LDPC Ensembles that Universally Achieve Capacity under Belief Propagation Decoding

A Simple Derivation

Anatoly Khina

Joint work with: Yair Yona, Uri Erez

Tel Aviv University

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Requirements of a "Good Code"

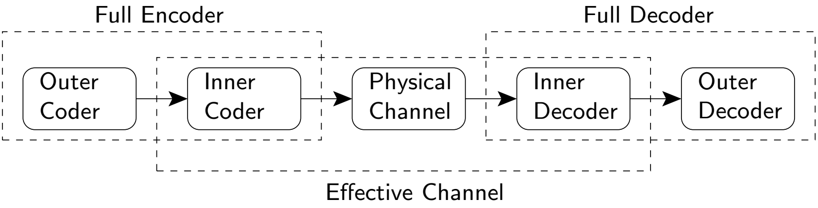
- Capacity achieving (universally?)
- Low computational complexity
- Good performance in practice (BP?)

Existing Code Constructions

Random codes [Shannon '48]

- Capacity achieving ✓
- Universal for **B**inary-Input **M**emoryless Output-**S**ymmetric (BMS) channels with same capacity ☺
- Exponential encoding/decoding complexity X
- Impractical ☹

Existing Code Constructions



Concatenated codes [Forney '61]

- Approach capacity ✓
- Can be extended to work universally over BMS channels
- Reduced (polynomial) complexity
- Works well in practice ✓

Existing Code Constructions

Expander codes [Barg–Zémor ’04][Guruswami-Indyk ’05]

- Approach capacity ✓
- Linear complexity ✓
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Polar codes [Arıkan '09]

- Approach capacity ✓
- Low complexity: $\mathcal{O}(N \log N)$
- Practical...? – The verdict is still out

Existing Code Constructions

Recent entry: Convolutional/spatially-coupled LDPC codes [Felström–Zigangirov '99]

- Linear decoding complexity under BP decoding ✓
- Approach capacity over BMS channels [Kudekar *et al.* ISIT'12]
- Performance in practice? Good for long blocklengths
- Overall code performance under BP decoding

  =

  Shorter regular LDPC code performance under ML decoding

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  - LDPC over BEC behavior guarantees suffice [Khandekar '02][Miller–Burshtein '02]

New Code Construction

Bonus

- **Regular LDPC** codes can be used
- IRA codes can be used $\Rightarrow$ **Systematic representation** **Linear encoding**

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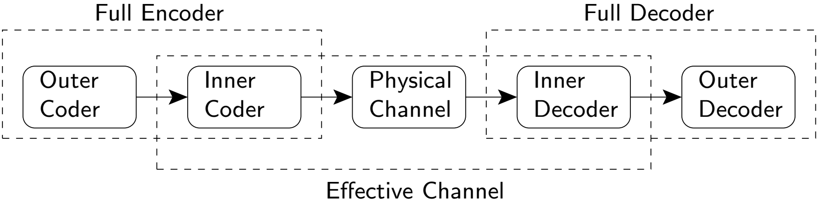
- Practical?
  - Good reasons to believe it is
  - Currently under research

First Scheme: Two-Stage Message-Passing Decoding



Concatenated encoder

- Encode info. bits using an (outer) LDPC code of length n
- [Interleave LDPC coded bits]
- Encode LDPC coded bits using an (inner) zero-terminated convolutional code of length L

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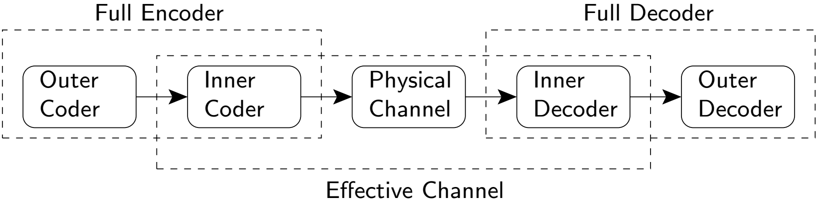


Two-stage message passing decoder

- Apply BCJR decoding of the inner convolutional code
  $\Rightarrow$ Calculate LLRs of each input bit
- [De-interleave LLRs]
- Apply BP decoding of LDPC code over induced LLR channel
- [The induced channel is memoryless due to the interleaver]

(Full) BP decoding

BP decoder outperforms two-stage decoder (under tree assump.)

Universality of Block Codes

Exponential upper bound on block error probability [Gallager '68]

For a BMS c of capacity C:

$$P_b^{(c)} \leq e^{-N E_G(R)}$$

- Achievable by a random (block) code
- $E_G(R) > 0$ for $R < C$
- Upper bounds also the BER

Extremes of Error Exponents (EE) [Guillen i Fabregas *et al.* '13]

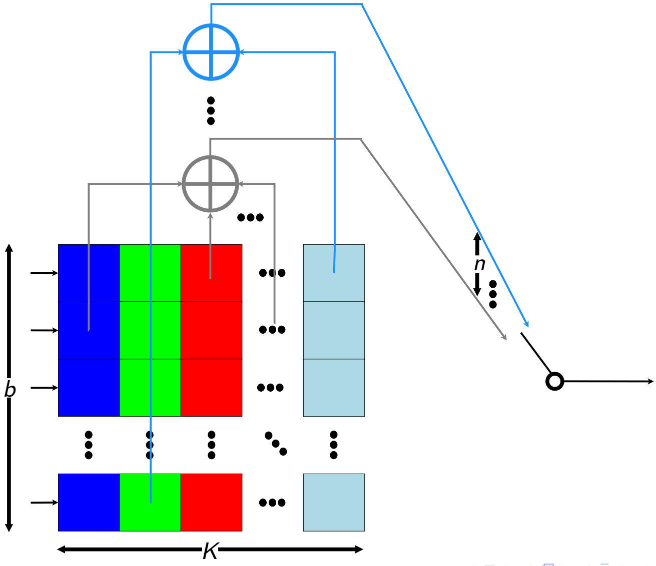
EE of BSC is worse than EE of any other BMS with same capacity:

$$E_G^{(c)}(R) \geq E_G^{\text{BSC}}(R)$$

Conclusion for random block codes

Random codes designed for BSC(C) achieve better BER over all BMS(C)

