LDPC Code Ensembles that Universally Achieve Capacity under Belief Propagation Decoding

A Simple Derivation

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PARADISE Workshop 2016
Pasadena, CA, USA
February 3, 2016
Requirements of a “Good Code”

- Capacity achieving (universally?)
- Low computational complexity for encoding/decoding (linear)?
- Good performance in practice (BP?)
Random codes [Shannon ’48]

- Capacity achieving ✓
- Universal for Binary-Input Memoryless Output-Symmetric (BMS) channels with same capacity 😊
- Exponential encoding/decoding complexity X
- Definitely impractical 😞
Known Code Constructions

Concatenated codes [Forney '61]

- Approach capacity ✓
- Can be extended to work universally over BMS channels
- Reduced (polynomial) complexity
- Works well in practice ✓
Known Code Constructions

**Concatenated + Outer Expander Code**
[Barg–Zémor ’04][Guruswami-Indyk ’05]
- Approach capacity ✓
- Linear complexity ✓
- Can be extended to work universally over BMS channels
- Considered impractical ✗

**Polar codes [Arıkan ’09]**
- Approach capacity ✓
- Low complexity: $O(N \log N)$
- Practical? – Getting there...
Known Code Constructions

Recent (re-)entry: Convolutional/spatially-coupled LDPC codes [Felström–Zigangirov ’99]

- Linear decoding complexity under BP decoding ✓
- Approach capacity over BMS channels [Kudekar et al. ISIT’12]
- Performance in practice? Good for long blocklengths
- Overall code performance under BP decoding
  - Shorter regular LDPC code performance under ML decoding
- Threshold saturation [Kudekar et al. ISIT’12]
Recapitulation

Several different goals

- Capacity achieving (for a given BMS)
- Universality
- Low complexity in practice (BP?)
Concatenated Convolutional/LDPC-BC: Overview

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  - Inner code – should have good ML performance over BSC
  \[\Downarrow\]
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  random ensemble + extremal properties of BSC ✓
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  - High rate outer LDPC code – should be good for the BEC
LDPC ensemble (different from spatially-coupled)

Achieves capacity under BP (linear complexity) ✓

Simple “black box” analysis

Follows classical code-concatenation approach

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    ⇓

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LDPC behavior over BEC guarantees suffice

[Khandekar ’02][Burshtein–Miller ’02]
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- LDPC behavior over BEC guarantees suffice
  [Khandekar ’02][Burshtein–Miller ’02]
- Regular LDPC / IRA codes can be used
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Concatenated encoder

- Encode info. bits using an (outer) LDPC code of length $M$
- [Interleave LDPC coded bits]
- Encode LDPC coded bits using an (inner) zero-terminated convolutional code of length $L$
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LDPC Ensembles that Universally Achieve Capacity under BP
Decoder: BP Decoding over Overall Factor Graph

Example of $M = L = 3$:

- Factor graphs of LDPC codes
- Factor graphs of convolutional codes (state-space representation)

LDPC Var. Nodes

LDPC Check Eq.

Conv. Codes

LDPC 1

Conv. code 1

LDPC 2

Conv. code 2

LDPC 3

Conv. code 3

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LDPC Ensembles that Universally Achieve Capacity under BP
“Degraded” Decoder

**Overview Encoder Interleaver Decoder**

**Scheme**

**Inner Outer Analysis Conclusions**

**Overview**

Two-stage message passing decoder

- Apply BCJR decoding to the inner convolutional code
  ⇒ Calculate LLRs of each input bit
- Apply slicer to get hard decisions (for sake of analysis)
- [De-interleave LLRs]
- Apply BP decoding of LDPC code over induced BSC channel
  [The induced channel is memoryless due to the interleaver]

**Decoder**

(Full) BP decoding

BP decoder outperforms two-stage decoder (under tree assumption)
Concatenated Convolutional/LDPC-BC: Questions

- Capacity achieving?
- Universal?
- Practical?
Universality of (Capacity-Achieving) Block Codes

### Exponential upper bound on block error probability [Gallager '68]

For a BMS $c$ of capacity $C$:

$$P_b^{(c)} \leq e^{-NE_G(r)}$$

- Achievable by a random (block) code
- $E_G(r) > 0$ for $r < C$
- Upper bounds also the BER

### Extremes of Error Exponents (EE) [Guillen i Fabregas et al. '13]

EE of BSC is worse than EE of any other BMS with same capacity:

$$E_G^{(c)}(r) \geq E_G^{\text{BSC}}(r)$$

### Conclusion for random block codes

Random codes designed for BSC(C) achieve better BER over all BMS(C)
Universality of Convolutional Codes

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Universality of Convolutional Codes

Upper bound on BER of convolutional codes [Yudkin '65][Viterbi '67]

