended the view to networks where the noises are colored, and specifically to the important case of BW mismatch between the BC and MAC sections, by introducing a new relaying strategy, a JSCC approach named *Rematch and Forward* (R&F): The encoder uses a codebook of the MAC section BW; Between the encoder and the relays, a JSCC scheme suitable for BW mismatch translates the BC into an equivalent BC with the MAC BW and a "mutual-information-preserving" SNR. We presented the approach using a variant of the Analog Matching scheme [6], which uses modulo lattice modulation, in conjunction with prediction and equalization.

In this paper we present some improvements and extensions to the R&F approach. We start in Section 2 by presenting R&F, defining the requirements for a JSCC scheme to be adequate. In Section 3 we bring examples of such schemes. In Section 4 we use the R&F strategy as a building block in more general networks. Finally, in Section 5 we discuss the possibility of applying a JSCC scheme in the MAC section of the parallel relay network.

2. THE REMATCH AND FORWARD STRATEGY

In the network model of interest, depicted in Fig. 1, a message W needs to be recovered at the decoder. We assume a discrete time model, where the encoder and each relay transmit an n-dimensional block, and such blocks are denoted in bold. The additive mutually-independent noise sequences $\mathbf{Z}_1, \cdots, \mathbf{Z}_M, \mathbf{Z}_{\text{MAC}}$ are taken from stationary Gaussian processes of spectra $\mathcal{S}_1(\mathrm{e}^{j2\pi f}), \cdots, \mathcal{S}_M(\mathrm{e}^{j2\pi f}), \mathcal{S}_{\text{MAC}}(\mathrm{e}^{j2\pi f})$, respectively. The encoder and the relays are subject to average power constraints:

$$\begin{aligned} &\frac{1}{n} E\|\mathbf{X}_{\text{BC}}\|^2 \le P_{\text{BC}} \\ &\frac{1}{n} E\|\mathbf{X}_m\|^2 \le P_m \ \ , m = 1, \cdots, M \end{aligned} \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean norm.

A rate R is admissible if a coding scheme exists where each of $\lceil 2^{nR} \rceil$ equiprobable messages can be reconstructed with arbitrarily low probability of error, and the network capacity C is the supremum of all such rates. We will concentrate on the *symmetric* case:

$$\begin{aligned} \mathcal{S}_1\left(\mathrm{e}^{j2\pi f}\right) = \cdots = \mathcal{S}_M\left(\mathrm{e}^{j2\pi f}\right) &\triangleq \mathcal{S}_{\text{BC}}\left(\mathrm{e}^{j2\pi f}\right) \\ P_1 = \cdots = P_M &\triangleq P_{\text{MAC}}. \end{aligned} \tag{2}$$

Without loss of generality, assume that the noise spectra of both sections are monotonically increasing[1]. The BW of both sections are defined as the maximal frequencies in which the noise spectra are finite[2]. To simplify notations, we assume perfect low pass filters (LPFs) at the end of each channel section of the aforementioned BW ($G_{\text{BC}}\left(\mathrm{e}^{j2\pi f}\right)$ and $G_{\text{MAC}}\left(\mathrm{e}^{j2\pi f}\right)$ in Fig. 2), setting the outband noise to zero. We denote the (inband) SNRs of the BC and MAC sections by S_{BC} and S_{MAC} respectively, where the MAC SNR is defined w.r.t. the total power of all the relays:

$$\begin{aligned} S_{\text{BC}} &\triangleq \frac{P_{\text{BC}}}{\int \mathcal{S}_{\text{BC}}\left(\mathrm{e}^{j2\pi f}\right) df} \\ S_{\text{MAC}} &\triangleq \frac{M P_{\text{MAC}}}{\int \mathcal{S}_{\text{MAC}}\left(\mathrm{e}^{j2\pi f}\right) df} \ . \end{aligned} \tag{3}$$

We denote by ρ the ratio between the BC BW and the MAC BW. Equivalently, ρ uses of the BC section per each use of the MAC section are allowed. All rates in this paper are per MAC section use. Let $C(S) \triangleq 1/2 \log(1+S)$.

In the simple white equal-BW case, we can use an AWGN codebook in conjunction with the A&F strategy. Consequently, the decoder receives:

$$\mathbf{Y}_{\text{MAC}} = \sum_{m=1}^{M} X_m + Z_{\text{MAC}} = \gamma(M\mathbf{X}_{\text{BC}} + \sum_{m=1}^{M} Z_m) + Z_{\text{MAC}}$$

where γ is the relay amplification factor. The resulting achievable rate of the network is:

$$C_{\text{AF}} = C\left(\frac{M S_{\text{MAC}} S_{\text{BC}}}{S_{\text{MAC}} + S_{\text{BC}} + 1}\right). \tag{4}$$

This rate reflects the coherence gain w.r.t. the noises of both sections, since it uses the MAC addition to sum the signal components coherently. However, when we leave the white equal-BW case, this strategy fails to fully exploit the capacity of the individual channels; For example, it is restricted to the minimal BW of the two sections, since it is *fully* analog.

