The Dirty MIMO Multiple-Access Channel

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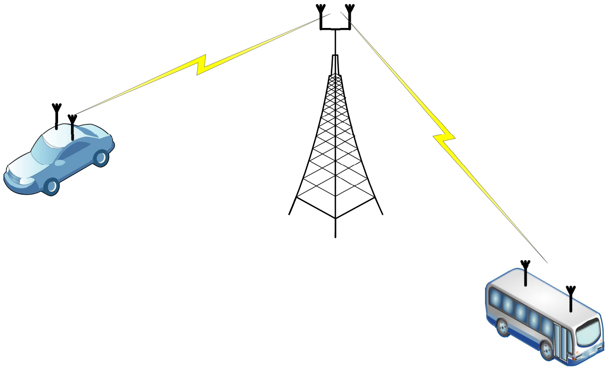
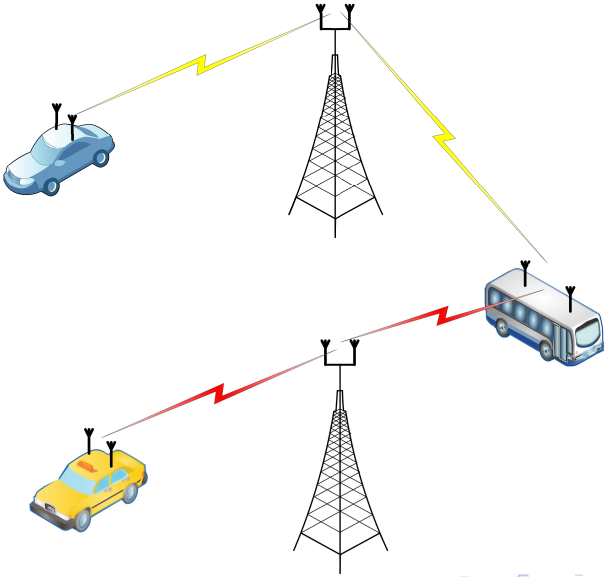
Motivation: MIMO Dirty-Paper Channel

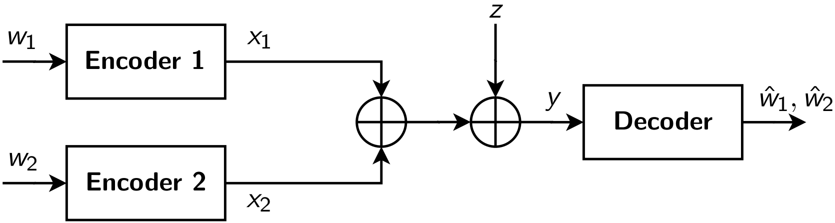


Motivation: MIMO Dirty Multiple-Access Channel



Multiple-Access Channel (MAC) Scenarios

Single-Input Single-Output (SISO) MAC



$$y = x_1 + x_2 + z$$

- x_i – Scalar input of Encoder i of power P_i
- y – Scalar output
- z – White Gaussian noise $\sim \mathcal{N}(0, 1)$

SISO Multiple-Access Channel

Capacity region [Ahleswede'71][Liao'72]

$$R_1 \le \frac{1}{2}\log\left(1 + P_1\right)$$
$$R_2 \le \frac{1}{2}\log\left(1 + P_2\right)$$
$$R_1 + R_2 \le \frac{1}{2}\log\left(1 + P_1 + P_2\right)$$