For a BMS $c$ of capacity $C$ and any $0 < \epsilon < 1$:

$$P_b \leq \left(2^b - 1 \right) \frac{2^{-K \frac{b}{r} E_{VY}(r, \epsilon)}}{\left[1 - 2^{-\epsilon \frac{b}{r} E_{VY}(r, \epsilon)}\right]^2} \equiv P^\text{UB}_b$$

- Achievable by a random time-varying convolutional code
- $E_{VY}(r, \epsilon) > 0$ for $r < C(1 - \epsilon)$
- Register length $K$ plays the role of the blocklength

Crude lower bound on Viterbi–Yudkin EE

BSC VY EE is the worst and lower bounded by BSC block-code EE:

$$E^{(c)}_{VY}(r, \epsilon) \geq E^{BSC}_{VY}(r, \epsilon) \geq E_G^{BSC} \left(\frac{r}{1 - \epsilon}\right) > 0$$

Conclusion for random (time-varying) convolutional codes

Random codes designed for BSC(C) achieve better BER over all BMS(C)
Performance of LDPC codes over BMS

**LDPC over BEC**

LDPC ensembles that approach capacity over the BEC under BP decoding are known [Luby et al. ’97][Shokrollahi ’01], ...

**LDPC performance over BMS [Khandekar ’02]**

BER of LDPC ensemble over BMS with Bhattacharyya param. $B \leq$ BER of LDPC ensemble over BEC with erasure probability $B$

**Conclusion**

LDPC ensembles for BEC guarantee performance over BMS with same $B$
Analysis of Two-Stage HDD + Message-Passing Decoder

- Total rate: $R = C - \Delta$, for any $\Delta > 0$
- Convolutional code rate: $r \in (R, C)$
- Register length $K$ is taken long enough (but fixed!), s.t.

$$0 < 2\sqrt{P_{b}^{\text{Hard}}[1 - P_{b}^{\text{Hard}}]} \triangleq B^{\text{Hard}} < 1 - \frac{R}{r} - \delta$$

Bhattacharyya of BSC($P_{b}^{\text{Hard}}$)

Threshold over BEC

- Use LDPC ensemble of
  - Rate = $R/r$
  - Threshold over BEC > $B^{\text{Hard}}$
  - Long enough LDPC length $M \Rightarrow$ Arbitrarily small BER

Conclusion

Achieves universally capacity with linear decoding complexity over BMS(C)!
Using **regular** LDPC codes

- Regular LDPC ensemble:
  - Variable- and check-nodes have degrees $d_v$ and $d_c$
- Rate $= 1 - d_v/d_c$
- Has a threshold $T^{\text{BEC}}$ bounded away from zero
- High rate code $\rightarrow$ Approaches capacity

**Conclusion:**

- Take long enough register length $K$ s.t. $B^\text{Hard} < T^{\text{BEC}}$
- Achieves performance at least as good as LDPC of blocklength length over BEC

Using IRA codes [Jin-Khandekar-McEliece ISIT’00]

- Performance guarantee via Bhattacharyya parameter holds for IRA
- Using IRA codes achieves $\Rightarrow$ **Systematic representation**
  - Linear encoding
Analysis of BP Decoding over Overall Factor Graph

Example of $n = L = 3$:

- Factor graphs of LDPC codes
- Factor graphs of convolutional codes
### Analysis of BP Decoding over Overall Factor Graph

**Extended tree assumption**

\(\ell\)-depth extended tree of variable node shares no loops with 
\(\ell\)-depth trees of other variable nodes.

- "Standard" tree assumption for each LDPC code
- "Extension": No loops with variable nodes that belong to same convolutional code

**Lemma**

Extended tree assumption is satisfied w.h.p. for large enough \(n\).

- Requires longer \(n\) than "standard" tree assumption

**BP decoding universally achieves capacity**

- Under extended tree assumption (guaranteed by long enough \(n\))
- BP over the factor graph is optimal
  
  \(\Rightarrow\) Outperforms two-stage decoding
Conclusions

- Capacity-achieving with BP ✓
- Universal ✓
- Practical?
  - Who does the heavy lifting? Inner code or outer code...?
  - In practice: Use one long CC instead of multiple copies
  - Construction is actually a variation of serial turbo codes [Benedetto et al. ’98]
    - Outer CC is replaced with LDPC code
Puncturing: Using Low-Rate LDPC Codes

The presented scheme uses:
- High-rate LDPC code (rate ≈ 1)
- CC of rate ≈ C

"Rate switch"
- Can start with low-rate LDPC code and puncture it
- Declare LDPC symbols in pre-determined positions as “erasures”
- Encode the rest using the convolutional coder
- Inner code = CC + “punctures”: Rate ≈ 1
- Outer LDPC code of rate ≈ C