New Matrix Decompostions for Gaussian Communication Networks

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Black Box Approach to Point-to-Point MIMO Channels: $\boldsymbol{y} = \mathbf{H}\boldsymbol{x} + \boldsymbol{z}, \quad \boldsymbol{z} \sim \mathcal{CN}(0, \mathbf{I})$

Singular Value Decomposition (SVD)

- $\mathbf{H} = \mathbf{Q}\mathbf{D}\mathbf{V}^\dagger$
- $\mathbf{Q}$ and $\mathbf{V}$ — unitary
- Apply $\mathbf{V}$ at Tx and $\mathbf{Q}$ at Rx
- $\mathbf{D} = \begin{pmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \sigma_{n-1} & 0 \\ 0 & \cdots & 0 & 0 & \sigma_n \end{pmatrix} \Rightarrow \begin{matrix} y_1 = \sigma_1 x_1 + z_1 \\ y_2 = \sigma_2 x_2 + z_2 \\ \vdots \\ y_n = \sigma_n x_n + z_n \end{matrix}$
- Results in parallel scalar sub-channels (each sub-channel has a different SNR)

QR Decomposition

- $\mathbf{H} = \mathbf{Q}\mathbf{T}$
- $\mathbf{Q}$ — unitary
- Apply $\mathbf{Q}$ at Rx (no SP is required at Tx)
- $\mathbf{T} = \begin{pmatrix} t_1 & * & * & \cdots & * \\ 0 & t_2 & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{n-1} & * \\ 0 & 0 & \cdots & 0 & t_n \end{pmatrix} \Rightarrow \begin{matrix} y_1^{\text{eff}} = t_1 x_1 + z_1 \\ y_2^{\text{eff}} = t_2 x_2 + z_2 \\ \vdots \\ y_n^{\text{eff}} = t_n x_n + z_n \end{matrix}$
- Off-diagonal elements are canceled via successive interference cancellation (SIC)

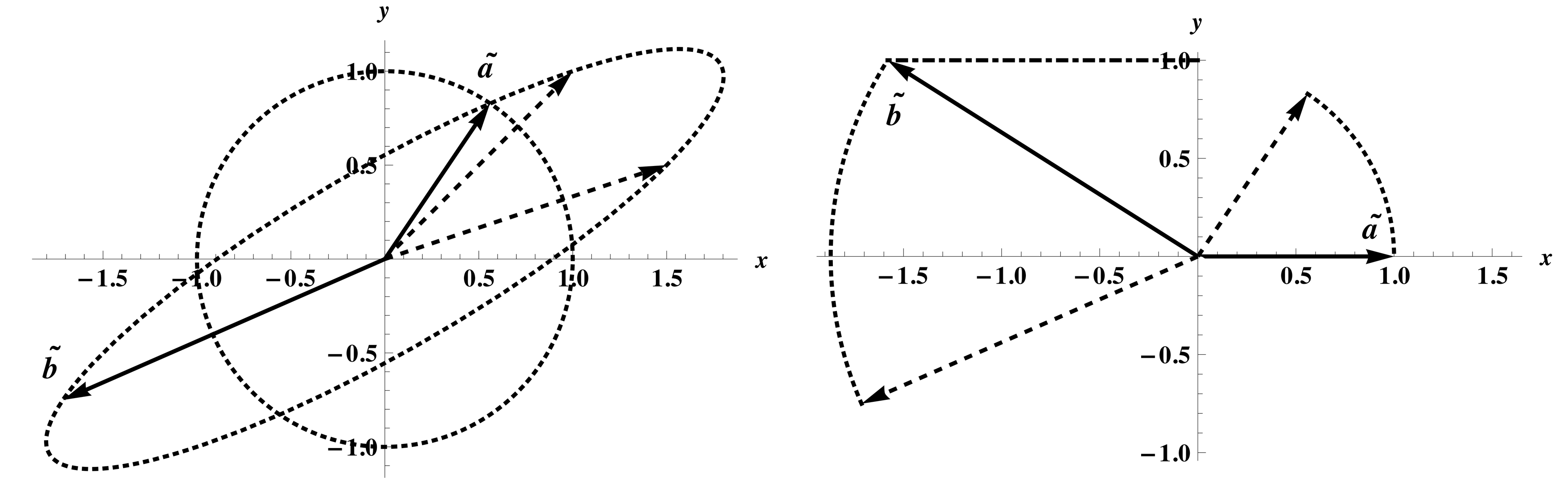
Geometric Mean Decomposition (GMD)