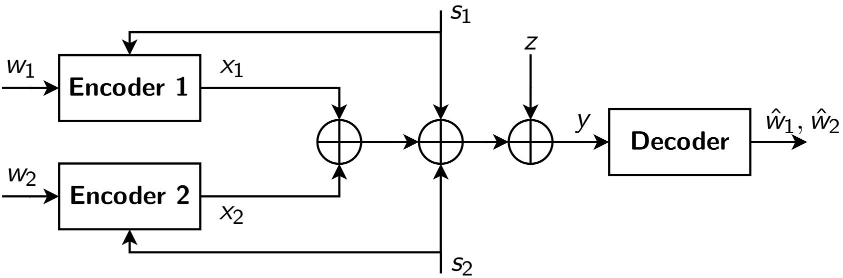
Capacity region at high SNRs

$$R_1 \lesssim \frac{1}{2}\log\left(P_1\right)$$
$$R_2 \lesssim \frac{1}{2}\log\left(P_2\right)$$
$$R_1 + R_2 \lesssim \frac{1}{2}\log\left(P_1 + P_2\right)$$

- Sum-capacity strictly greater than individual capacities

The Dirty MAC

SISO Dirty Multiple-Access Channel



$$y = x_1 + x_2 + s_1 + s_2 + z$$

- x_i – Scalar input of Encoder i of power P_i
- s_i – Arbitrary side-information sequence known to Encoder i
- y – Scalar output
- z – White Gaussian noise $\sim \mathcal{N}(0, 1)$

SISO Dirty Multiple-Access Channel

- Random binning schemes are bad

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  - Costa-like auxiliaries $U_i = X_i + \alpha S_i$ achieve *zero rate*! ☹

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Achievable rate region using lattices [Philosof et al. IT'11]

$$R_1 + R_2 \le \frac{1}{2} \log \left(\frac{1}{2} + \min\{P_1, P_2\} \right)$$

Outer region [Philosof et al. IT'11]

$$R_1 + R_2 \le \frac{1}{2} \log \left(1 + \min\{P_1, P_2\} \right)$$

DMAC capacity region at high SNR

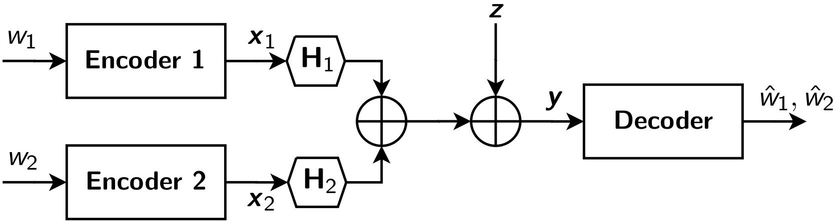
$$R_1 + R_2 \lesssim \frac{1}{2} \log \left(\min\{P_1, P_2\} \right)$$

MAC sum-capacity at high SNR

$$R_1 + R_2 \lesssim \frac{1}{2} \log \left(P_1 + P_2 \right)$$

What about Multiple-Input Multiple-Output (MIMO)?

MIMO MAC



$$\boldsymbol{y} = \mathbf{H}_1\boldsymbol{x}_1 + \mathbf{H}_2\boldsymbol{x}_2 + \boldsymbol{z}$$

- $\boldsymbol{x}_i$ – $N \times 1$ input vector of Encoder i of power P_i
- $\mathbf{H}_i$ – $N \times N$ unit-determinant channel matrix of Encoder i
- $\boldsymbol{y}$ – $N \times 1$ output vector
- $\boldsymbol{z}$ – $N \times 1$ white Gaussian noise $\sim \mathcal{N}(\mathbf{0}, \mathbf{I}_N)$

MIMO MAC

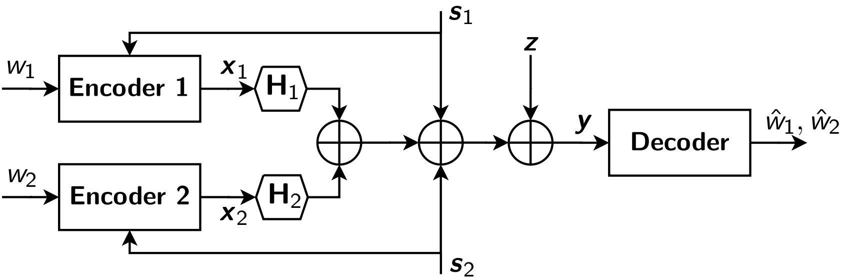
Capacity region [Yu PhD'02]

