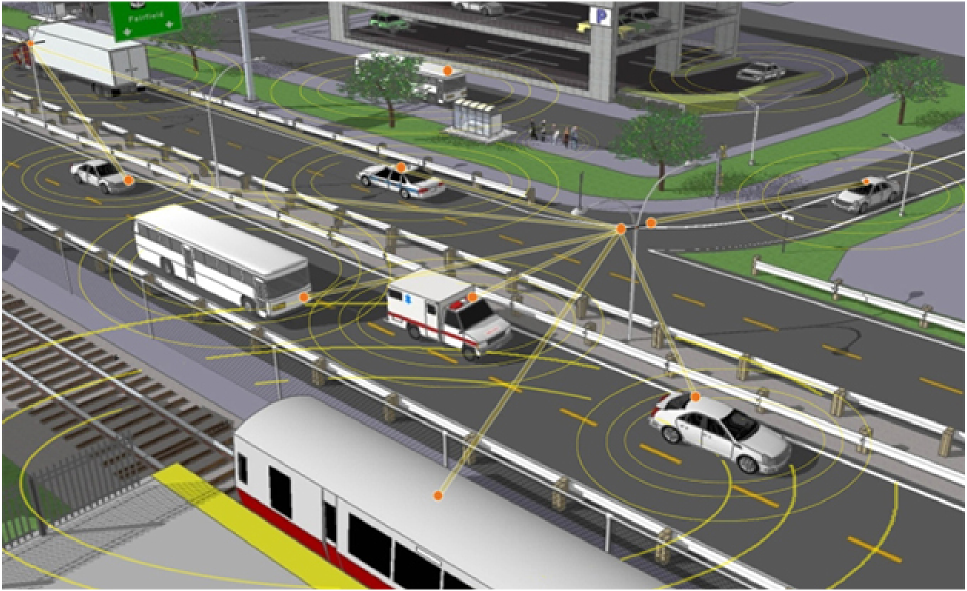
Control over Noisy Communication Media

Anatoly Khina

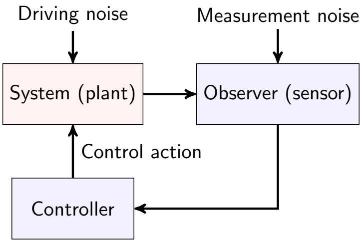
Please **DO NOT** *turn off your cell phones!*

Self-Driving Cars: Vehicle-to-Vehicle Communication

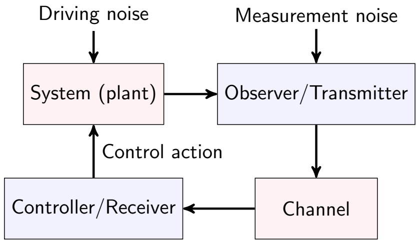


Traditional versus Networked Control

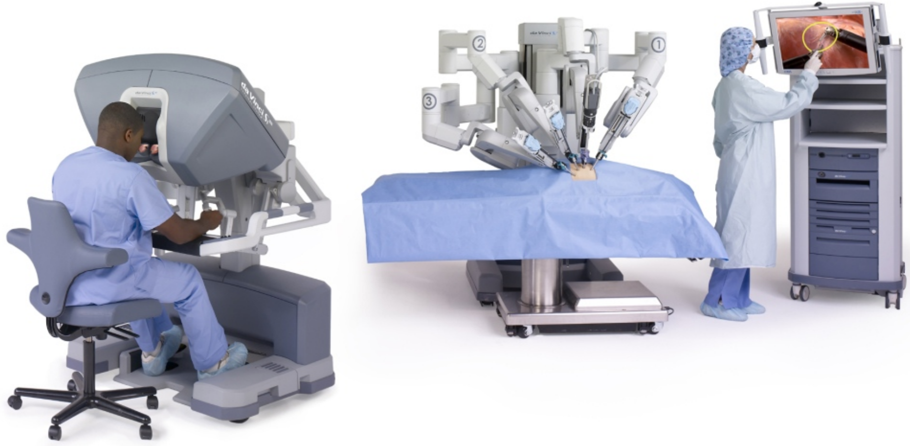
Traditional control



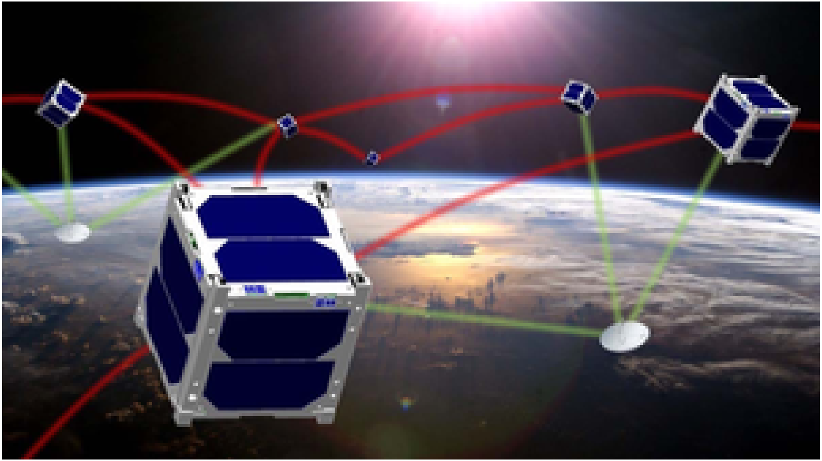
Networked control



Remote Surgery



Pico-Satellites



- In Israel: Genesis Consortium

Neuroscience: Resolution ⇔ Delay Tradeoff

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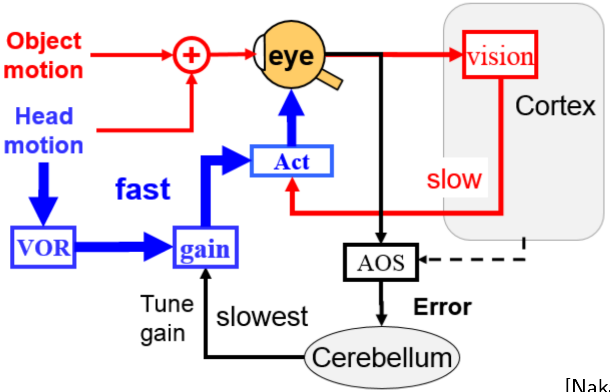
Now please turn off your cell phones...

Neuroscience: Macro-level



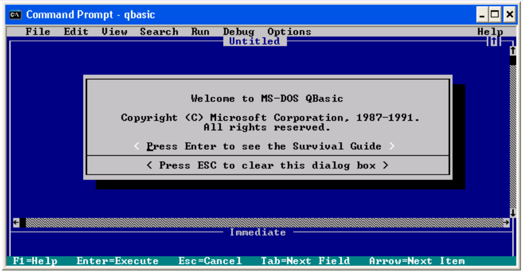
[Nakahira et al. CDC'15]

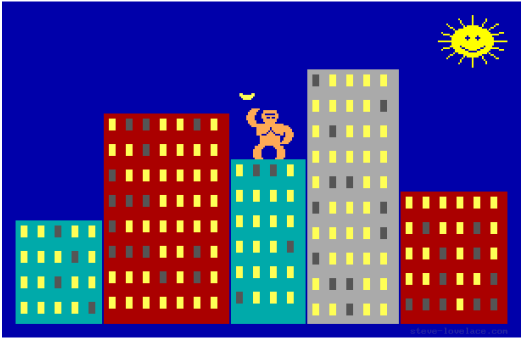
Visual system (delay $\geq$ 200ms, high res.)

VOR = Vestibulo-Ocular Reflex (delay $\approx$ 10ms, low res.)

AOS = Accessory Optical System

Back to Basics...



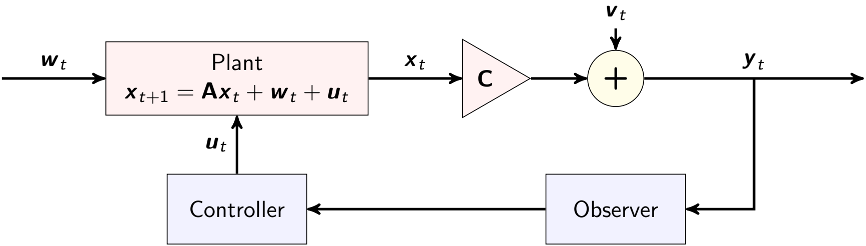


Linear Quadratic Gaussian (LQG) Control

LQG system

$$\boldsymbol{x}_{t+1} = \mathbf{A}\boldsymbol{x}_t + \mathbf{B}\boldsymbol{u}_t + \boldsymbol{w}_t, \qquad \boldsymbol{w}_t \sim \text{ i.i.d. } \mathcal{N}(0, \mathbf{W})$$

$$\boldsymbol{y}_t = \mathbf{C}\boldsymbol{x}_t + \boldsymbol{v}_t, \qquad \boldsymbol{v}_t \sim \text{ i.i.d. } \mathcal{N}(0, \mathbf{V})$$



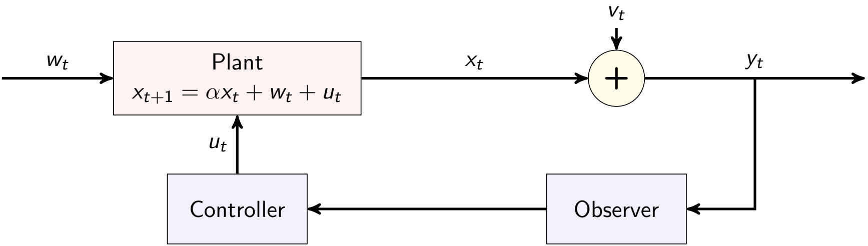
LQG cost

$$\bar{J}_T = \mathbb{E}\left[\sum_{t=1}^{T-1}\left(\boldsymbol{x}_t^T\mathbf{Q}_t\boldsymbol{x}_t + \boldsymbol{u}_t^T\mathbf{R}_t\boldsymbol{u}_t\right) + \boldsymbol{x}_T^T\mathbf{Q}_T\boldsymbol{x}_T\right]$$

Linear Quadratic Gaussian (LQG) Control

Scalar LQG system

$$x_{t+1} = \alpha x_t + u_t + w_t, \qquad w_t \sim \text{ i.i.d. } \mathcal{N}(0, W), \qquad |\alpha| > 1$$
$$y_t = x_t + v_t, \qquad v_t \sim \text{ i.i.d. } \mathcal{N}(0, V)$$



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$$\bar{J}_T = \mathbb{E}\left[\sum_{t=1}^{T-1}\left(Q_t x_t^2 + R_t u_t^2\right) + Q_T x_T^2\right]$$

Linear Quadratic Gaussian (LQG) Control

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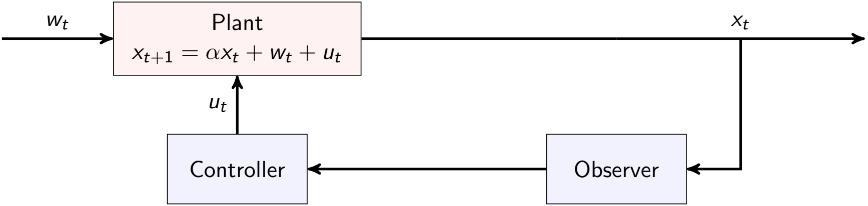

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Linear Quadratic Gaussian (LQG) Control over Noisy Channels

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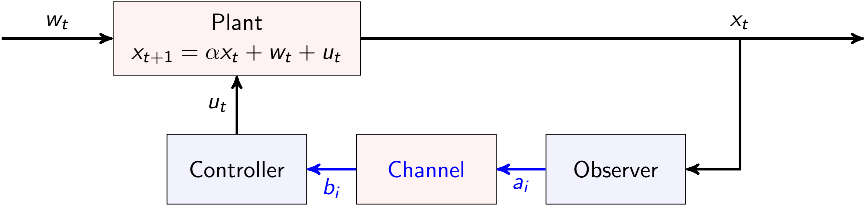

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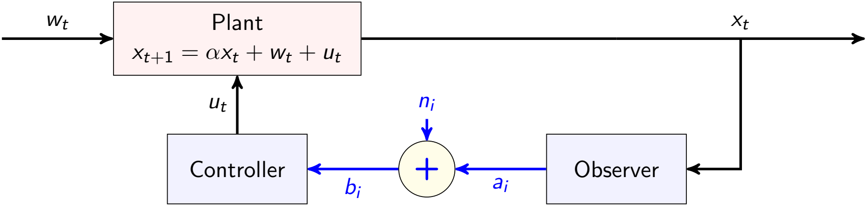

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Linear Quadratic Gaussian Control over Gaussian Channels

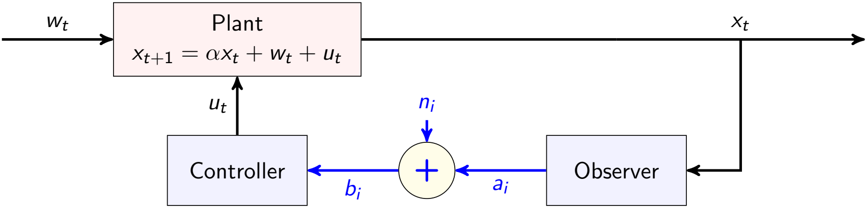
Scalar LQG system

$$x_{t+1} = \alpha x_t + u_t + w_t$$

Scalar AWGN channel

$$b_i = a_i + n_i, \qquad n_i \sim \mathcal{N}$$

Power constraint: $\mathbb{E}\left[a_i^2\right] \leq P$



Linear Quadratic Gaussian Control over Gaussian Channels

Control rate $\neq$ Communication rate!

- Assume N channel uses per one control sample

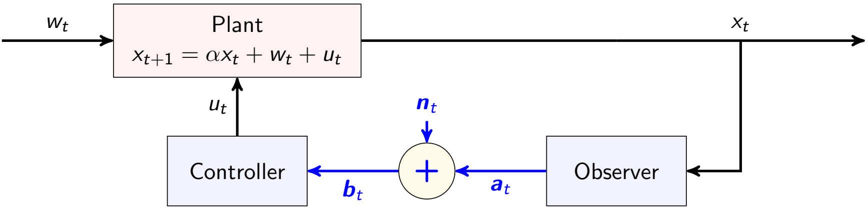
Scalar LQG system

$$x_{t+1} = \alpha x_t + u_t + w_t$$

Scalar AWGN channel

$$\boldsymbol{b}_t = \boldsymbol{a}_t\left(x^t, u^{t-1}\right) + \boldsymbol{n}_t$$

Power constraint: $\mathbb{E}\left[\boldsymbol{a}_t^2\right] \leq NP$



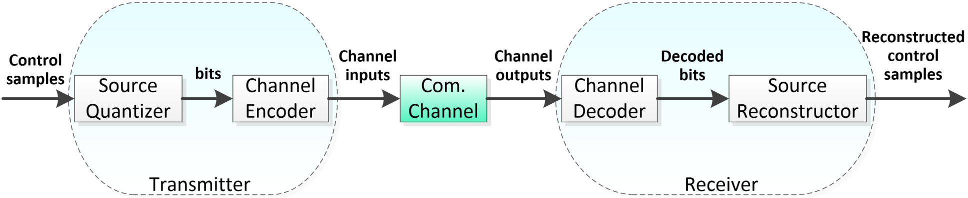
Control Sampling Rate vs. Communication Signaling Rate

- How fast the plant dynamic is $\Rightarrow$ Control sampling rate
- Bandwidth available $\Rightarrow$ Communication signaling rate
- Communication rate can be much higher in practice

How to benefit from excess signaling rate (bandwidth)?