We extend this property to the colored (possibly BW mismatched) case, by using a JSCC scheme to turn the BC channel into an equivalent white additive BC channel (of the MAC BW) from the encoder to the relays:

$$\hat{\mathbf{V}}_m = \mathbf{V} + \tilde{\mathbf{Z}}_m \tag{5}$$

This channel has an equivalent SNR:

$$\tilde{S}_{\text{BC}} \triangleq P_{\text{BC}}/\mathrm{Var}(\tilde{\mathrm{Z}}_{\mathrm{m}}) \ . \tag{6}$$

We would like this equivalent channel to be *mutual-information preserving* w.r.t. the original channel, i.e. to have:

$$C(\tilde{S}_{\text{BC}}) = C_{\text{BC}} \ , \tag{7}$$

where C_{BC} is the point-to-point capacity of the channel from the encoder to a single relay, given by the water-pouring solution. Moreover, to obtain the coherence gain w.r.t. these equivalent noises, they need to be mutually independent. Since the only mutual dependence might be introduced by the encoder, this condition is assured if the JSCC scheme used produces an equivalent noise which is independent of the source. Formally we have the following definitions.

Definition 1 (λ-Analog Reconstruction-Error JSCC Scheme) *A JSCC scheme for a white Gaussian source with bandwidth B and an additive (possibly colored) Gaussian noise channel of bandwidth ρB is a "λ-analog reconstruction error JSCC scheme", if the* unbiased *reconstruction error*[3] *is independent of the source signal, given the encoding and decoding functions, for all frequencies within a BW of λB.*

One cannot hope to achieve independent reconstruction error beyond the degrees of freedom supplied by the source and channel. This leads to the following definition.

Definition 2 (MAR JSCC Scheme) *A "maximally analog reconstruction-error (MAR) JSCC scheme", is a λ-analog reconstruction-error JSCC scheme with $\lambda = \min\{1, \rho\}$.*

[1] This is done for convenience of the definition of bandwidth only.

[2] In practice, these frequencies arise from the sampling frequencies used, thus they are always finite.

[3] unbiased in the strong sense, viz., given any source signal value.

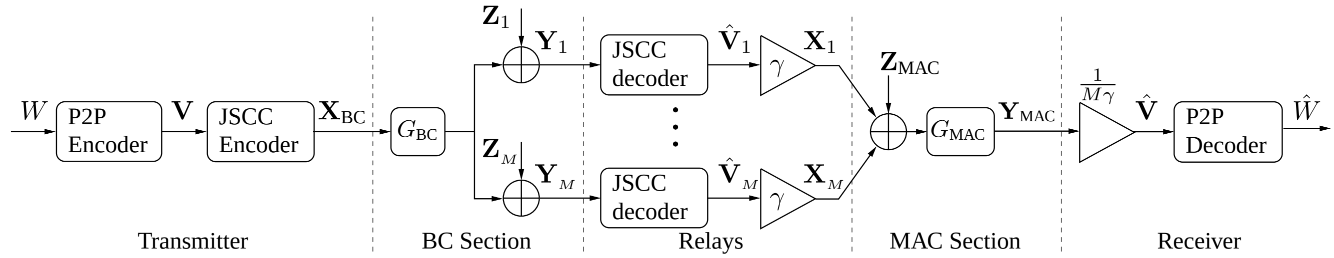

Figure 2: R&F Strategy Applied to the Parallel Relay Network (P2P stands for "point-to-point")

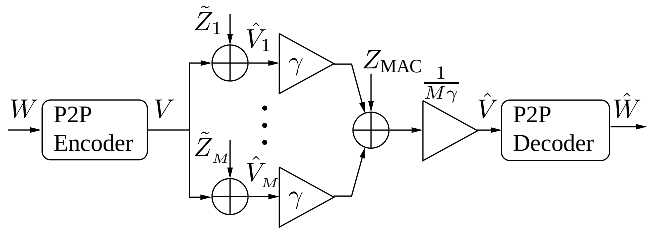

Figure 3: Equivalent A&F Channel

In order to clarify these definitions, consider the white equal-BW case ($\rho = 1$). For this case, applying only a power adjustment of the source power to the channel input, proves to be an optimal JSCC scheme [3]. Moreover, the unbiased reconstruction error is composed of only the channel noise, which is independent of the transmitted signal and hence maximally analog. On the other hand, for any separation-based scheme, the reconstruction error is made entirely of the quantization error, which depends on the actual source signal value, given the encoding and decoding functions.