$$\bigcup_{\substack{\mathrm{tr}\{\boldsymbol{C_{x_i}}\} \leq P_i \\ \boldsymbol{C_{x_i}} \succeq 0}} (R_1, R_2): \quad \begin{aligned} R_1 &\leq \frac{1}{2} \log \left| \mathbf{I} + \mathbf{H}_1^T \boldsymbol{C_{x_1}} \mathbf{H}_1^T \right| \\ R_2 &\leq \frac{1}{2} \log \left| \mathbf{I} + \mathbf{H}_2^T \boldsymbol{C_{x_2}} \mathbf{H}_2^T \right| \\ R_1 + R_2 &\leq \frac{1}{2} \log \left| \mathbf{I} + \mathbf{H}_1^T \boldsymbol{C_{x_1}} \mathbf{H}_1^T + \mathbf{H}_2^T \boldsymbol{C_{x_2}} \mathbf{H}_2^T \right| \end{aligned}$$

MIMO MAC

- Capacity is achieved using random coding
- Practical schemes based on matrix decompositions can be used
  - SVD, QR decomposition, GMD,...
  - Reduces MIMO coding task to coding over parallel SISO links
  - Use good SISO codes to approach capacity

Dirty MIMO MAC



$$\mathbf{y} = \mathbf{H}_1\mathbf{x}_1 + \mathbf{H}_2\mathbf{x}_2 + \mathbf{s}_1 + \mathbf{s}_2 + \mathbf{z}$$

- $\mathbf{x}_i$ – $N \times 1$ input vector of Encoder i of power P_i
- $\mathbf{H}_i$ – $N \times N$ unit-determinant channel matrix of Encoder i
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- $\mathbf{y}$ – output of length N
- $\mathbf{z}$ – white Gaussian noise $\sim \mathcal{N}(\mathbf{0}, \mathbf{I}_N)$

Dirty MIMO MAC

Outer bound

- In SISO case: Sum-capacity $\leq$ Individual capacities

$$R_1 + R_2 \leq \min\left\{\frac{1}{2}\log\left(1+P_1\right), \frac{1}{2}\log\left(1+P_2\right)\right\}$$

- Straightforward adaptation for the MIMO case:

$$R_1 + R_2 \leq \min\left\{\begin{array}{l}\max\limits_{C_{x_1}:\mathrm{tr}\left(C_{x_1}\right)\leq P_1} \frac{1}{2}\log\left|\mathbf{I}+\mathbf{H}_1 C_{x_1}\mathbf{H}_1^T\right|, \\ \max\limits_{C_{x_1}:\mathrm{tr}\left(C_{x_1}\right)\leq P_1} \frac{1}{2}\log\left|\mathbf{I}+\mathbf{H}_2 C_{x_2}\mathbf{H}_2^T\right|\end{array}\right\}$$

- **High SNR:** White inputs become optimal

$$R_1 + R_2 \lesssim \frac{1}{2}\log\min\left\{\left|\frac{P_1}{N}\mathbf{H}_1\mathbf{H}_1^T\right|, \left|\frac{P_2}{N}\mathbf{H}_2\mathbf{H}_2^T\right|\right\}$$

$$= \frac{N}{2}\log\left(\frac{\min\{P_1, P_2\}}{N}\right)$$

Dirty MIMO MAC

- How to approach the outer bound at high SNRs?
- Cannot use random coding/binning
  - Achieves zero rate even in SISO case...
- Need to incorporate structure

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Idea

Use matrix decompositions + SISO lattice-based scheme

Singular Value Decomposition (SVD) Based Scheme?

$$\mathbf{H}_1 = \mathbf{Q}_1 \mathbf{D}_1 \mathbf{V}_1^T$$
$$\mathbf{H}_2 = \mathbf{Q}_2 \mathbf{D}_2 \mathbf{V}_2^T$$