Networked Control Approaches

Networked Control Approaches: Information-Theoretic Separation

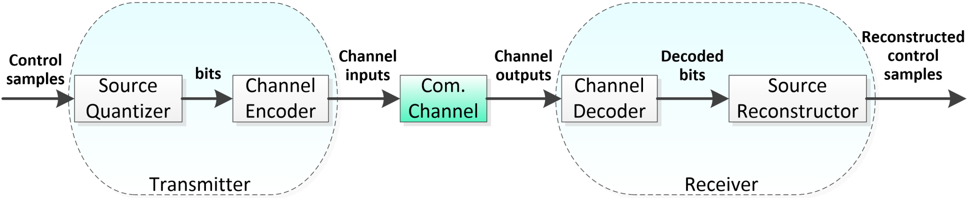


Communications is broken into two tasks:

1. Source quantization: Batch of source samples $\rightarrow$ Block of bits
2. Channel coding: Block of bits $\rightarrow$ Batch of channel uses

- Bits serve as an interface

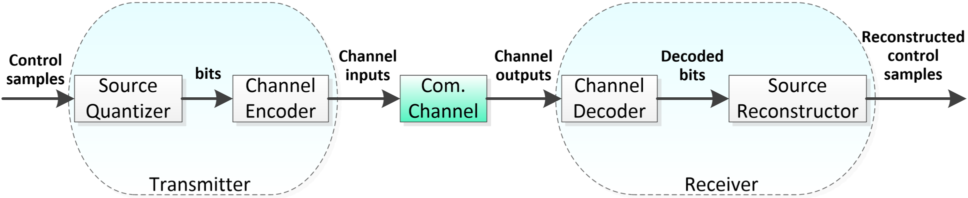
Networked Control Approaches: Information-Theoretic Separation



Advantages

- Breaks down design and analysis tasks into two simpler tasks
- Implementation: "Two different specializations"
- Breakthrough in analysis of either task $\to$ Better overall analysis
- Becomes optimal when block lengths **(=delay!)** go to infinity

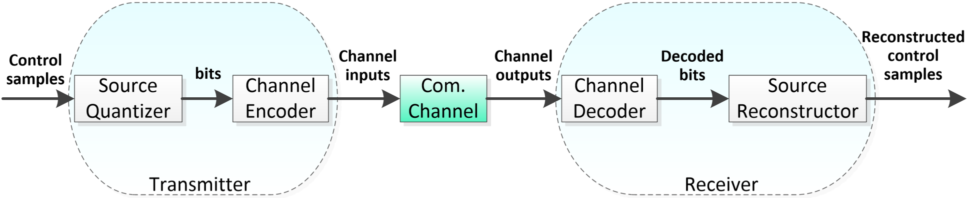
Networked Control Approaches: Information-Theoretic Separation



Shortcomings

- Requires large blocks **(delay!)** of source samples and channel uses
- **Suboptimal for control!**
- Requires codes with strong "anytime reliability" properties [Schulman IT'96][Sahai-Mitter IT'06][Sukhavasi-Hassibi AC'16]
- Problematic in practice: Convolutional code with infinite memory [Kh.-Halbawi-Hassibi ISIT'16, submitted IT'17]
- Requires proper **adaptive** quantizers: Static quantizers fail!

Networked Control Approaches: Information-Theoretic Separation



Packetizing: Extreme case

- Assumes communication rate $\gg$ control rate, very good SNR
- Problem reduces to control-oriented quantization
- Bad channel events are translated to packet drops / delays

Networked Control Approaches: Joint Source–Channel Coding (JSCC)

- What to do when control and communication rates are close?
- Can we do better than IT-separation?

Less familiar IT avenue

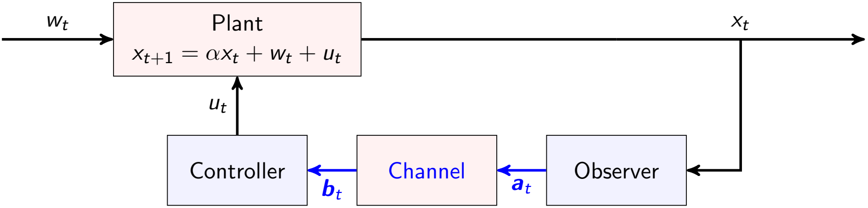
- Low-delay joint source–channel coding (JSCC)
- Analog mappings (no going through bits!)
- Control sample corresponds to source sample

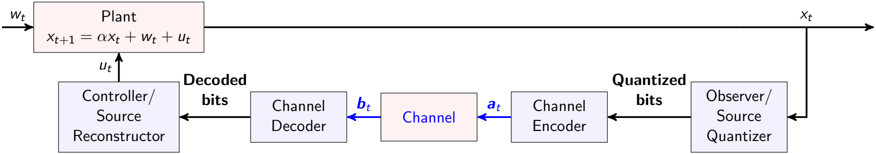
More general concept

- Use control loop as communications feedback

Source–Channel Separation

Source–Channel Separation

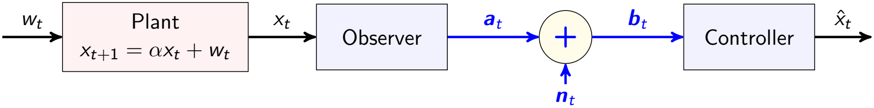




Motivating Example: Tracking a Random Walk [Sahai PhD'01]

$$x_{t+1} = \alpha x_t + w_t$$

- $|\alpha| > 1 \implies$ not stable!
- $w_t \in \{\pm 1\}$ — quantized bits representing the control state
- We wish to track x_t with bounded expected distortion
- If tracking is possible $\Rightarrow$ Stability



Distortion requirement

$$\mathbb{E}\left[(x_t - \hat{x}_t)^2\right] < \infty, \qquad \forall t$$

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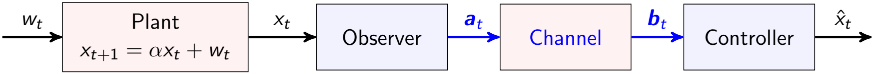

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Motivating Example: Tracking a Random Walk [Sahai PhD'01]

- $\hat{w}_{t-d|t}$ – Estimate of w_{t-d} at time t
- Probability of first error event at time $t-d$:
  $P_e(t,d) \triangleq \Pr\left(w_{t-d} \neq \hat{w}_{t-d|t}, \forall \delta > d, w_{t-\delta} = \hat{w}_{t-\delta|t}\right)$

$$\mathbb{E}\left[\left(x_t - \hat{x}_{t|t}\right)^2\right] \propto \sum_{d=1}^{t} P_e(t,d)\alpha^{2d} = \sum_{d=1}^{t} P_e(t,d) 2^{2\log\alpha \cdot d} < \infty$$

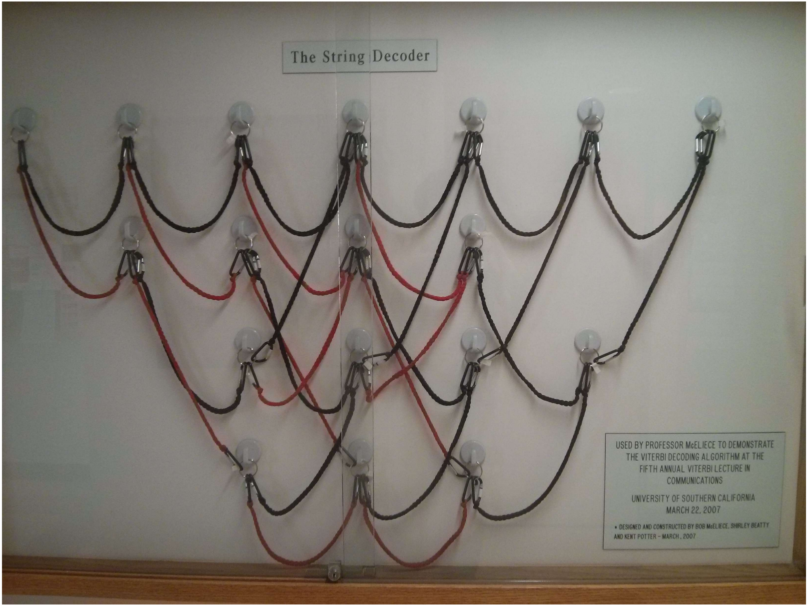
Error probability profile: Anytime-reliable code

$$P_e(t,d) < A 2^{-(2\log\alpha + \epsilon)d}, \qquad \forall t, \forall d$$

Higher-order moments

Higher exponent $\Longrightarrow$ Cannot stabilize all moments!

Tree Codes [Schulman IT'96]



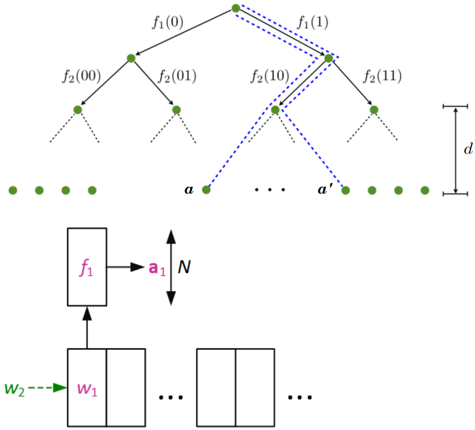
Adaptation to Control: Anytime-Reliable Codes [Sahai-Mitter IT'06]

Error probability profile

$$P_e(t,d) < A2^{-(2\log\alpha+\epsilon)d}\,, \qquad \forall t, \forall d$$

How to generate such a code?

$$\begin{aligned} a_1 &= f_1(w_1) \\ a_2 &= f_2(w_1, w_2) \\ &\vdots \\ a_t &= f_t(w_1, w_2, \ldots, w_t) \\ &\vdots \end{aligned}$$



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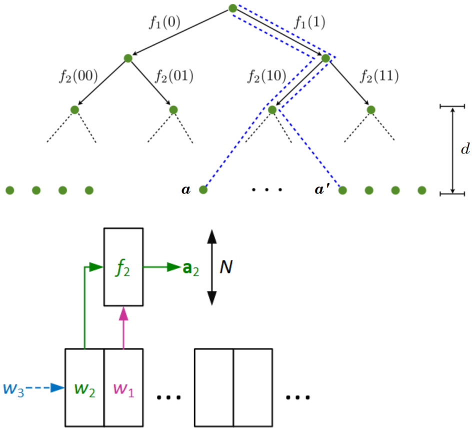

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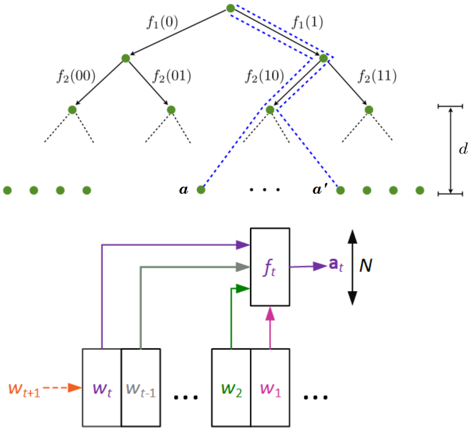


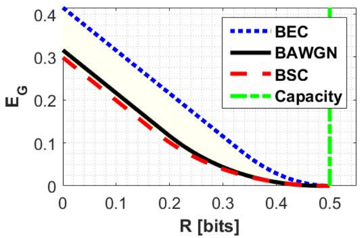
Anytime-Reliable Codes as Convolutional Codes

Random **time-varying** convolutional-code ensemble [Viterbi, Yudkin, Zigangirov, Schulman–Feder, ...]

- Most results assume infinite stream ($\gg$ delay-line length)
- We wish to recover a bit using subsequent Nd output symbols
- The random time-varying CC ensemble achieves:

$$\mathbb{E}\left[P_e(t,d)\right] \leq 2^{-E_G(R)Nd}$$

- $E_G(R) > 0$ for $R < C$ – Gallager's error exponent



Anytime-Reliable Codes as Convolutional Codes

Good ensemble performance ⇒ Good specific code performance?

- $\mathbb{E}\left[P_e(t,d)\right] \leq 2^{-E_G(R)Nd} \overset{?}{\Rightarrow} P_e(t,d) \leq A2^{-E_G(R)Nd}$
- Yes, with high probability, for **specific** t and d
- Anytime reliable-code?
- Needs to hold $\forall d$ and $\forall t$!
- Such a code exists [Schulman IT'96], but **not w.h.p.** ☹
  (Proof requires min-distance $\propto$ delay)

- LDPC-based constructions: [Grosjaen et al. IT'14] [Noor-A-Rahim et al. COM'15][Zhang et al. IT'16]
- Explicit constructions: [Gelles-Moitra-Sahai, FOCS'11, IT'14] [Moore-Schulman ITS'14][Pudlák LinAlg&Apps'16]

Anytime-Reliable Codes as Convolutional Codes

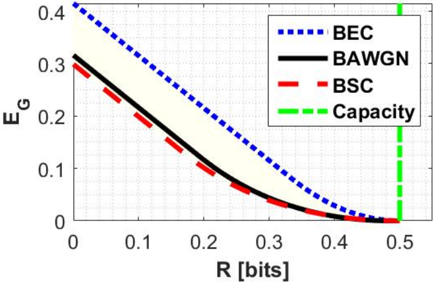
Linear time-invariant codes [Sukhavasi-Hassibi AC'16]

- Time invariance ⇒ No dependence on t: $P_e(t,d) \equiv P_e(d)$
- Proof simply follows by the union bound
- [Kh.-Halbawi-Hassibi ISIT'16, submitted IT'17]:
  Easily proved by viewing as CC + [S~~c~~hulman-Feder IT'00]
- Better results for lower rates using linear codes [Barg-Forney IT'02]

Universality [Kh.-Halbawi-Hassibi, submitted IT'17]

- High probability proof ⇒ Universality result w.r.t. channel
- Similar to the universal LDPC code construction of [Kh.-Yona-Erez ISIT'15]

Anytime-Reliable Codes as Convolutional Codes



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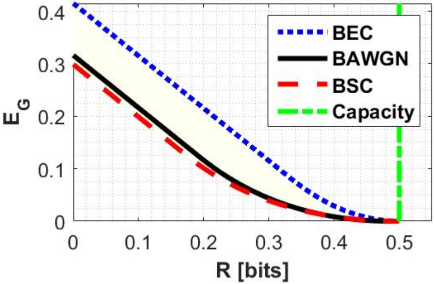

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What about decoding?