Universality of Convolutional Codes



Universality of Convolutional Codes

Upper bound on BER of convolutional codes [Yudkin '65][Viterbi '67]

For a BMS c of capacity C and any $0 < \epsilon < 1$:

$$P_b \leq \left(2^b - 1\right) \frac{2^{-K \frac{b}{R} E_{\text{VY}}(R,\epsilon)}}{\left[1 - 2^{-\epsilon \frac{b}{R} E_{\text{VY}}(R,\epsilon)}\right]^2} \triangleq P_b^{\text{UB}}$$

- Achievable by a random time-varying convolutional code
- $E_{\text{VY}}(R,\epsilon) > 0$ for $R < C(1-\epsilon)$
- Constraint length K takes the role of the blocklength

Lower bound on Viterbi–Yudkin EE

BSC block-coding EE serves as lower bound:

$$E_{\text{VY}}^{(c)}(R,\epsilon) \geq E_G^{(c)}\left(\frac{R}{1-\epsilon}\right) \geq E_G^{\text{BSC}}\left(\frac{R}{1-\epsilon}\right) > 0$$

Conclusion for random (time-varying) convolutional codes

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Universality of Convolutional Codes

Bhattacharyya parameter B [Guillen i Fabregas '13]

B of any BMS c is bounded by those of BEC(C) and BSC(C):

$$P(\text{Erasure}) \equiv B^{\text{BEC}} \leq B^{(c)} \leq B^{\text{BSC}} \equiv 2\sqrt{BER \cdot (1 - BER)}$$

- Will serve as a guarantee for performance of outer LDPC code
- $B^{(c)}$ can always be upper bounded by B^{BSC} after “slicing”

Performance of LDPC codes over BMS

LDPC over BEC

- LDPC ensembles that approach capacity over the BEC under BP decoding are known [Luby *et al.* "97][Shokrollahi '01], ...
- True also for IRA codes [Khandekar '02]

LDPC performance over BMS [Khandekar '02]

BER of LDPC ensemble over BEC with Bhattacharyya param. B

$$\geq$$

BER of LDPC ensemble over BMS with Bhattacharyya param. B

Conclusion

LDPC ensembles for BEC guarantee performance over BMS with same B

Analysis of Two-Stage Message-Passing Decoder

- Total rate: $R = C - \Delta$, for any $\Delta > 0$
- Convolutional code rate: $r \in (R, C)$
- Constraint length K is taken long enough (but fixed!), s.t.

$$0 < 2\sqrt{P_b^{\mathrm{UB}}\left[1 - P_b^{\mathrm{UB}}\right]} \triangleq B^{\mathrm{UB}} < 1 - \frac{R}{r}$$

- Use LDPC ensemble of rate $\frac{R}{r}$ with threshold $> B^{\mathrm{UB}}$ over BEC
- Long enough LDPC ensemble $n \Rightarrow$ Arbitrarily small BER

Conclusion

Achieves universally capacity with linear complexity over BMS(C)!

Analysis of Two-Stage Message-Passing Decoder

Using **regular** LDPC codes

- A regular LDPC ensemble
- Varaible- and check-node degrees d_v and d_c
- Has a threshold T bounded away from zero

Conclusion:

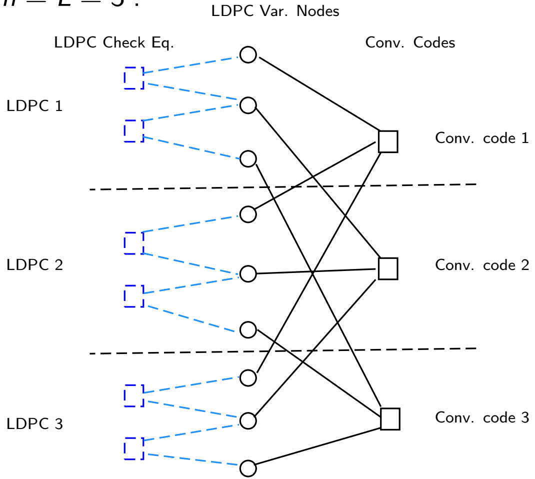
- Take long enough constraint length K s.t. $T > B^{\mathrm{UB}}$
- BER can be made arbitrarily small by enlarging LDPC length n

Using IRA codes

- Performance guarantee via Bhattacharyya parameter holds for IRA
- Using IRA codes achieves $\Rightarrow$ **Systematic representation** **Linear encoding**

Analysis of BP Decoding over Overall Factor Graph

Example of $n = L = 3$:



- Factor graphs of LDPC codes
- Factor graphs of convolutional codes

Analysis of BP Decoding over Overall Factor Graph

Extended tree assumption

ℓ-depth extended tree of variable node shares no loops with ℓ-depth trees of other variable nodes.

- "Standard" tree assumption for each LDPC code
- **"Extension":** No loops with variable nodes that belong to same convolutional code

Lemma

Extended tree assumption is satisfied w.h.p. for large enough n.

- Requires longer n than "standard" tree assumption

BP decoding universally achieves capacity

- Under extended tree assumption (guaranteed by long enough n)
- BP over the factor graph is optimal
- $\Rightarrow$ Outperforms two-stage decoding

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- Regular LDPC, or systematic rep. and linear enc. via IRA