- $\mathbf{H} = \mathbf{Q}\mathbf{T}\mathbf{V}^\dagger$
- $\mathbf{Q}$ and $\mathbf{V}$ — unitary
- Apply $\mathbf{V}$ at Tx and $\mathbf{Q}$ at Rx
- $\mathbf{T} = \begin{pmatrix} t & * & * & \cdots & * \\ 0 & t & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & t & * \\ 0 & 0 & \cdots & 0 & t \end{pmatrix} \Rightarrow \begin{matrix} y_1^{\text{eff}} = t x_1 + z_1 \\ y_2^{\text{eff}} = t x_2 + z_2 \\ \vdots \\ y_n^{\text{eff}} = t x_n + z_n \end{matrix}$
- Off-diagonal elements are canceled via SIC
- **Constant diag.** $\Rightarrow$ same code over all sub-channels

Generalized Triangular Decomposition (GTD)

- Weyl condition: $\boldsymbol{\sigma} \succeq \boldsymbol{r} : \prod_{i=1}^{\ell} \sigma_i \geq \prod_{i=1}^{\ell} |t_i| , \quad i = 1, \ldots, N ; \quad \prod_{i=1}^{N} \sigma_i = \prod_{i=1}^{N} |t_i|$

$$\mathbf{Q}^\dagger\mathbf{H}\mathbf{V} = \begin{bmatrix} \cos\theta_\ell & -\sin\theta_\ell \\ \sin\theta_\ell & \cos\theta_\ell \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & \cos\theta_r \end{bmatrix} = \begin{bmatrix} \cos\theta_\ell & -\sin\theta_\ell \\ \sin\theta_\ell & \cos\theta_\ell \end{bmatrix} \begin{bmatrix} a_1\cos\theta_r + b_1\sin\theta_r & a_1\cos(\theta_r + \frac{\pi}{2}) + b_1\sin(\theta_r + \frac{\pi}{2}) \\ a_2\cos\theta_r + b_2\sin\theta_r & a_2\cos(\theta_r + \frac{\pi}{2}) + b_2\sin(\theta_r + \frac{\pi}{2}) \end{bmatrix}$$



Black Box Approach to Physical-Layer MIMO Multicast Channels: $\boldsymbol{y}_i = \mathbf{H}_i\boldsymbol{x} + \boldsymbol{z}_i, \quad \boldsymbol{z}_i \sim \mathcal{CN}(0, \mathbf{I}), \quad i = 1, \ldots, K$

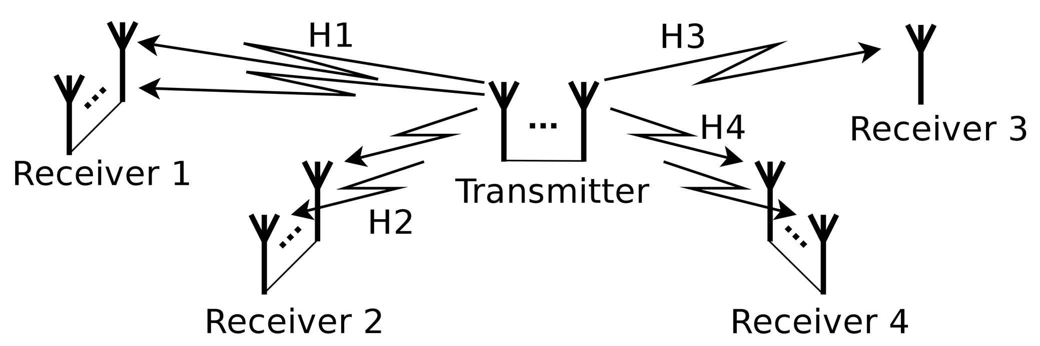
All three approaches reduce to the **"black box"** task:

- **Coding over single-user AWGN scalar channels**
- **Any "off-the-shelf" (fixed SNR) single-user codes**

How to implement MIMO multicast via a black box approach?