Decoding of LTI Anytime-Reliable Codes

- All results assumed maximum-likelihood (ML) decoding
- ML complexity rises exponentially with t

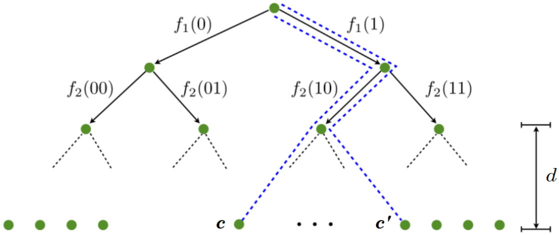
Binary Erasure Channel (BEC)

- For LTI codes: ML $=$ Solving linear equations

- What about other channels?

Sequential Decoding

- Before Viterbi algo.: Sequential decoding *de facto* standard
- Sequential decoding = class of algorithms
- Introduced originally in [Wozencraft '57] for **tree codes**
- Common to all: Explore only subset of (likely) codewords
- Most prominent variants: Stack and Fano's algorithms
- Proposed for general tree ensembles in [Schulman IT'96][Sahai-Palaiyanur Allerton'05]

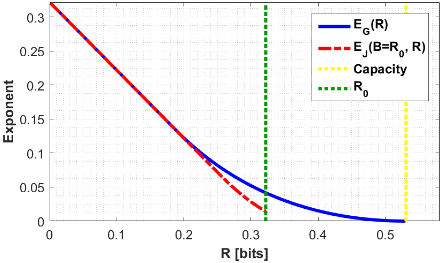


Sequential Decoding: Error Probability

Error probability of **general** conv. ensemble [Jelinek's Book '68]

$$\mathbb{E}\left[P_e(t,d)\right] \le A\, 2^{-E_J(B,R)Nd}$$

- A is finite for $B < R_0$
- $E_J(B,R) \le E_G(R)$
- $E_J(B,R) \xrightarrow{B \to R_0} E_G(R)$, for $R < R_{\text{crit}}$
- Does not guarantee a good **specific code** w.h.p.



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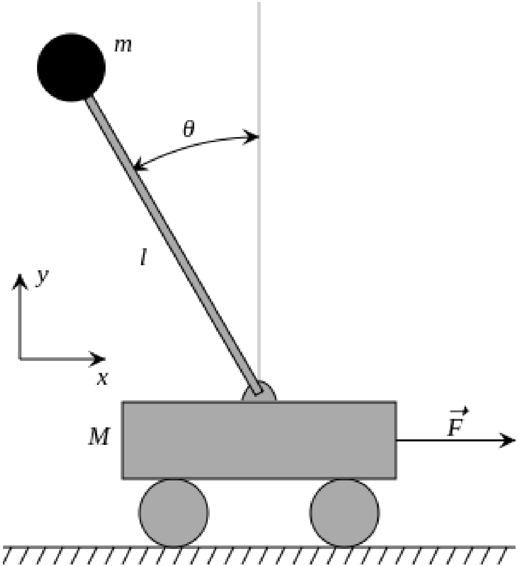
BER of LTI tree codes [Kh.-Halbawi-Hassibi ISIT'16, submitted IT'17]

- BER guarantees extend to LTI codes
- Anytime reliable w.h.p. for a **specific code**
- Universal for channels with given capacity
- Design for the BSC

Simulation: Cart–Stick over BSC(0.01)



Simulation: Cart–Stick over BSC(0.01)

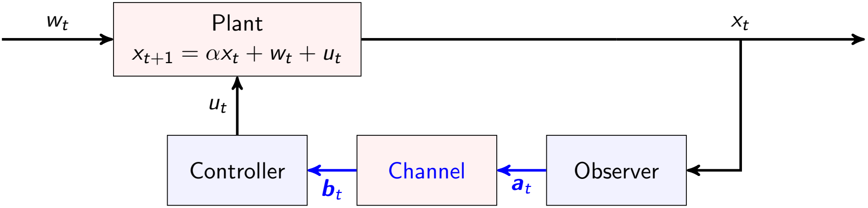
- Cart–stick system model [Franklin-Powell-Emami-Naeini Book]
- BSC(0.01)
- For this setting [Sukhavasi–Hassibi ISIT'11]: $E_{\text{min}} = 0.21$

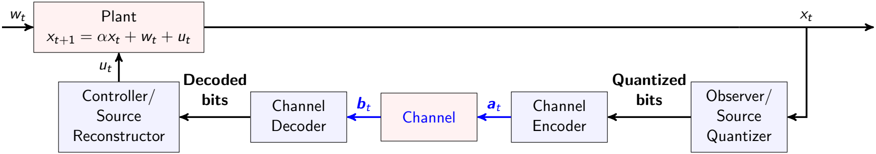
$$E = 0.54 \qquad E = 0.24 \qquad E = 0$$

Quantization

- Channel error correction ✓
- What about quantization?

Source–Channel Separation





Linear Quadratic Gaussian Control over Gaussian Channels

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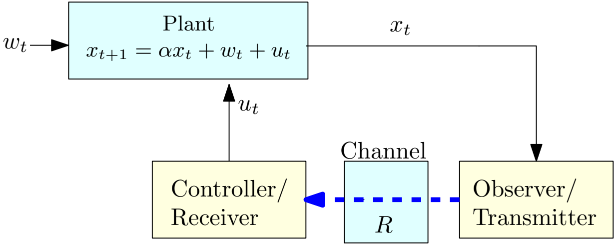

Noiseless finite-rate channel of rate R

Fixed rate: Exactly R bits are available at every time step t

Variable rate: R bits are available **on average** at every t

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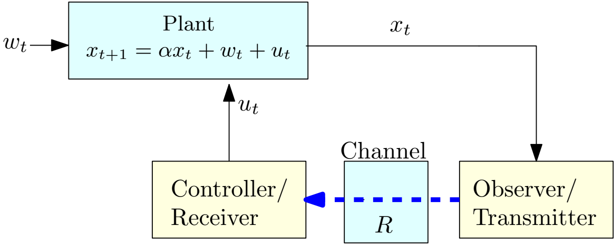

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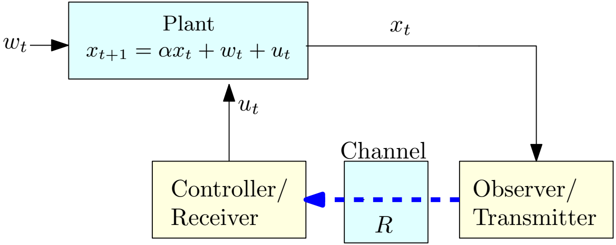

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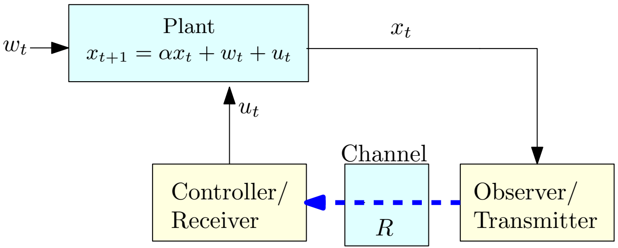

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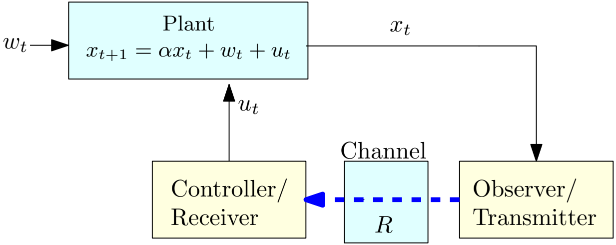

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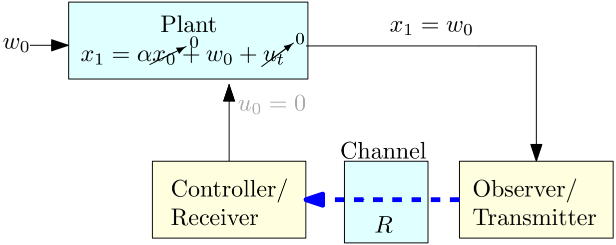

LQG cost

$$J = \mathbb{E}\left[\sum_{t=1}^{T}\left[Q_t x_t^2 + R_t u_t^2\right] + Q_T x_{T+1}^2\right]$$

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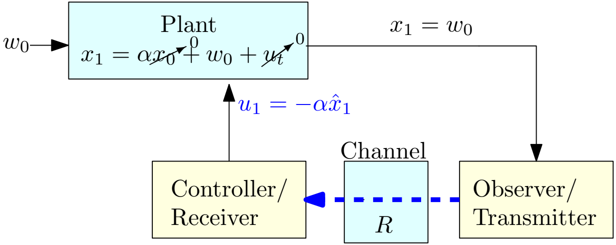

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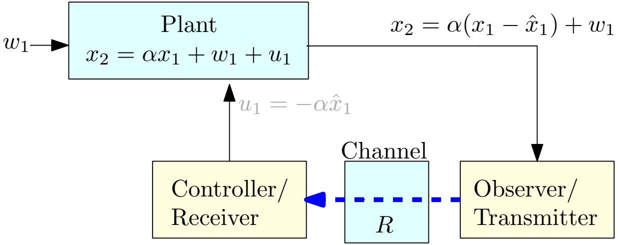

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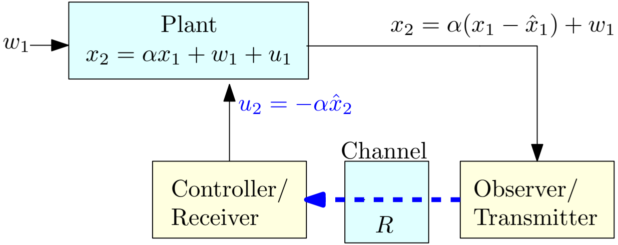

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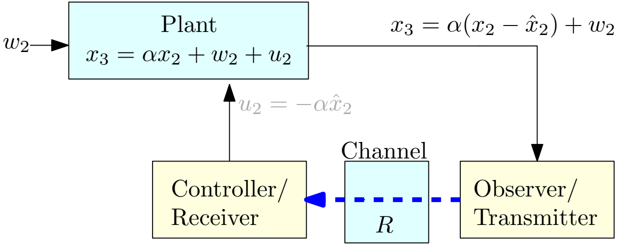

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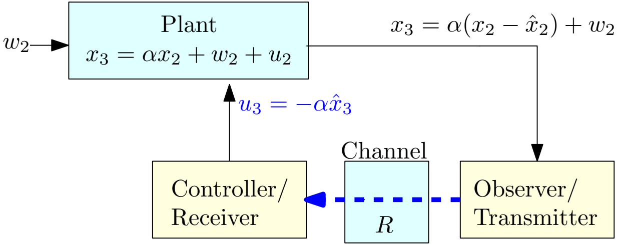

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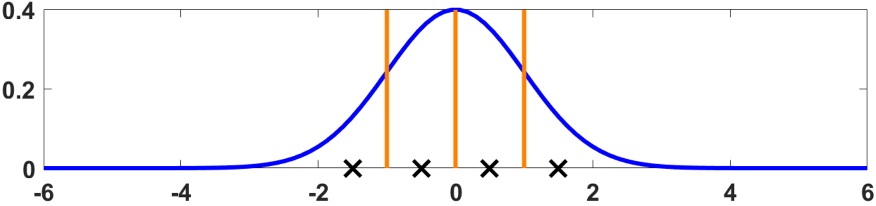
Adaptive Fixed-Rate Quantizer



- Use an adjusted quantizer to the input p.d.f.

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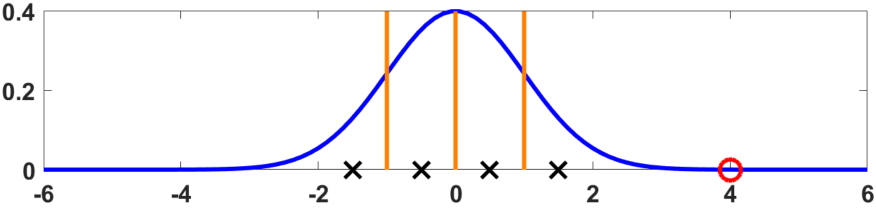

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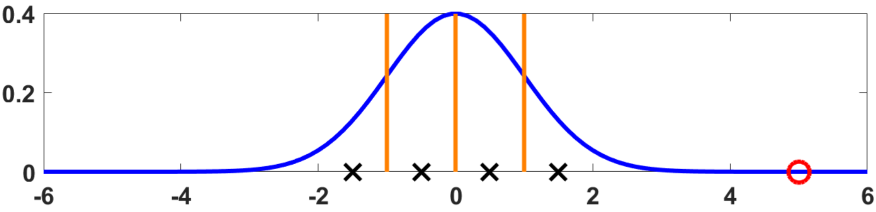

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- At some point a (rare) event will happen
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- Next time step: Input will be even larger!
- **Avalanche effect**

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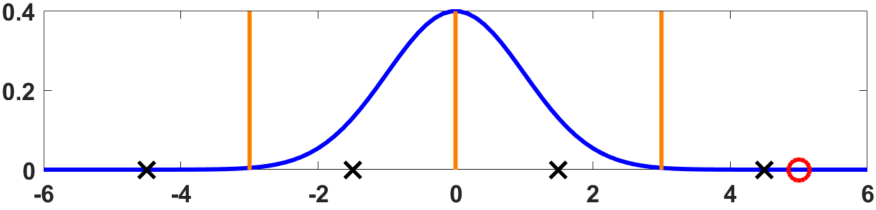

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- Next time step: Input will be even larger!
- **Avalanche effect**
- To avoid this $\Rightarrow$ Quantizer needs to be **adaptive**

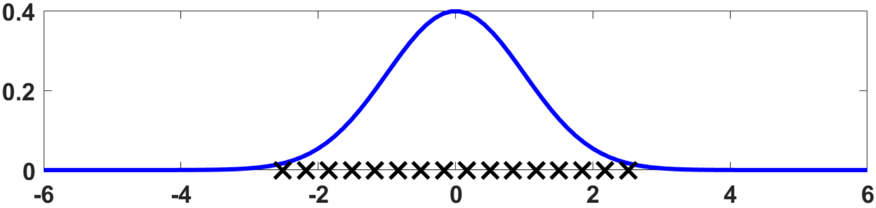
Adaptive Optimal Fixed-Rate Quantizer?