- $\mathbf{Q}_i, \mathbf{V}_i$ – Orthogonal matrices
- $\mathbf{D}_i$ – Diagonal matrices

Problem 1: **Common** left operation at decoder

- Decoder matrix $\mathbf{Q}_i$ depends on channel matrix $\mathbf{H}_i$
- But $\mathbf{Q}$ is shared by both users:

$$\mathbf{Q}^T \boldsymbol{y} = \underbrace{\mathbf{Q}^T \mathbf{H}_1 \mathbf{V}_1}_{\stackrel{?}{=} D_1} \boldsymbol{x}_1 + \underbrace{\mathbf{Q}^T \mathbf{H}_2 \mathbf{V}_2}_{\stackrel{?}{=} D_2} \boldsymbol{x}_2 + \mathbf{Q}\boldsymbol{s}_1 + \mathbf{Q}\boldsymbol{s}_2 + \mathbf{Q}\boldsymbol{z}$$

Problem 2: Rate/SNR bottleneck problem

Even if $\mathbf{H}_i$ are diagonal $\Rightarrow$ Rate limited by minimal diagonal value

Bottleneck Problem: Example

$$P_1 = 60,\ \mathbf{H}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P_2 = 60,\ \mathbf{H}_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$$

Dirty MIMO point-to-point setting

$$R_1 \approx \frac{1}{2}\log\left(30 \times 1/2^2\right) \approx 1.45$$
$$R_2 \approx \frac{1}{2}\log\left(30 \times 2^2\right) \approx 3.45$$
$$R_{\text{tot}} = R_1 + R_2 \approx 4.90$$

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Dirty MIMO MAC setting

$$R_1 \approx \frac{1}{2}\log\left(\min\left\{3.529 \times 2^2, 56.471 \times 1/2^2\right\}\right) \approx 1.91$$
$$R_2 \approx \frac{1}{2}\log\left(\min\left\{56.471 \times 1/2^2, 3.529 \times 2^2\right\}\right) \approx 1.91$$
$$R_{\text{tot}} = R_1 + R_2 \approx 2 \times 1.91 = 3.82 < 4.90$$

Resolving Problem 1 (Common Left Operation)

✗ Joint diagonalization is not possible in general...

LQ decomposition (QR decomposition transposed)

$$\mathbf{H}_1 = \mathbf{T}_1 \mathbf{V}_1^T$$
$$\mathbf{H}_2 = \mathbf{T}_2 \mathbf{V}_2^T$$

- $\mathbf{T}_i$ – Triangular matrices
- $\mathbf{V}_i$ – Orthogonal matrices
- Off-diagonal values treated as part of the side information

✓ Always possible

✓ No left matrix is needed $\Rightarrow$ **Problem 1 solved!**

✗ But... $\text{diag}(\mathbf{T}_1) \neq \text{diag}(\mathbf{T}_2) \Rightarrow$ **Bottleneck problem!**

- Can we make the diagonals equal?

Resolving Problem 1 (Common Left Operation)

✗ Joint diagonalization is not possible in general...

LQ decomposition (QR decomposition transposed)

$$\mathbf{H}_1 = \mathbf{I}_N \mathbf{T}_1 \mathbf{V}_1^T$$
$$\mathbf{H}_2 = \mathbf{I}_N \mathbf{T}_2 \mathbf{V}_2^T$$

- $\mathbf{T}_i$ – Triangular matrices
- $\mathbf{V}_i$ – Orthogonal matrices
- Off-diagonal values treated as part of the side information

✓ Always possible

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Resolving the Bottleneck Problem

- LQ decomposition did not use any left orthogonal matrix
- Use left orthogonal matrix to make diagonals equal!