Physical-Layer MIMO Multicast

- Same information is transmitted to all users



Joint Unitary Triangularization

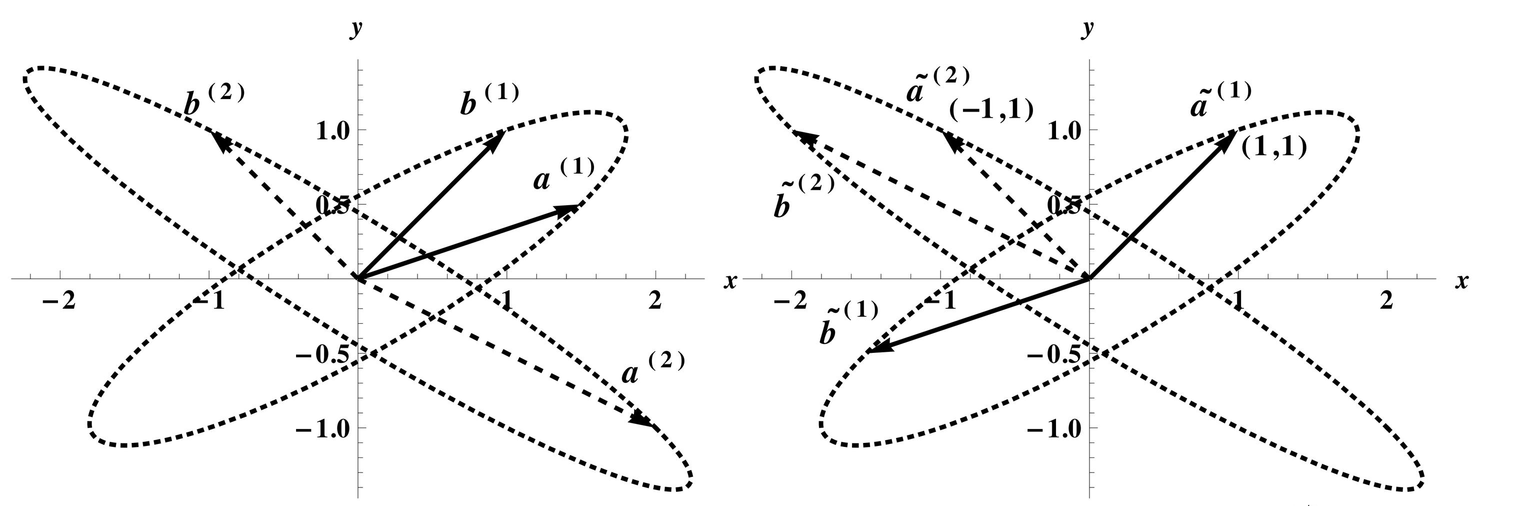
- $\mathbf{H}_1$ and $\mathbf{H}_2$ – $N \times N$
- $\det(\mathbf{H}_1) = \det(\mathbf{H}_2)$
- $\mathbf{H}_1$ and $\mathbf{H}_2$ can be jointly decomposed as:

$$\mathbf{H}_1 = \mathbf{Q}_1\mathbf{T}_1\mathbf{V}^\dagger$$
$$\mathbf{H}_2 = \mathbf{Q}_2\mathbf{T}_2\mathbf{V}^\dagger$$

- $\mathbf{Q}_1, \mathbf{Q}_2, V$ – unitary; $\mathbf{T}_1, \mathbf{T}_2$ – Triangular
- $\mu(\mathbf{H}_1, \mathbf{H}_2)$ — Generalized singular values vector
- Generalized Weyl: $\text{diag}(\mathbf{T}_1)/\text{diag}(\mathbf{T}_2) \preceq \mu(\mathbf{H}_1, \mathbf{H}_2)$

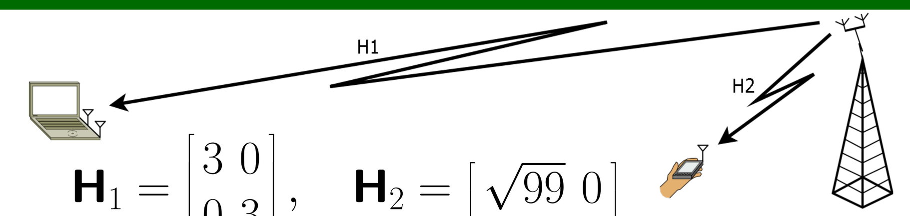
Joint Equi-Diagonal Triangularization (JET)