- Adaptive uniform quantizer [Yüksel AC'10]
  - Based on Jayant's adaptive quantizer [Jayant '73]
  - Similar idea in [Brockett-Liberzon AC'00]: "Zooming in/out"
- Adaptive exponential quantizer [Nair-Evans SICON'04][Minero et al. AC'09]
- Both results prove condition on stabilizability: $R > \log \alpha$
- But no cost optimality claims...
- Other notable contributions: [Borkar-Mitter '97] [Tatikonda-Sahai-Mitter AC'04] [Matveev-Savkin '04] [Tsumura-Maciejowski CDC'03], ...

How to optimize cost?

Optimal Quantizer for One Time Step

- Let $x \sim \mathcal{N}(0, 1)$
- R bits $\Rightarrow 2^R$ quantization points
- Uniform quantizer is suboptimal
- How to construct an optimal quantizer?



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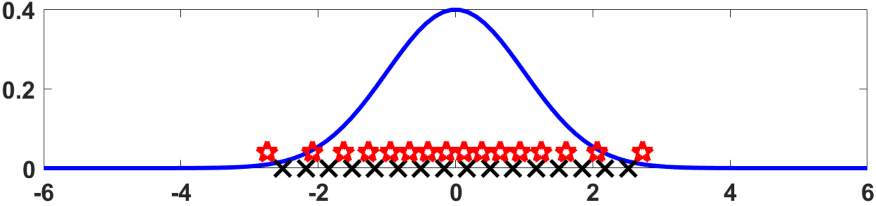

- In general a hard (NP-hard) problem
- Necessary conditions by Lloyd ['57, IT'82] and Max [IT '60]
  - Also known in machine learning as "k-means" clustering

Lloyd-Max Algorithm

Nearest Neighbor: Given reconstruction points, find optimal cells

$$\text{Cell } i = \left\{ x \middle| (x - \hat{x}_i)^2 < (x - \hat{x}_j)^2, \ \forall j \neq i \right\}$$

Centroid: Given quant. cells, find optimal reconstruction points

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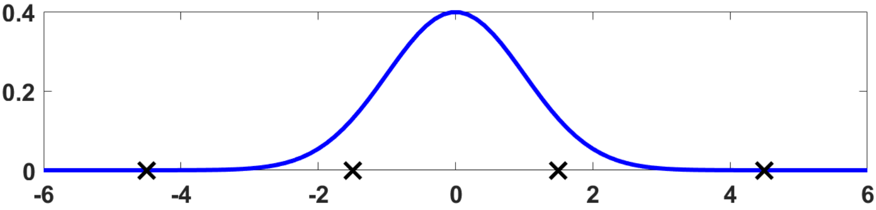

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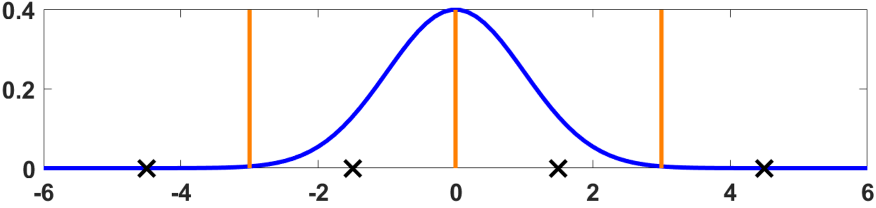

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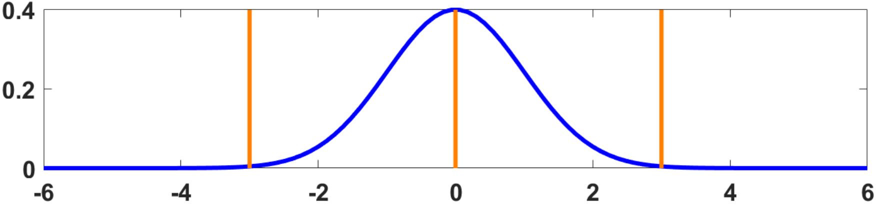

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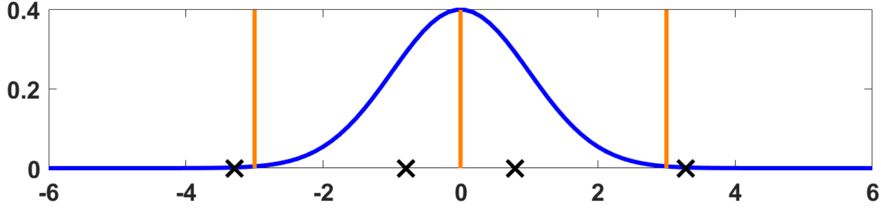

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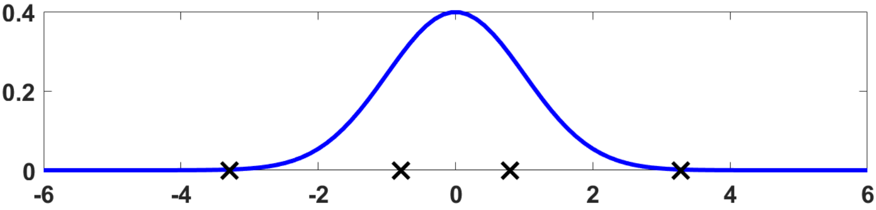

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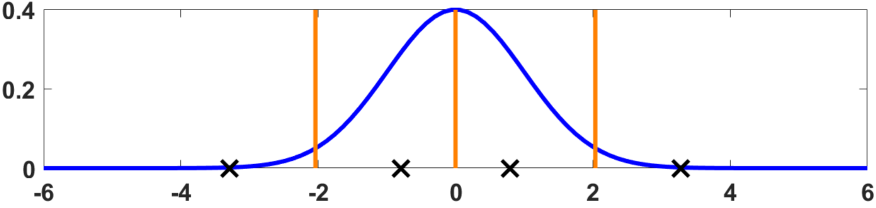

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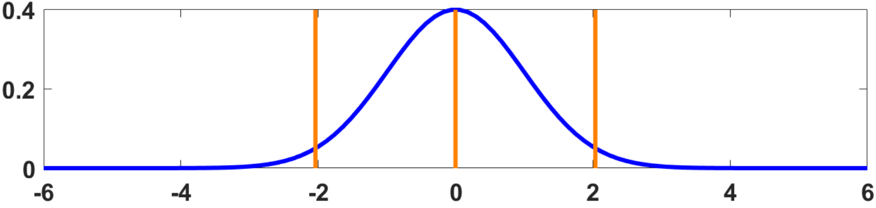

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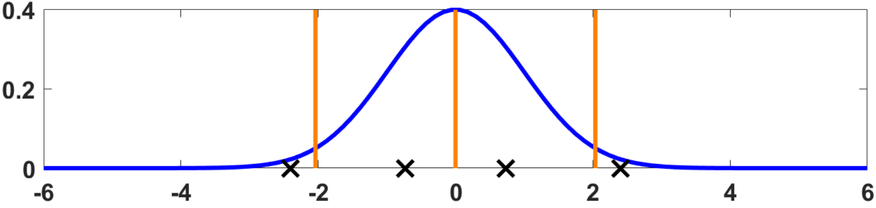

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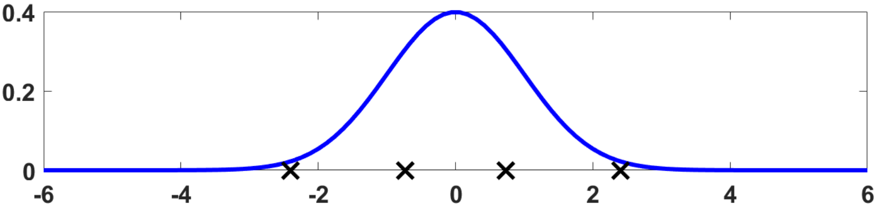

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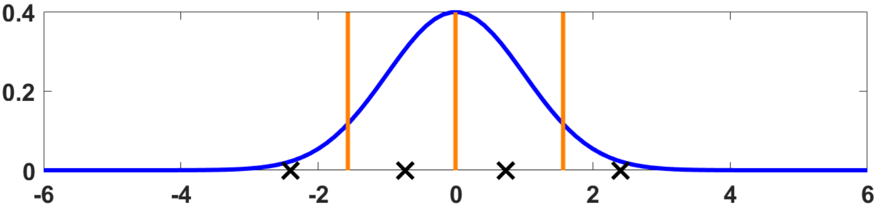

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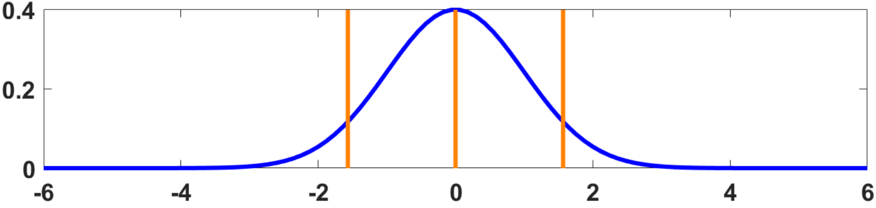

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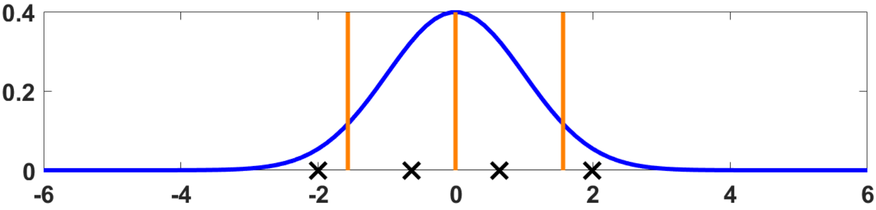

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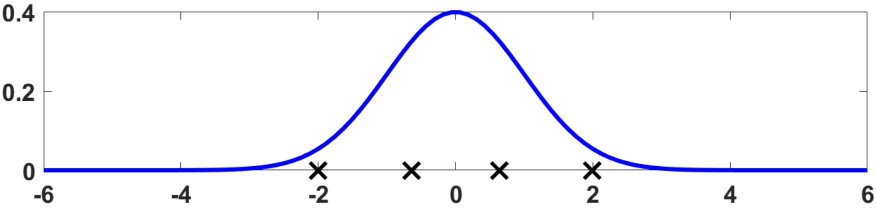

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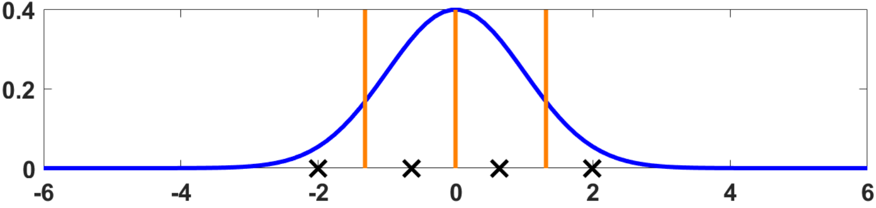

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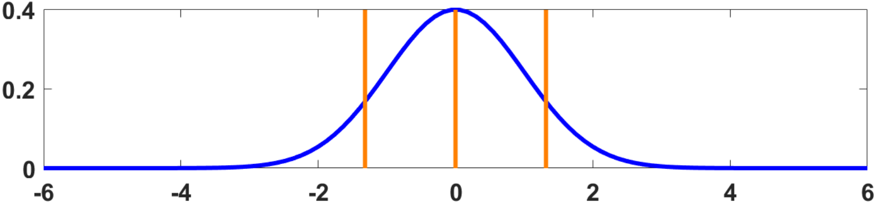

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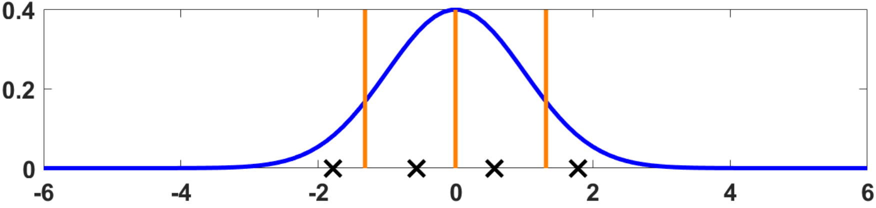

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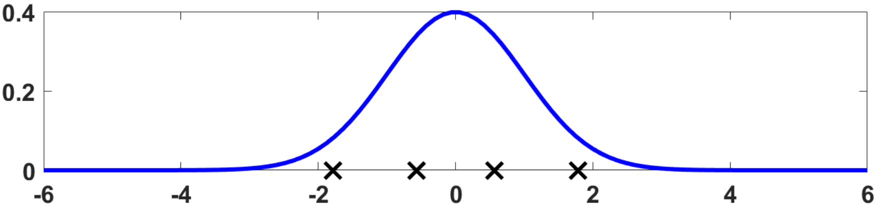

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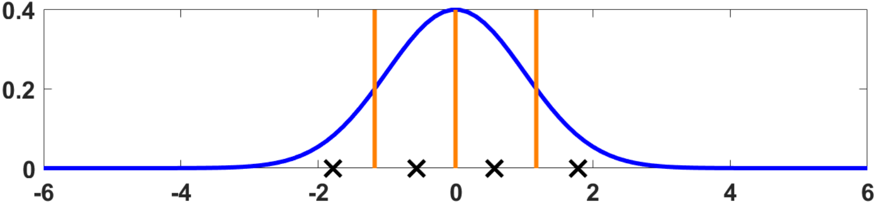

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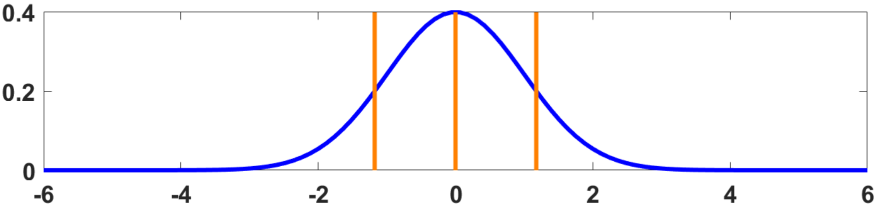

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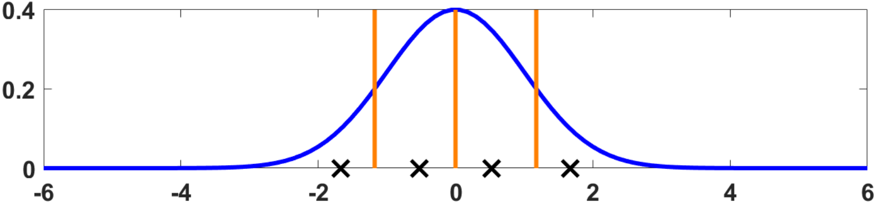