Joint Equi-diagonal Triangularization (JET) [Kh.-Kochman-Erez SP'12]

- $|\mathbf{H}_1| = |\mathbf{H}_2|$

$$\mathbf{H}_1 = \mathbf{Q}\mathbf{T}_1\mathbf{V}_1^T$$

$$\mathbf{H}_2 = \mathbf{Q}\mathbf{T}_2\mathbf{V}_2^T$$

- $\mathbf{Q}, \mathbf{V}_1, \mathbf{V}_2$ – Orthogonal matrices
- $\mathbf{T}_1, \mathbf{T}_2$ – Triangular matrices
- $\mathrm{diag}(\mathbf{T}_1) = \mathrm{diag}(\mathbf{T}_2)$

Resolving the Bottleneck Problem: Example

$$\mathbf{H}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} = \overbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}^{\mathbf{Q}} \overbrace{\begin{pmatrix} {\color{blue}1.458} & 0 \\ -1.286 & {\color{red}0.686} \end{pmatrix}}^{\mathbf{T}_1} \overbrace{\begin{pmatrix} 0.243 & 0.970 \\ -0.970 & 0.243 \end{pmatrix}}^{\mathbf{V}_1^T}$$

$$\mathbf{H}_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} = \overbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}^{\mathbf{Q}} \overbrace{\begin{pmatrix} {\color{blue}1.458} & 0 \\ 1.286 & {\color{red}0.686} \end{pmatrix}}^{\mathbf{T}_2} \overbrace{\begin{pmatrix} 0.243 & 0.970 \\ -0.970 & 0.243 \end{pmatrix}}^{\mathbf{V}_2^T}$$

Achievable Region

- Apply the JET to the channel matrices: $(t_1, \ldots, t_N) \triangleq \text{diag}(T_1) = \text{diag}(T_2)$

Achievable rate region

All (R_1, R_2) satisfying: $R_1 + R_2 \le \frac{N}{2} \log\left(\frac{\min\{P_1, P_2\}}{N}\right)$

Proof

$$
\begin{aligned}
R_1 + R_2 &= \sum_{j=1}^{N} (r_{1;j} + r_{2;j}) \\
&\ge \sum_{j=1}^{N} \frac{1}{2} \log\left(\frac{\min\left\{P_1 t_j^2, P_2 t_j^2\right\}}{N}\right) \\
&= \frac{N}{2} \log\left(\frac{\min\{P_1, P_2\}}{N}\right) + \log \underbrace{\prod_{j=1}^{N} t_j}_{=|H_i|=1}
\end{aligned}
$$

Achievable Region

- Apply the JET to the channel matrices: $(t_1, \ldots, t_N) \triangleq \text{diag}(T_1) = \text{diag}(T_2)$

Achievable rate region

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Outer bound at high SNRs

$$R_1 + R_2 \lesssim \frac{N}{2} \log\left(\frac{\min\{P_1, P_2\}}{N}\right)$$

Capacity region at high SNRs

All (R_1, R_2) satisfying: $R_1 + R_2 \lesssim \frac{N}{2} \log\left(\frac{\min\{P_1, P_2\}}{N}\right)$

K-user Dirty MAC

SISO capacity at high SNRs [Philosof et al. IT'11]

$$\sum_{i=1}^{K} R_i \lesssim \frac{1}{2} \log \left(\min_{i=1,\dots,K} P_i \right)$$

MIMO outer bound at high SNRs

Again, sum-capacity $\leq$ individual capacities

$$\sum_{i=1}^{K} R_i \lesssim \frac{N}{2} \log \left(\frac{\min_{i=1,\dots,K} P_i}{N} \right)$$

Achievable rate region

Problem: JET not possible for $K > 2$ matrices, in general

K-user JET via Space–Time Coding [Kh.-Livni-Hitron-Erez IT'15]