We now phrase the R&F strategy as follows, see Fig. 2:

1. Use an encoder, suitable for a point-to-point channel of capacity C_{RF}, which produces an output of the MAC BW.
2. At the encoder and the relays, employ a MAR JSCC scheme to obtain, at each relay, an unbiased estimate $\hat{V}_m$ of the chosen codeword V. Thus we establish an equivalent A&F channel as depicted in Fig. 3.
3. Use the A&F strategy w.r.t. these estimates, in order to obtain an estimate $\hat{V}$ of V at the decoder.
4. Recover the message W, using a point-to-point decoder.

Theorem 1 *The capacity of the Gaussian parallel relay network with expansion factor ρ satisfies for $\rho \geq 1$:*

$$C \geq C\left(\frac{M S_{MAC}\tilde{S}_{BC}}{\tilde{S}_{BC} + S_{MAC} + 1}\right), \tag{8}$$

and for $\rho < \mathbf{1}$:

$$C \geq C\left(\frac{M S_{MAC}\tilde{S}_{BC}}{\left(\tilde{S}_{BC} + S_{MAC} + 1\right)^{\rho}\left(\tilde{S}_{BC} + M S_{MAC} + 1\right)^{1-\rho}}\right), \tag{9}$$

where $\tilde{S}_{BC}$ is the equivalent SNR of (7).

Proof outline: We use the R&F strategy described above. If we show that $I(V;\hat{V}) = R$ is achievable, than a rate R can be achieved using an appropriate point-to-point encoder/deoder pair[4]. For $\rho \geq 1$ all the equivalent noises $\{\tilde{Z}_m\}$ are independent, hence a mutual-information of (8) is attained. For $\rho < 1$, the noises in the MAC outband are added coherently, and none-coherently inband, resulting in a colored spectrum and a corresponding mutual-information of (9).

We now consider to special cases of the colored case.

The Bandwidth Mismatch Case: In this case both the BC and the MAC sections are white, but of different BW. Using a mutual-information preserving scheme, would result in an equivalent mutual-information preserving SNR $\tilde{S}_{\text{BC}} = (1 + S_{\text{BC}})^{\rho} - 1$. For a detailed analysis of the rate obtained by the R&F approach for this case and comparison to other possible strategies, see [4].

The Tight SUB Case: The Shannon upper bound (SUB) on the channel capacity states the following:

$$C \leq C_{\text{UB}} \triangleq 1/2 \log\big(\Gamma(1 + \text{SNR})\big), \tag{10}$$

where Γ is the channel (noise) prediction gain, .i.e., the ratio between the noise power and its power entropy. Consequently. if the noise spectrum is bounded from above in all frequencies within the channel BW, then (10) holds with equality for sufficiently high SNR[5] (see [6]). If the SUB is tight in the BC section, (6) reduces to

$$\tilde{S}_{\text{BC}} = [\Gamma(1 + S_{\text{BC}})]^{\rho} - 1. \tag{11}$$

Remarks:

1. For $\rho < 1$, one might expect to achieve an expression similar to (8), with M replaced by M^{ρ}, instead of the inferior (9). This loss comes from the fact, that we can only compensate for the color of the equivalent BC noise $\sum_{m=1}^{M} \tilde{Z}_m$ after adding the MAC noise. This could be overcome by working with a codebook of the BC section BW, see Section 5.
2. A mutual-information preserving JSCC scheme uses a channel BW according to the water-pouring solution, which may be narrower than the physical BW, inducing an effective BW-expansion factor ρ'. Consequently in the general case, the performance is according to (11), with ρ replaced by ρ' and Γ calculated w.r.t. the resulting inband. This choice of BW does not necessarily maximize the overall mutual-information in the network, and some other choice between ρ' and ρ may improve the bound of Theorem 1.
3. In the scheme, we do not address the matter of the MAC noise color, as we do not employ a mutual-information preserving JSCC scheme for the MAC section. If that noise is

[4] Since the channel from V to $\hat{V}$ may be colored, an AWGN codebook in conjunction with any ISI treatment approach, such as DMT or MMSE FFE/DFE, is applicable.