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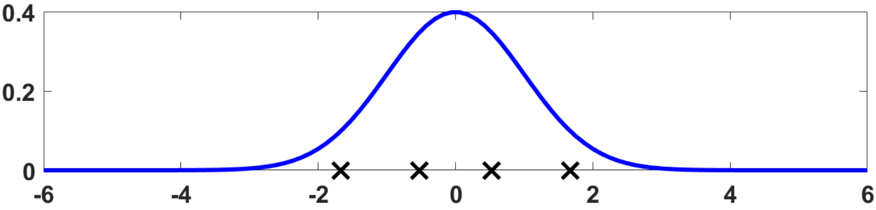

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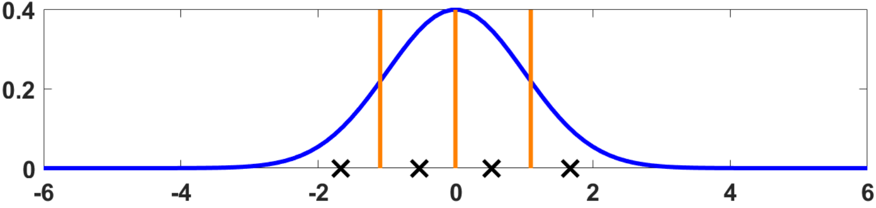

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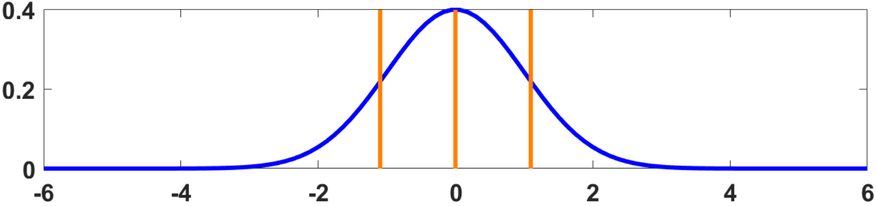

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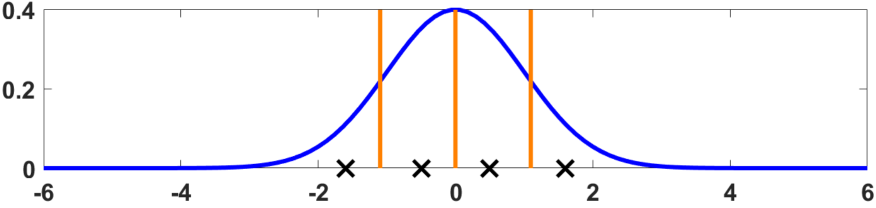

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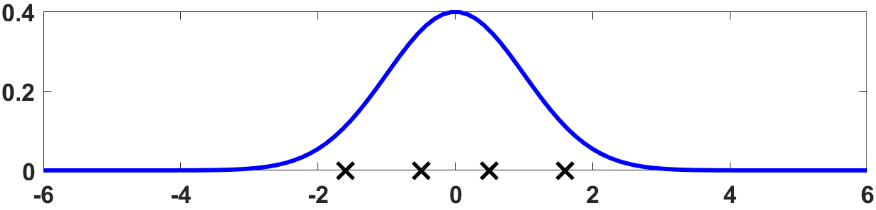

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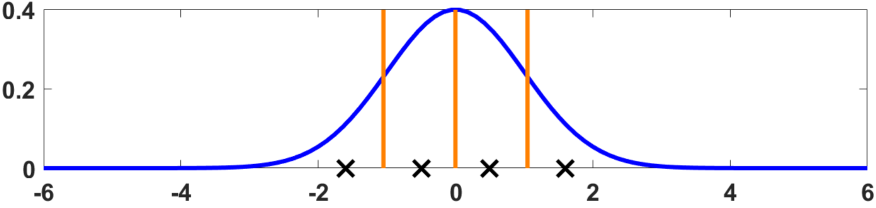

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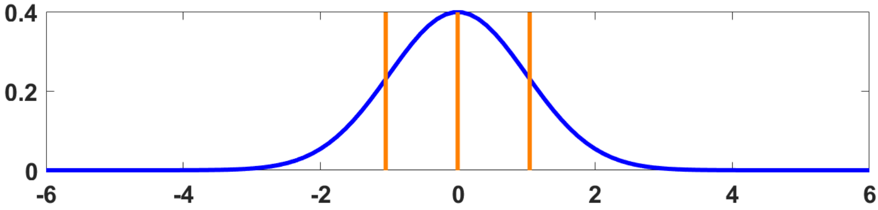

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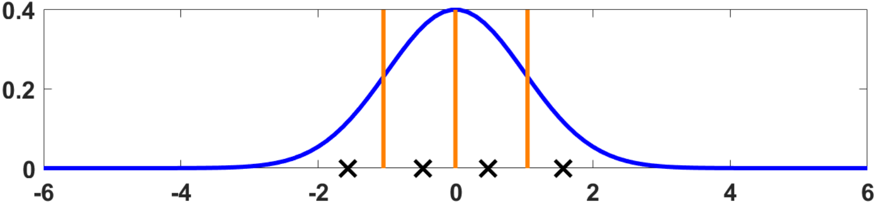

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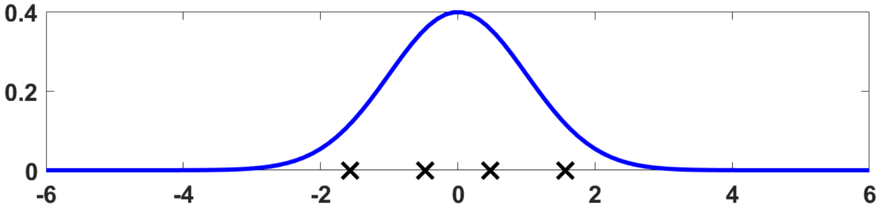

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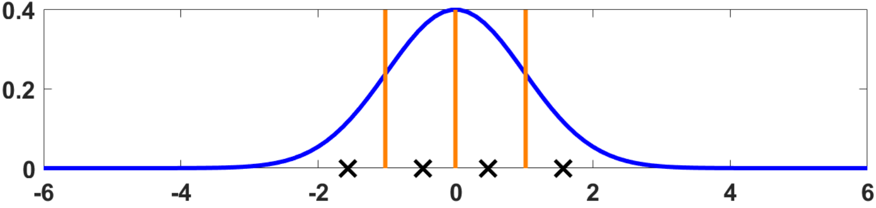

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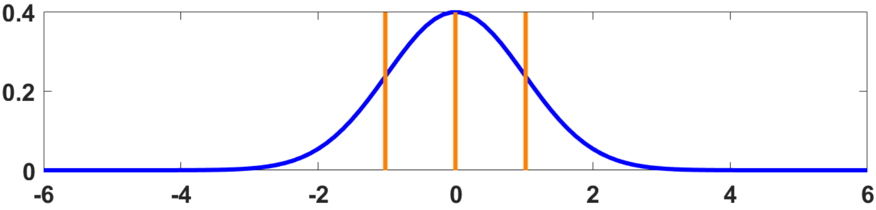

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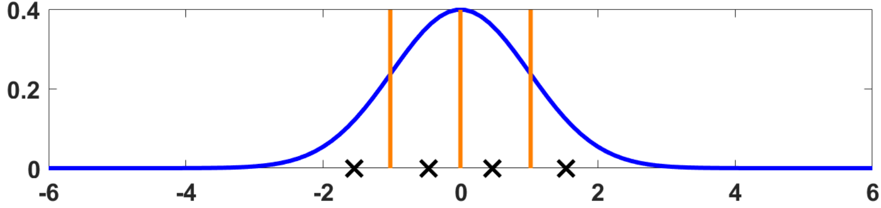

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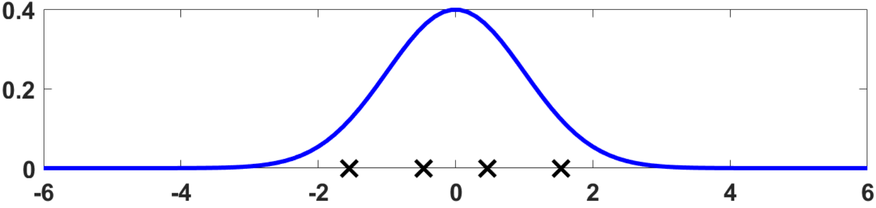

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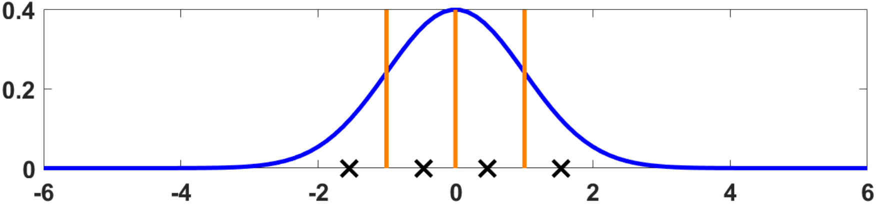

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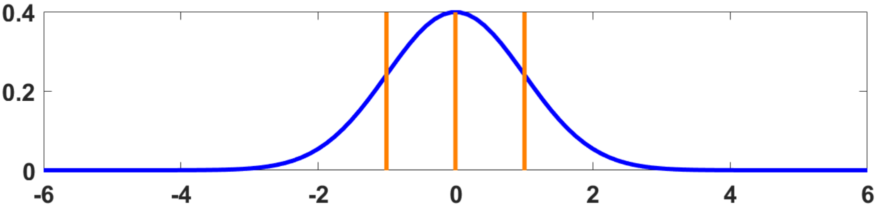

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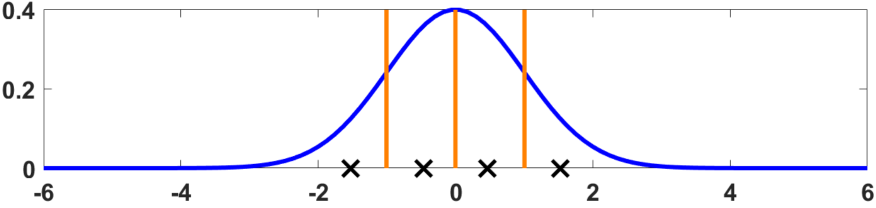

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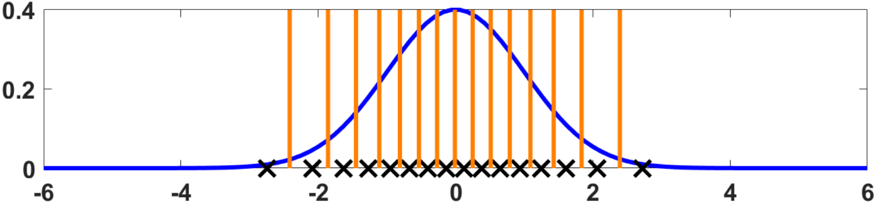
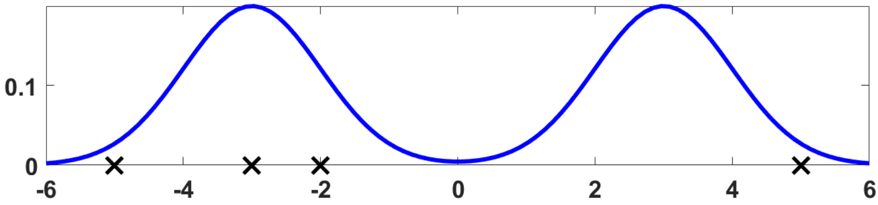


Lloyd-Max Algorithm

- Optimal quantizer necessarily satisfies *Centroid* and *NN*
- But... They are not sufficient in general! ☹
- Lloyd-Max algorithm might converge to a local optimum...



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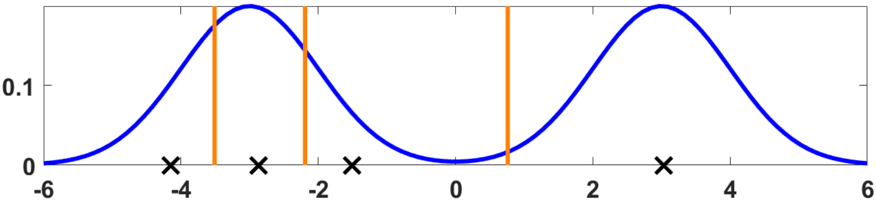


Lloyd-Max Algorithm

When does Lloyd-Max converge to global optimum?
[Fleischer '64][Trushkin IT'82][Kieffer-Jahns-Obuljen IT'88]

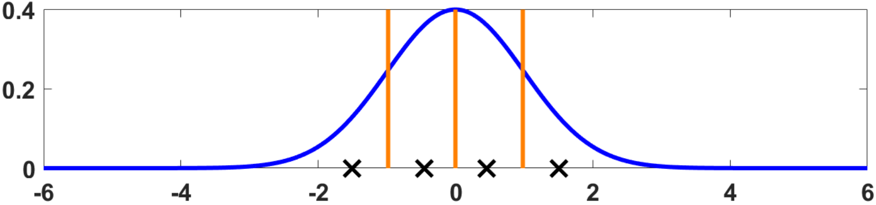
- Conditions for existence of only one local optimum $\Rightarrow$ **Global**
- **Log-concave** distributions satisfy these conditions
- Important special case: **Gaussian distribution** ☺

- One time step of LQG with finite-rate noiseless channel ✓

What about more time step?