- $\text{diag}(\mathbf{T}_1) = \text{diag}(\mathbf{T}_2)$



- Can be extended to genral SNR and non-square matrices by decomposing $\begin{pmatrix} \mathsf{H}_i C_x^{1/2} \\ \mathsf{I} \end{pmatrix}$

Example: DoF Mismtach / Rateless Coding



$\mathbf{H}_1 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{H}_2 = \begin{bmatrix} \sqrt{99} & 0 \end{bmatrix}$

- $C_1^{\text{WI}} = 2\log(1 + 3^2) = \log\left(1 + (\sqrt{99})^2\right) = C_2^{\text{WI}}$
- Send same signal over both antennas
  $\Rightarrow$ **Losses half of the rate at High SNR!**

Using JET (MMSE variant)

Matrix **V** is applied to $\begin{bmatrix} \mathsf{H}_i \\ \mathsf{I}_{N_t} \end{bmatrix}$ (MMSE variant for general SNR):

$$\begin{bmatrix} \mathsf{H}_1 \\ \mathsf{I}_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \overbrace{\begin{bmatrix} 0.286 & -0.905 \\ 0.905 & 0.286 \\ 0.095 & -0.301 \\ 0.301 & 0.095 \end{bmatrix}}^{\mathsf{Q}_1} \overbrace{\begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{10} \end{bmatrix}}^{\mathsf{T}_1} \overbrace{\begin{bmatrix} 0.302 & 0.954 \\ -0.954 & 0.302 \end{bmatrix}}^{\mathsf{V}^\dagger}$$

$$\begin{bmatrix} \mathsf{H}_2 \\ \mathsf{I}_2 \end{bmatrix} = \begin{bmatrix} \sqrt{99} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \overbrace{\begin{bmatrix} 0.949 & -0.300 \\ 0.905 & -0.030 \\ 0.302 & 0.954 \end{bmatrix}}^{\mathsf{Q}_2} \overbrace{\begin{bmatrix} \sqrt{10} & -9 \\ 0 & \sqrt{10} \end{bmatrix}}^{\mathsf{T}_2} \overbrace{\begin{bmatrix} 0.302 & 0.954 \\ -0.954 & 0.302 \end{bmatrix}}^{\mathsf{V}^\dagger}$$

JET/GMD for $K > 2$ Users

$\mathbf{H}_i = \mathbf{Q}_i\mathbf{T}_i\mathbf{V}^\dagger$ ✗

- **Bunch two channel uses together:**

$$\overbrace{\begin{pmatrix} \mathbf{H}_i & 0 \\ 0 & \mathbf{H}_i \end{pmatrix}}^{\mathcal{H}_i} = \overbrace{\begin{pmatrix} \mathbf{Q}_i & 0 \\ 0 & \mathbf{Q}_i \end{pmatrix}}^{\mathcal{Q}_i} \overbrace{\begin{pmatrix} \mathbf{T}_i & 0 \\ 0 & \mathbf{T}_i \end{pmatrix}}^{\mathcal{T}_i} \overbrace{\begin{pmatrix} \mathbf{V}^\dagger & 0 \\ 0 & \mathbf{V}^\dagger \end{pmatrix}}^{\mathcal{V}} \quad ✗$$

- $\mathcal{H}_i$ have a block-diagonal structure
- Use general $\mathcal{Q}_i$, $\mathcal{V}$ (*not* block-diagonal):

$$\overbrace{\begin{pmatrix} \mathbf{H}_i & 0 \\ 0 & \mathbf{H}_i \end{pmatrix}}^{\mathcal{H}_i} = \left(\mathcal{Q}_i\right)\left(\mathcal{T}_i\right)\left(\mathcal{V}\right)^\dagger \quad ✓$$

- Exploiting off-diagonal 0s $\Rightarrow$ JET of **more users!**
- Process jointly N^{K-1} symbols
- Prefix–suffix loss of N^{K-1} symbols total
- Numerical evidence: Can be greatly improved!

Multicast is (Almost) Everywhere...