[5] If the noise spectrum is unbounded, still it holds asymptotically under the Paley-Wiener condition.

colored, then the mutual-information is higher than guaranteed by Theorem 1. This can be further improved in the following ways. First, it may be beneficial to assume a lower MAC BW, abandoning frequencies where the noise spectrum is high. Second, LTI filtering (shaping) at the encoder and the relays can be added. Finding the optimal filters can be done by solving the optimization problem of maximizing the mutual-information between V and $\hat{V}$.

3. JOINT SOURCE/CHANNEL SCHEMES WITH ANALOG RECONSTRUCTION ERROR

In this section, we show that there indeed exist JSCC schemes which fit into the R&F framework. While certainly not all schemes are suitable, we show that a variety of schemes are. We first present Hybrid Digital-Analog (HDA) schemes, then modulo-lattice modulation (MLM) ones.

3.1. Hybrid Digital-Analog Schemes

HDA is the general name for a family of schemes, all of which share the basic idea of dividing the transmission resources (power, BW) between analog and digital components. These schemes were originally presented in the context of broadcasting, i.e., transmission for two receivers of different SNRs.

For the special cases of BW expansion and compression, systems 3 and 4 of Mittal and Phamdo [7], respectively, when optimized for the lower SNR, are both mutual-information preserving and maximally analog.

For BW expansion, this solution consists of quantization of a block of n block samples with rate $\frac{\rho-1}{\rho}C$, where C is the capacity per channel use. Then $(\rho-1)n$ channel uses are devoted to digitally transmitting the quantized source, while in the remaining n uses the quantization error is transmitted in an analog manner. This is mutual-information conserving in the limit of large n since the Gaussian source is successively refinable; The reconstruction error is clearly analog, since it is the channel error for the n analog channel uses.

For BW compression, consider source blocks of length $\frac{n}{\rho}$. Of each such block, n samples are sent in an analog manner, in superposition with digital transmission of quantization of the other $\frac{\rho-1}{\rho}n$ samples. Again it can be verified that there is no loss of mutual-information, and the samples which are transmitted in an analog manner (and, accordingly, have analog reconstruction error) consist of a portion of the source samples according to the channel BW.

The work by Prabhakaran et al. [8] generalizes this approach to the general colored case. They take a frequency-domain approach: The signal is divided into "thin" frequency bins. In a version of the scheme that they call "weak-user optimal", which is mutual-information preserving for a specific SNR, the signal components which correspond with frequencies inside the channel BW are transmitted either in an analog manner, or by sending digitally the quantization and then the quantization error in an analog manner; This results again in an analog reconstruction error inside the channel BW, thus this is a MAR scheme.

3.2. Modulo-Lattice Modulation Schemes

The MLM approach is different from HDA, since it has no digital data component. Rather, an analog signal is transmitted modulo a lattice, which plays the role of a "shaping code". Since there is no quantization involved, this approach combines naturally with the demand for an analog reconstruction error. This approach was first used for BW expansion by an integer factor in [9], and then generalized to any source and channel color in the Analog Matching (AM) scheme of [6].

For a white source V, the unbiased source estimate obtained by the AM scheme is given by:

$$\hat{V}[n] = V[n] + \frac{1}{\beta} P_C[n] * \Big(Z[n] * H[n] - X[n] * (1 - H[n]) \Big) \ ,$$

where $X[n]$ and $Z[n]$ are the channel input and noise, respectively, $*$ denotes convolution, and β, $H[n]$ and $P_C[n]$ are the "zoom-in factor", linear filter and predictor, respectively, used by the scheme. It follows, that the reconstruction error is analog only in frequencies where $H(\mathrm{e}^{j2\pi f}) = 1$. For the scheme to be mutual-information preserving, $H[n]$ and $P_C[n]$ have to be the linear equalizer and and noise predictor of an optimum MMSE receiver, and consequently at all the frequencies $|H(\mathrm{e}^{j2\pi f})| < 1$. However, at high-SNR conditions, the loss due to the use of a unity linear equalizer in the inband frequencies vanishes, as stated in the following.

Lemma 1 *Consider transmission of a white Gaussian source through an additive Gaussian channel with BW expansion factor ρ, SNR S and prediction gain Γ. With an appropriate choice of filters, the Analog Matching scheme is MAR and produces an equivalent AWGN of $\tilde{S} = (\Gamma S)^\rho - 1$, as long as the SUB is tight and $S > 1$.*

Comparing with (11), we see that the MAR variant of the AM scheme becomes mutual-information preserving in the high-SNR limit. The loss for general SNR stems from seeing the predictable source and noise components as interference, and using methods for canceling any arbitrary interference.