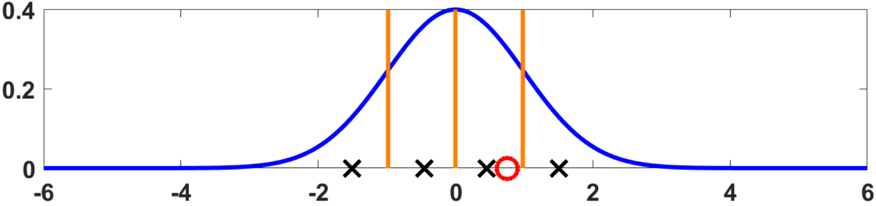
Multi-Step Control with Finite-Rate Feedback [Kh. et al. CDC'17]

- First input $x_1 = w_0$ is Gaussian $\Rightarrow$ Log-concave pdf
- Lloyd-Max quantizer is optimal



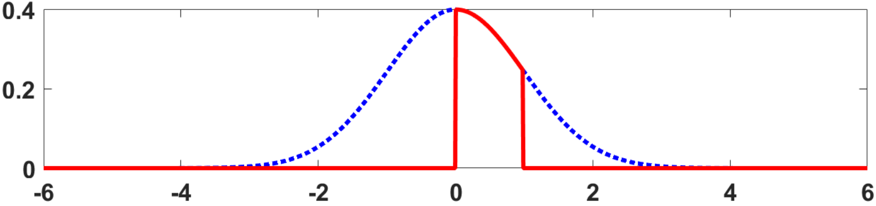
Multi-Step Control with Finite-Rate Feedback [Kh. et al. CDC'17]

- First input x_1 arrives
- Chooses cell: cell i
- Chooses reconstruction point: $\hat{x}_i$



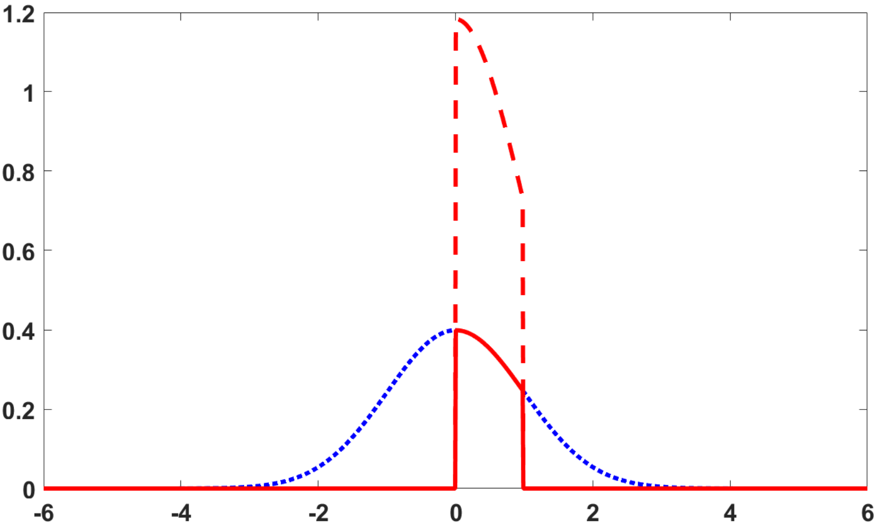
Multi-Step Control with Finite-Rate Feedback [Kh. et al. CDC'17]

- pdf given hit cell i $p(x_1 | x_1 \in \text{cell } i)$ = truncated original pdf $p(x_1)$



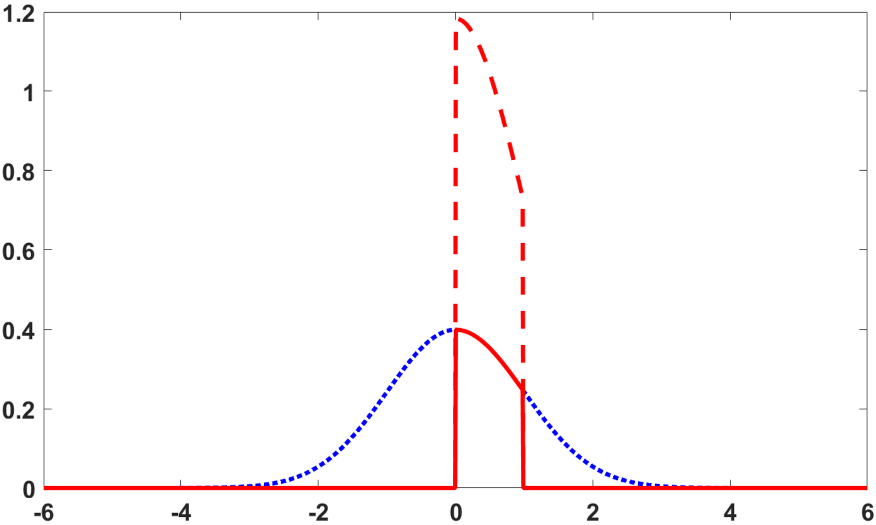
Multi-Step Control with Finite-Rate Feedback [Kh. et al. CDC'17]

- Up to scaling...



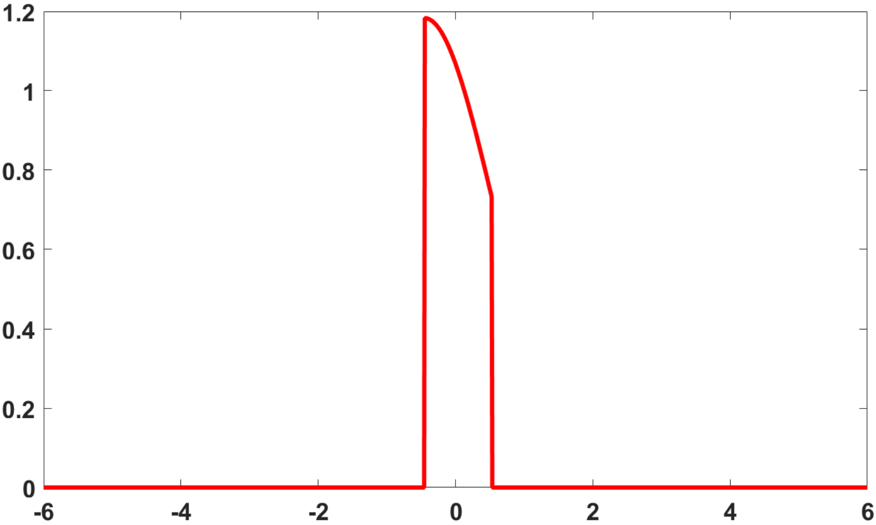
Multi-Step Control with Finite-Rate Feedback [Kh. et al. CDC'17]

- Truncated log-concave pdf is **log-concave**!



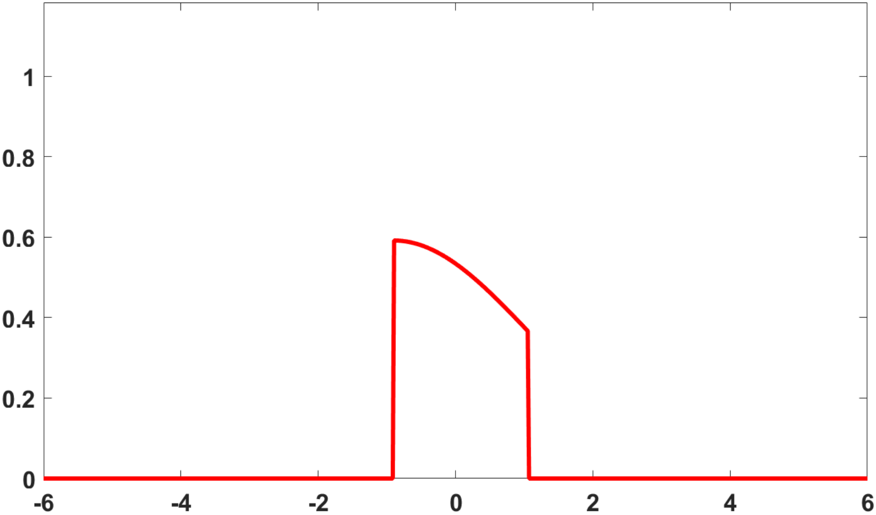
Multi-Step Control with Finite-Rate Feedback [Kh. et al. CDC'17]

- pdf of quantization noise $p(x_1 - \hat{x}_1 | x_1 \in \text{ cell } i)$



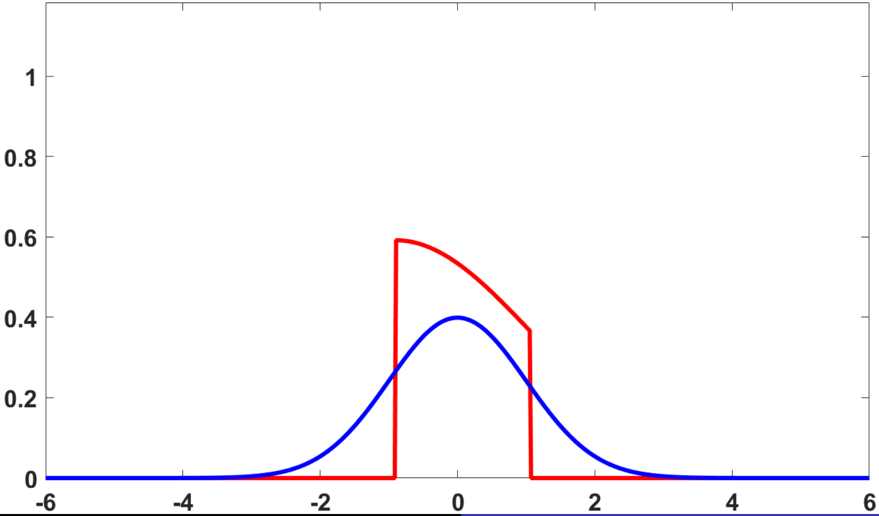
Multi-Step Control with Finite-Rate Feedback [Kh. et al. CDC'17]

- Quantization noise inflated by α: $\alpha(x_1 - \hat{x}_1)$



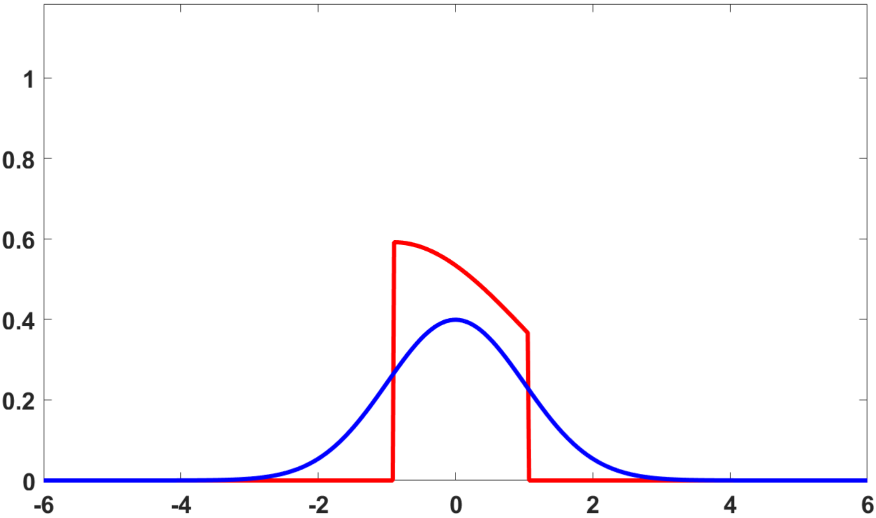
Multi-Step Control with Finite-Rate Feedback [Kh. et al. CDC'17]

- New w_t added: $\alpha(x_1 - \hat{x}_1) + w_1 \Rightarrow$ Convolution of pdfs



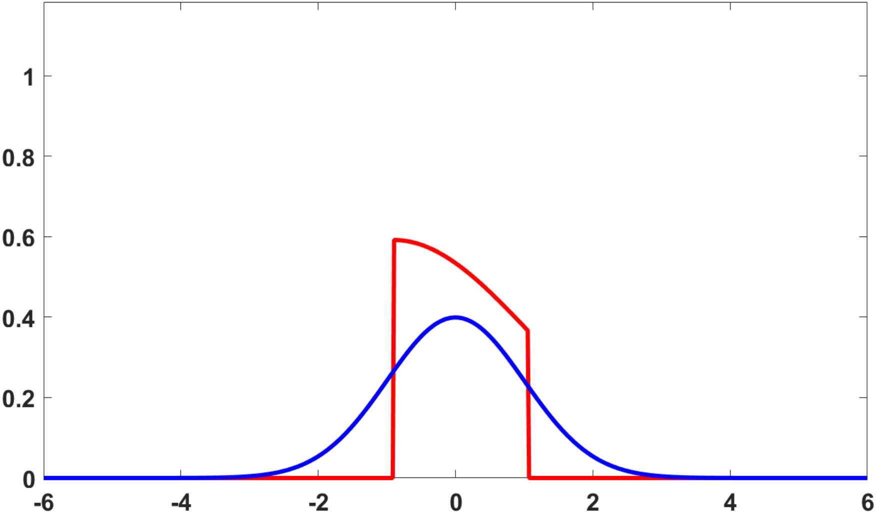
Multi-Step Control with Finite-Rate Feedback [Kh. et al. CDC'17]

- $w_t \sim \mathcal{N}(0, W) \ast$ log-concave quantization error



Multi-Step Control with Finite-Rate Feedback [Kh. et al. CDC'17]

- Convolution of log-concave functions is also **log-concave**!



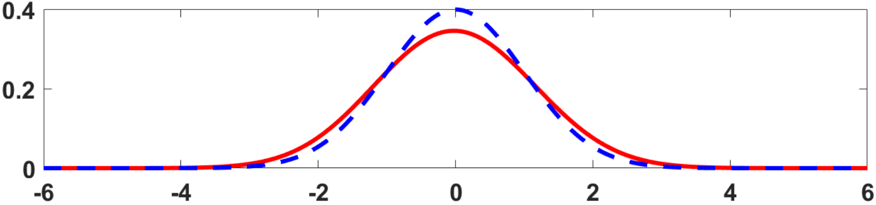
Multi-Step Control with Finite-Rate Feedback [Kh. et al. CDC'17]

Resulting pdf (in red)

- Depends on cell index chosen in previous step(s)
- Log-concave

Applying Lloyd-Max quantization in second step is optimal!

- First-step pdf (in blue) for comparison



Optimal Greedy Algorithm

- Lloyd-Max quantization minimizes squared error of that step
- Lloyd-Max quantization = Optimal **greedy** algorithm
- But... It is not necessarily globally optimal...
- Quantizer used affects pdf of future time steps
- Quantizer should be chosen according to the dynamic program (take into account the "cost-to-go")

Greedy optimal vs. Globally optimal

Low rates: Negligible loss $\sim 1\% - 2\%$

High rates: Can be proved to be optimal via Bennett's rule

Back to the Gaussian Channel Setting

We developed two ingredients:

1. **Tree code** transform the problem:
   Noisy channel $\Rightarrow$ Noiseless channel with random delay
2. **Lloyd–Max-based scheme**
   used over the resulting noiseless channel

Separation-based scheme

Encoder:

- Applies Lloyd-Max-based scheme
- Encodes quantized bits using a tree code

Decoder:

- Recovers all coded bits
- If error is detected $\rightarrow$ rerun LM from that point

Joint Source–Channel Coding

Linear Quadratic Gaussian Control over Gaussian Channels

Control rate $\neq$ Communication rate!