$$\mathbf{H}_i = \mathbf{Q}\mathbf{T}_i\mathbf{V}_i^\dagger \qquad \text{X}$$

- **Bunch two channel uses together:**

$$\overbrace{\begin{pmatrix}\mathbf{H}_i & \mathbf{0}\\ \mathbf{0} & \mathbf{H}_i\end{pmatrix}}^{\mathcal{H}_i} = \overbrace{\begin{pmatrix}\mathbf{Q} & \mathbf{0}\\ \mathbf{0} & \mathbf{Q}\end{pmatrix}}^{\mathcal{Q}} \overbrace{\begin{pmatrix}\mathbf{T}_i & \mathbf{0}\\ \mathbf{0} & \mathbf{T}_i\end{pmatrix}}^{\mathcal{T}_i} \overbrace{\begin{pmatrix}\mathbf{V}_i^\dagger & \mathbf{0}\\ \mathbf{0} & \mathbf{V}_i^\dagger\end{pmatrix}}^{\mathcal{V}_i} \qquad \text{X}$$

- $\mathcal{H}_i$ have a block-diagonal structure
- Use general $\mathcal{Q}$, $\mathcal{V}_i$ (*not* block-diagonal):

$$\overbrace{\begin{pmatrix}\mathbf{H}_i & \mathbf{0}\\ \mathbf{0} & \mathbf{H}_i\end{pmatrix}}^{\mathcal{H}_i} = \left(\mathcal{Q}\right)\left(\mathcal{T}_i\right)\left(\mathcal{V}_i\right)^\dagger \qquad \checkmark$$

- Exploiting off-diagonal $\mathbf{0}$s enables JET of **more users!**

X Edge effect: Fixed number of unbalanced parallel channels

✓ Negligible by processing together large number of channel uses

K-user Dirty MIMO MAC

Achievable rate region

All $(R_1, \ldots, R_K)$ satisfying: $\sum_{i=1}^{K} R_i \leq \frac{N}{2} \log \left(\frac{\min_{i=1,\ldots,K} P_i}{N} \right)$

K-user Dirty MIMO MAC

Achievable rate region

All $(R_1, \ldots, R_K)$ satisfying: $\sum_{i=1}^{K} R_i \leq \frac{N}{2} \log \left(\frac{\min_{i=1,\ldots,K} P_i}{N} \right)$

Outer bound at high SNRs

$$\sum_{i=1}^{K} R_i \lesssim \frac{N}{2} \log \left(\frac{\min_{i=1,\ldots,K} P_i}{N} \right)$$

Capacity region at high SNRs

All (R_1, R_2) satisfying: $\sum_{i=1}^{K} R_i \lesssim \frac{N}{2} \log \left(\frac{\min_{i=1,\ldots,K} P_i}{N} \right)$

General Matrices

- Thus far, we considered $N \times N$ full-rank matrices
- What about the more general case?

Case 1: Interference and signal "live" in same subspace

$$\boldsymbol{y} = \mathbf{H}_1 (\boldsymbol{x}_1 + \boldsymbol{s}_1) + \mathbf{H}_2 (\boldsymbol{x}_2 + \boldsymbol{s}_2) + \boldsymbol{z}$$

- The case of the two base-station motivating example
- Simple extension of the discussed solution

Case 2: Signal limited to subspace; interference is not

$$\boldsymbol{y} = \mathbf{H}_1 \boldsymbol{x}_1 + \mathbf{H}_2 \boldsymbol{x}_2 + \boldsymbol{s}_1 + \boldsymbol{s}_2 + \boldsymbol{z}$$

- Constructing a tight outer bound is more challenging
- Sum-rate bounding by the individual capacities seems loose

General Matrices

Example

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_1 = 100, \qquad \mathbf{H}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = 50^2$$

$$C_1 \approx 2 \times \frac{1}{2} \log\left(\frac{100}{2}\right) = 2 \times \frac{1}{2} \log(50) \qquad C_2 \approx \frac{1}{2} \log(50^2) \approx C_1$$

- Current outer bound: $R_1 + R_2 \leq \min\{C_1, C_2\} \approx 2 \times \frac{1}{2} \log(50)$
- Clearly not achievable...

Proposed inner bound

1. Project onto common subspace
2. Apply previous result

In example: Projection $y_1 + y_2$ achieves $R_1 + R_2 \approx \frac{1}{2} \log(2 \times 50)$