Remarks:

1. If the SUB is not tight, then the Lemma is still true with ρ being w.r.t. the BW of the water-pouring solution, and Γ defined within that frequency band; See Remark 2 in Section 2.

2. The loss w.r.t. (11) can be somewhat reduced, by using a choice of filters which leads to a scheme which is not strictly MAR; The optimal filters will strike a balance between a high effective SNR and low correlation between the noises, such that the overall mutual-information is maximized.

4. EXTENSION TO LAYERED NETWORKS

The RF strategy for the parallel relay network can be used as a building stone for more general networks. We consider *layered networks*, which are directed acyclic graphs (DAGs) where the nodes can be divided into layers, and nodes in each layer receive the (noisy) sum of transmissions from the adjacent preceding layer only. We index the relay layers as $l =$

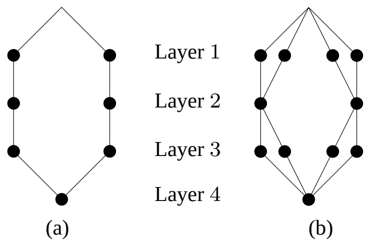

Figure 4: BW-mismatched symmetric layered networks. Each node denoted by a full circle contains a MAC channel.

$1, \cdots, L$, where layer L consists of the destinations. Specifically we consider the symmetric case, where the noise spectra, as well as the number of received transmissions ("fan-in") and the number of destinations in the next layer ("fan-out") are fixed for all nodes in the same layer. In the white symmetric case with BW mismatch between layers, each layer is characterized by its SNR S_l and BW ρ_l. Fig. 4 shows two examples of networks which fall under this category. In the sequel, we show how combining the RF and CF strategies is beneficial in the first example, while recursive use of the RF strategy is the key to the treatment of the second one.

Consider the network of Fig. 4a. We use a codebook BW according to layer 4, i.e. $\rho_4 = 1$. From layer 1 to layer 3 there are no MACs, thus analog transmission produces no coherence gain. Hence, the noise accumulation in these layers can be avoided by having each of the nodes in layer 1 compress their estimation of $\mathbf{V}$ according to the rate $\min\big(\rho_2 C(S_2), \rho_3 C(S_3)\big)$ and send it digitally, making sure that the resulting quantization errors are mutually independent (c.f. by using mutually-independent dither in each node). For the outer layers, we use the RF scheme. The decoder "sees" a parallel relay network, where the MAC noise is the sum of the noise of layer 4 and the quantization noises. Thus we achieve the coherence gain w.r.t. all the network noises.

Next consider the network of Fig. 4b, which contains MACs in both layer 2 and layer 4. On one hand, the coherence gain is only known to be achieved for analog transmission over the MACs, but on the other hand using analog transmission for both does not enable to utilize the full BW if $\rho_2 \neq \rho_4$. This can be circumvented by using two information-bearing signals of different BW. We use again a codebook BW according to layer 4, applying a JSCC method to re-match it to BW ρ_2. This re-matched codeword is sent using the RF scheme to layer 2, where instead of the codebook decoder, the "source signal" associated with the outer JSCC scheme is reproduced. Note that this way the estimation errors in both relays of layer 2 are mutually independent. Next we use RF again to transmit these reproductions to relay 4. In the overall result, again the full coherence gain is achieved.

5. DISCUSSION: JSCC OVER MACS

In the definition of R&F, we always take a white codebook with the BW of the MAC section. However, as can be seen throughout the work, this choice imposes a few problems:
1. For $\rho < 1$ the equivalent point-to-point channel is colored.
2. If the MAC is colored, we cannot achieve optimality.
3. It is difficult to generalize the approach to a setting with more than one MAC.

These issues could be avoided, if we had a JSCC scheme that could turn the MAC into an equivalent white one, thus we could also choose the codebook to have the BC section BW. This was the target of a recent work [5], but the task turns out to be complicated. Since each relay observes the codeword $\mathbf{V}$ with different noise, a scheme which includes any "digital" element (whether a quantized component in a HDA scheme or a modulo-lattice operation) faces the inherent problem that different relays might take different decisions, causing large errors; On the other hand, pure analog schemes cannot be mutual-information preserving for BW mismatch (and even for equal-BW colored cases).

The MLM approach does seem to have an advantage here, since in the absence of a data-bearing code, the relays only have to agree upon the "coarse" (shaping) lattice point. Indeed in [5] it is shown that if, before any modulo-lattice operation, the uncertainty about the source at each relay is smaller than the uncertainty at the decoder, then the relays can agree upon a lattice point with high probability, maintaining optimality. This direction is currently under further investigation.

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