- Assume N channel uses per one control sample

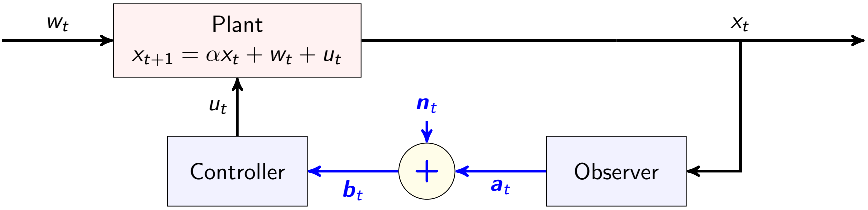
Scalar LQG system

$$x_{t+1} = \alpha x_t + u_t + w_t$$

Scalar AWGN channel

$$\boldsymbol{b}_t = \boldsymbol{a}_t\left(x^t, u^{t-1}\right) + \boldsymbol{n}_t$$

Power constraint: $\mathbb{E}\left[\boldsymbol{a}_t^2\right] \leq NP$



1 : 1 JSCC: Rate-Matched Case

- One AWGN channel use per one control sample

1 : 1 Optimal JSCC [Goblick IT'65]

1 : 1 optimal JSCC distortion = n : n optimal JSCC distortion

- No loss of performance
- **Analog scheme is optimal:** $a_t = \sqrt{\frac{P}{P_x}} x_t$

1 : 1 JSCC: Rate-Matched Case

Scheme

Observer/Transmitter:

- Generates the "source" signal: $s_t = x_t - \hat{x}_{t|t-1} = \tilde{x}_{t|t-1}$
- Adjusts power and transmits: $a_t = s_t / \sqrt{P_{t|t-1}}$

Controller/Receiver:

- Receives $b_t = a_t + n_t = \tilde{x}_{t|t-1} / \sqrt{P_{t|t-1}} + n_t$
- Applies Kalman filtering: $\begin{cases} \hat{x}_{t|t} = \hat{x}_{t|t-1} + \sqrt{P_{t|t-1}} \frac{\mathrm{SNR}}{1+\mathrm{SNR}} b_t \\ \hat{x}_{t|t-1} = \alpha \hat{x}_{t-1|t-1} + u_{t-1} \end{cases}$
- Generates LQG control signal: $u_t = -L_t \hat{x}_{t|t}$

1 : 1 JSCC: Rate-Matched Case

- We reduced the problem to that of classical LQG control

LQR coefficients

$$L_t = \frac{\alpha S_{t+1}}{S_{t+1} + R}\,,$$

$$S_t = \frac{\alpha^2 R\, S_{t+1}}{S_{t+1} + R} + Q\,,$$

$$S_T = F\,.$$

1 : 1 JSCC: Rate-Matched Case

LQG cost

- This schemes achieves optimal LQG cost
- Formally proved by applying
  - Shannon's lower bound
  - Entropy-power inequality
  - Tightness of both in Gaussian case
  - Optimality of "1 : 1 JSCC" scheme in the Gaussian case

  in the dynamic-programming solution
  (extension of [Kostina-Hassibi Allerton'16]
  [Kh. et al. ITW'17 & submitted TCNS'17])
- Recovers results of [Freudenberg-Middleton-Solo AC'10] as a special case

Conclusion: No coding is needed!

1 : 1 JSCC: Rate-Matched Case

Optimal infinite-horizon steady-state average-time LQG cost

$$\bar{J}^{\mathrm{r}} = \bar{J}^{\mathrm{t}} + \frac{Q + \left(\alpha^2 - 1\right) S}{1 + \mathsf{SNR} - \alpha^2} W$$

$$\bar{J}^{\mathrm{t}} = SW$$

- S is the positive solution of the DARE

$$S^2 - \left[Q + \left(\alpha^2 - 1\right) R\right] S - QR = 0$$

- System is stabilizable if and only if $\mathsf{SNR} > \alpha^2 - 1$
- This is in stark contrast to classical LQG

Conclusion: No coding is needed!

1 : 1 JSCC: Rate-Matched Case

Optimal infinite-horizon steady-state average-time LQG cost

$$\bar{J}^{\mathrm{r}} = \bar{J}^{\mathrm{t}} + \frac{Q + \left(\alpha^2 - 1\right) S}{1 + \mathsf{SNR} - \alpha^2} W$$
$$\bar{J}^{\mathrm{t}} = SW$$

- S is the positive solution of the DARE
$$S^2 - \left[Q + \left(\alpha^2 - 1\right) R\right] S - QR = 0$$
- System is stabilizable if and only if $\mathsf{SNR} > \alpha^2 - 1$
- This is in stark contrast to classical LQG

What about 1 : 2 case?

1 : 2 JSCC: Rate-Mismatched Case

- Two AWGN channel uses per one control sample

Naïve scheme: Repetition

Observer/Transmitter: $a_{t;1} = a_{t;2} = \tilde{x}_t / \sqrt{P_{t|t-1}}$

Controller/Receiver: $b_t^{\text{eff}} = \frac{b_{t;1} + b_{t;2}}{2}$

- Reduces to 1 : 1 JSCC with $\text{SNR}^{\text{eff}} = 2\text{SNR}$
- 3dB improvement comes from doubling total transmit power
- Same improvement is attained by
  - Using $2P$ during first channel use
  - Remaining silent during second channel use
- No real improvement due to extra degree of freedom...

1 : 2 JSCC: Rate-Mismatched Case

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Can we do better?

1 : 2 JSCC: Rate-Mismatched Case

Infinite blocklength: "n : 2n JSCC" for $n \to \infty$ [Shannon '48]

$$1 + \mathsf{SNR}^{\mathrm{eff}} = (1 + \mathsf{SNR})^2$$

- Much better than $\mathsf{SNR}^{\mathrm{eff}}_{\text{naïve}} = 2\mathsf{SNR}$ at high SNR

1 : 2 JSCC: Rate-Mismatched Case

Infinite blocklength: "n : 2n JSCC" for $n \to \infty$ [Shannon '48]

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- Much better than $\mathrm{SNR}^{\mathrm{eff}}_{\text{naïve}} = 2\mathrm{SNR}$ at high SNR

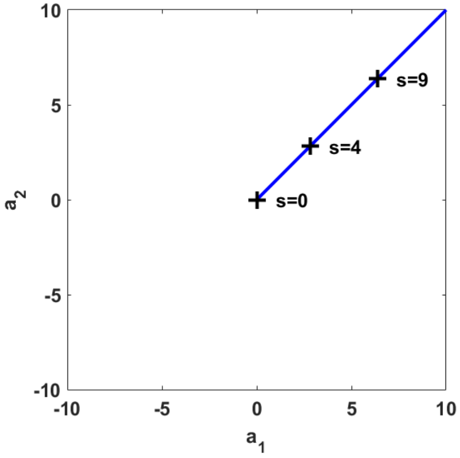
What about 1 : 2 JSCC?

Non-linear mappings can do better! [Kotel'nikov '47][Shannon '49]

1 : 2 JSCC: Rate-Mismatched Case

1 : 2 JSCC: Rate-Mismatched Case

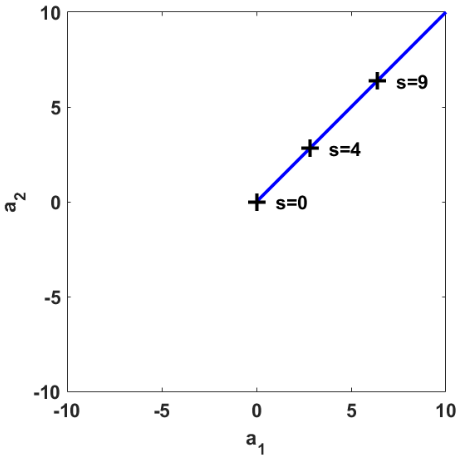
$$\begin{cases} a_1(s) = \frac{1}{\sqrt{2}} s \\ a_2(s) = \frac{1}{\sqrt{2}} s \end{cases}$$

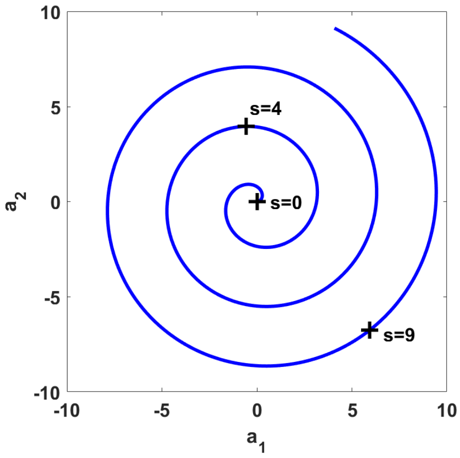


1 : 2 JSCC: Rate-Mismatched Case

$$\begin{cases} a_1(s) = \frac{1}{\sqrt{2}} s \\ a_2(s) = \frac{1}{\sqrt{2}} s \end{cases}$$

$$\begin{cases} a_1(s) = s\cos(2s) \\ a_2(s) = s\sin(2s) \end{cases}$$





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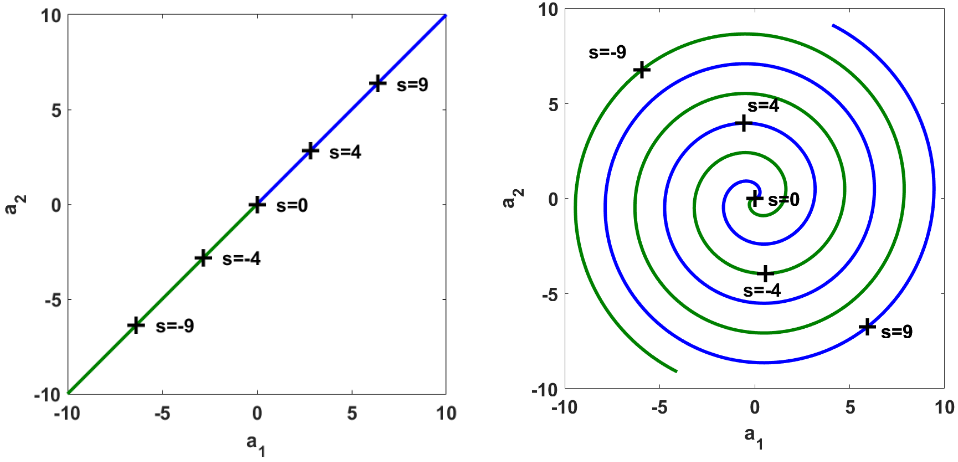
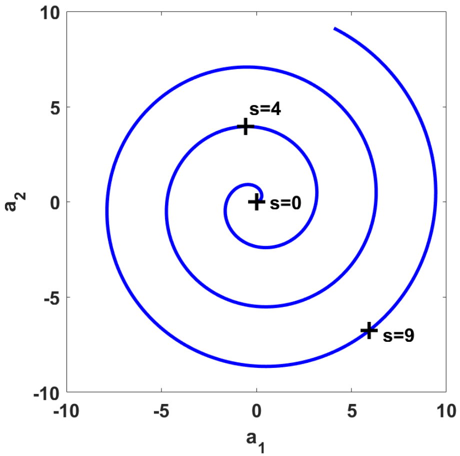
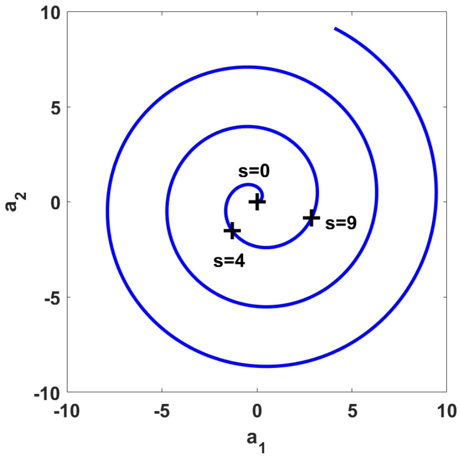


1 : 2 JSCC: Rate-Mismatched Case

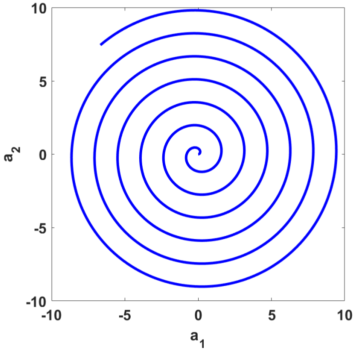
$$\begin{cases} a_1(s) = s\cos(2s) \\ a_2(s) = s\sin(2s) \end{cases}$$

$$\begin{cases} a_1(s) = \sqrt{s}\cos(2\sqrt{s}) \\ a_2(s) = \sqrt{s}\sin(2\sqrt{s}) \end{cases}$$





1 : 2 JSCC: Rate-Mismatched Case

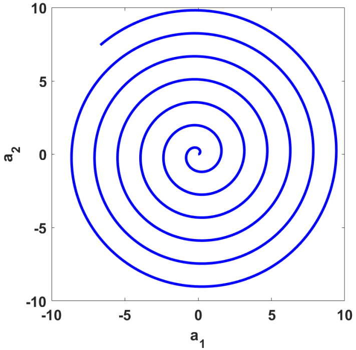


- Small distance between branches
  $\Rightarrow$ better for "weak noise"
- Large distance between branches
  $\Rightarrow$ better for "strong noise"

Standard spiral

$$\begin{cases} a_1(s) \propto s\cos(\omega s) & = |s|\cos(\omega|s|)\,\mathrm{sign}(s) \\ a_2(s) \propto s\sin(\omega s)\,\mathrm{sign}(s) & = |s|\sin(\omega|s|)\,\mathrm{sign}(s) \end{cases}$$

1 : 2 JSCC: Rate-Mismatched Case



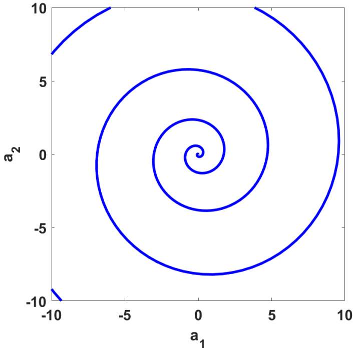
- Small distance between branches
  $\Rightarrow$ better for "weak noise"
- Large distance between branches
  $\Rightarrow$ better for "strong noise"

Stretched-source spiral

Stretch input before mapping to spiral: $s \to |s|^\lambda \operatorname{sign}(s)$

$$\begin{cases} a_1(s) \propto |s|^\lambda \cos\left(\omega |s|^\lambda\right) \operatorname{sign}(s) \\ a_2(s) \propto |s|^\lambda \sin\left(\omega |s|^\lambda\right) \operatorname{sign}(s) \end{cases}$$

1 : 2 JSCC: Rate-Mismatched Case



⟸ Control requirements

- Small distance between branches
  ⇒ better for "weak noise"
- Large distance between branches
  ⇒ better for "strong noise"

Bounded average distortion given any input

Avoid increase in distortion with $|s|$ ⇒ Slower rotation with $|s|$

$$\begin{cases} a_1(s) \propto |s|^{\lambda\beta} \cos\left(\omega|s|^{\lambda}\right) \operatorname{sign}(s) \\ a_2(s) \propto |s|^{\lambda\beta} \sin\left(\omega|s|^{\lambda}\right) \operatorname{sign}(s) \end{cases}$$

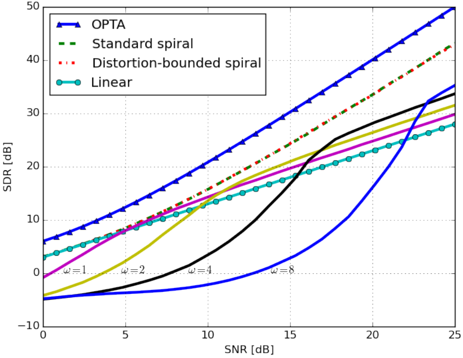
1 : 2 JSCC: Rate-Mismatched Case

⟸ Control requirements

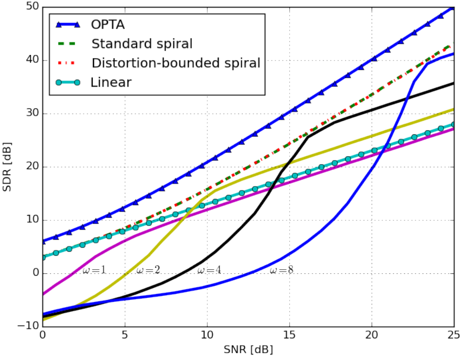
- Small distance between branches
  ⇒ better for "weak noise"
- Large distance between branches
  ⇒ better for "strong noise"

1 : 2 JSCC: Rate-Mismatched Case

- Average distortion given (almost) any s needs to be small!
- E.g., transmitters that truncate the signal do not perform well (avalanche effect)



(a) $\lambda = 1$.



(b) $\lambda = 0.5$.

1 : 2 JSCC: Rate-Mismatched Case

[Kh.-Riedel Gårding-Pettersson-Kostina-Hassibi CDC'16, submitted AC'17]

Inner bound: Black-box approach

Assume a JSCC scheme with bounded distortion $D = \frac{1}{\mathrm{SNR}^{\mathrm{eff}}}, \forall s$.

$$\bar{J}^{\mathrm{r}} \leq \bar{J}^{\mathrm{t}} + \frac{Q + (\alpha^2 - 1)\,S}{1 + \mathrm{SNR}^{\mathrm{eff}} - \alpha^2}(P_t^t - \bar{P}_t^t)$$

- **Improved stabilizability:** $\mathrm{SNR}^{\mathrm{eff}} \geq \alpha^2 - 1$

Outer bound: Extension of [Kostina-Hassibi Allerton'16], [Kh.-Kostina-Khisti-Hassibi ITW'17, submitted TCNS'17]

$$\bar{J}^{\mathrm{r}} \geq \bar{J}^{\mathrm{t}} + \frac{Q + (\alpha^2 - 1)\,S}{1 + \mathrm{SNR}^{\mathrm{eff}}_{n\to\infty} - \alpha^2}(P_t^t - \bar{P}_t^t)$$

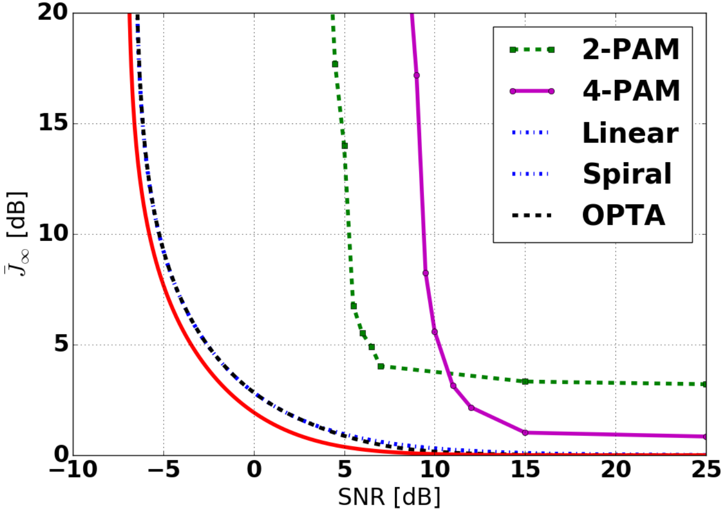
- $1 + \mathrm{SNR}^{\mathrm{eff}}_{n\to\infty} = (1 + \mathrm{SNR})^2$

- Difference between bounds is only due to effective SNR

Performance Comparison

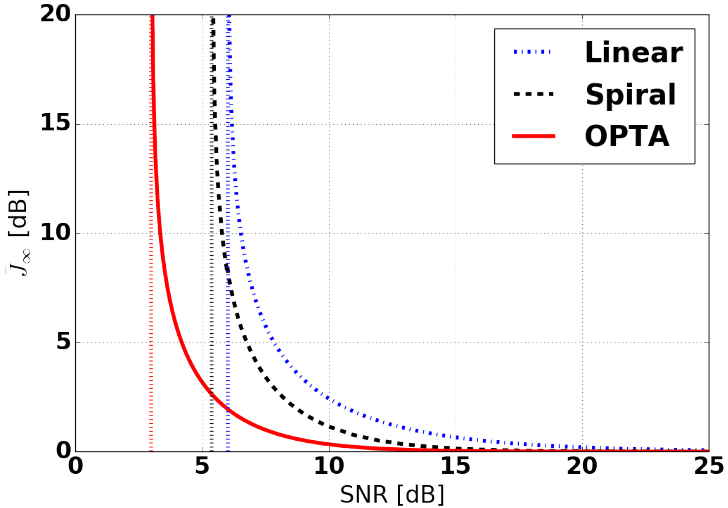
Performance Comparison

- $\alpha = 1.2, W_1 = 1, Q_t \equiv 1, R_t = 0$



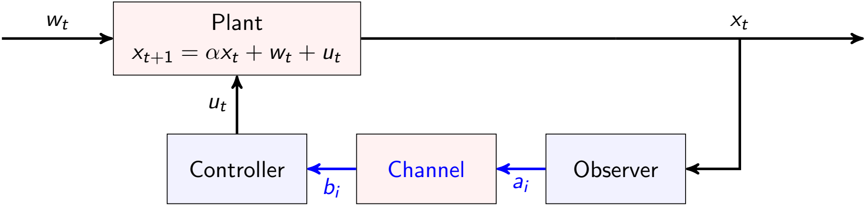
Performance Comparison

- $\alpha = 3, W_1 = 1, Q_t \equiv 1, R_t = 0$



Further Down the Road...

Outlook: Control Loop as Implicit Channel Feedback



- Control signal u_t is assumed known at observer/transmitter
- Can be used as implicit channel feedback
- Could be noisy
- Special instance: JSCC schemes over the AWGN channel

Outlook: Control Loop as Implicit Channel Feedback

Posterior Matching (PM) [Shayevitz-Feder IT'11, IT'16] [Li-El Gamal IT'15][Naghshvar-Javidi-Wigger IT'15], ...

- Fits an initial state (LQR) setting: $\begin{cases} x_{t+1} = \alpha x_t + u_t + \cancel{w_t} \\ x_0 \sim \text{random} \end{cases}$
- Assumes perfect instantaneous feedback is available
- Upon receiving channel output b_t recalculates posterior:

$$\Theta_0 = F_{X_0}(X_0), \quad A_1 = F_A^{-1}(\Theta_0)$$
$$A_{t+1} = F_A^{-1} \circ F_{\Theta_0|B^t}(\Theta_0|b^t)$$

- Transmitter knows b^t via feedback
- Can be calculated iteratively:

$$A_1 = F_A^{-1} \circ F_{X_0}(X_0)$$
$$A_{t+1} = F_A^{-1} \circ F_{A|B}(A_t|b_t)$$

Outlook: Control Loop as Implicit Channel Feedback

Posterior Matching (PM) [Shayevitz-Feder IT'11, IT'16] [Li-El Gamal IT'15][Naghshvar-Javidi-Wigger IT'15], ...

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$$A_1 = F_A^{-1} \circ F_{X_0}(X_0)$$
$$A_{t+1} = F_A^{-1} \circ F_{A|B}(A_t|b_t)$$

Problems

1. What to do for i.i.d. w_t? $\Rightarrow$ Causal variant of PM is needed!
2. We are interested in control-theoretic notions, say LQG cost

Outlook: Control Loop as Implicit Channel Feedback

BSC: Horstein's scheme [Horstein IT'63]

- Special case of PM scheme over the BSC
- At every step:
  - Calculates posterior
  - Sends whether the posterior of $X_0 \lessgtr$ Median

- Not bad for first moment minimization $\sum_{t=1}^{T} \mathbb{E}\left[\left|X_0 - \hat{X}_0(t)\right|\right]$
- Analysis (not tight!) in [Waeber-Frazier-Henderson SICON'13]
- What about LQG cost, say $\sum_{t=1}^{T} \mathbb{E}\left[\left|X_0 - \hat{X}_0(t)\right|^2\right]$?

Idea: Extend PM scheme to a wider class

- Compare to MMSE instead of median

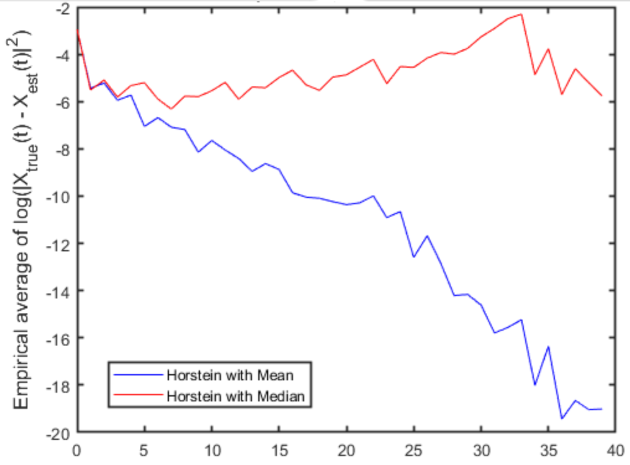
Performance of the Median- and MMSE-based Schemes

Plant

- $x_t = \alpha x_t + u_t + \cancel{w_t}$
- $\alpha = 1.2, x_0 \sim \mathcal{Unif}(0,1)$

Channel

- Binary Symmetric Channel
- Crossover probability 0.025



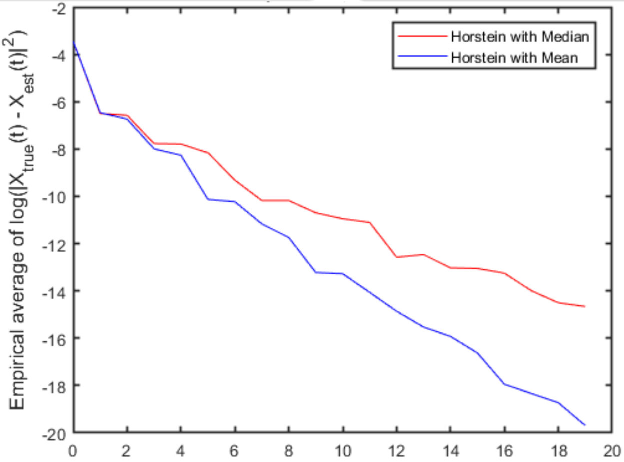
Performance of the Median- and MMSE-based Schemes

Plant

- $x_t = \alpha x_t + u_t + \cancel{w_t}$
- $\alpha = 1.01, x_0 \sim \mathcal{U}nif(0,1)$

Channel

- Binary Symmetric Channel
- Crossover probability 0.025



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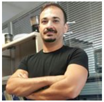

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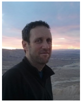

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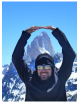

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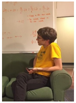

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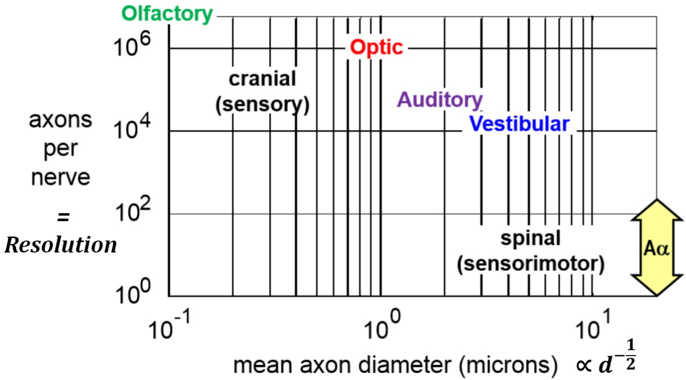
No relationships were ruined in the making of this presentation...

Backup Slides

Neuroscience: Micro-level [Perge et al. J. Neuroscience'12]

- Cranial and spinal nerves = bundles of fibers (=axons)
- Nerves connect collection of Neurons over long distances
- Have (roughly) the same cross sectional area



[Nakahira et al. CDC'15]

Ensemble Performance ⇒ Specific Code Performance?

Ensemble performance

$$\mathbb{E}\left[P_e(t,d)\right] \le 2^{-E_G(R)Nd}$$

Specific d and t

Using Markov's inequality:

$$\Pr\left(P_e(t,d) \ge 2^{-[E_G(R)-\epsilon]Nd}\right) \le \frac{\mathbb{E}\left[P_e(t,d)\right]}{2^{-[E_G(R)-\epsilon]Nd}} = 2^{-\epsilon Nd}$$

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All t and $d_0 < d \leq t$

Using the union bound:

$$\Pr\left(\bigcup_{t=1}^{\infty}\bigcup_{d=d_0}^{t} P_e(t,d) \geq 2^{-[E_G(R)-\epsilon]Nd}\right)$$

$$\leq \sum_{t=1}^{\infty}\sum_{d=d_0}^{t} \underbrace{\Pr\left(P_e(t,d) \geq 2^{-[E_G(R)-\epsilon]Nd}\right)}_{\leq 2^{-\epsilon Nd}} \leq \sum_{t=1}^{\infty} \text{const} \to \infty$$

Ensemble Performance ⇒ Specific Code Performance?

Ensemble performance

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Ensemble Performance ⇒ Specific Code Performance?

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$$\Pr\left(\bigcup_{d=d_0}^{\infty} P_e(d) \ge 2^{-[E_G(R)-\epsilon]Nd}\right) \le \sum_{d=d_0}^{\infty} 2^{-\epsilon Nd} = \frac{2^{-\epsilon Nd_0}}{1-2^{-\epsilon N}}$$

Analog Codes / JSCC: Further Results and Comments

- Inner bound can be improved: Optimization over curves, e.g. [Akyol-Vishwanatha-Rose-Ramstad IT'14]
- Outer bound for low-delay JSCC can be improved [Ziv-Zakai IT'73]
- High dimensional curves
- Other low-delay JSCC techniques: e.g., repetitive quantization [Kleiner-Rimoldi GLOBECOM'09]
  - Easy to generalize to higher dimensions
- Vector $\boldsymbol{x}$, vector $\boldsymbol{u}$, scalar y: *Simple extension of scalar setting!*
- Rate-matched case with vector $\boldsymbol{y}$: "n : 1 JSCC" is needed
  - Switch roles between Transmitter and Receiver
  - Improves over [Freudenberg-Middleton-Solo AC'10]