Incremental Coding over MIMO Channels

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Two-user Gaussian MIMO Broadcast — Channel Model

\[ y_i = H_i x + z_i, \quad i = 1, 2 \]

- \( x \) - \( N \times 1 \) Input vector of power \( P \).
- \( H_i \) - \( N \times N \) Channel matrix to user \( i \)
  (works for non-square matrices as well).
- \( y_i \) - \( N \times 1 \) Output vector of user \( i \).
- \( z_i \) - Channel noise \( \sim \mathcal{CN}(0, I_N) \).
- “Closed loop” (full channel knowledge everywhere).
Two-user Gaussian MIMO Broadcast — Goal

**Rateless setting**

- Transmit the same (common) message to both users.
- Each user aims to decode message after minimal number of channel uses.
- $k$ information bits ("message").
- $n_i$ channel uses needed by user $i$ to decode message.
- Effective rates: $R_i \triangleq \left\lceil \frac{k}{n_i} \right\rceil$.

**Goal**

Approach achievable region of all $(R_1, R_2)$ with **practical scheme**.
Achievable rate region [Shulman, Ph.D., 2004]

\((R_1, R_2)\) is achievable iff there exists a covariance matrix \(K\) with 
\[
\text{trace}\{K\} \leq P \quad \text{s.t.}: \\
\frac{R(H_i, K)}{R_i} + C_i \left[ \frac{1}{R_i} - \frac{1}{R_j} \right]^+ \geq 1, \quad i = 1, 2, \\
\]

where

\[
R(H, K) \triangleq \log \left| I_N + HKH^\dagger \right|
\]

and \(C_i\) is the point-to-point capacity of user \(i\).
Interpretation

1. Assume $R(H_1, K) \geq R(H_2, K)$ ($\Rightarrow n_1 < n_2$)
2. Use $K$ in first $n_1 = k/R(H_1, K)$ channel uses.
3. After $n_1$ channel uses, user 1 is able to decode.
4. Use optimal covariance matrix $K_2$ for user 2.
   $\Rightarrow$ User 2 needs additional
   
   \[ n_2 - n_1 = \frac{k - n_1 R(H_2, K)}{C_2} \]

   channel uses to decode.

Problem

How to achieve this using practical codes?
Different schemes were proposed for the SISO ("scalar") case:

- Superposition of good scalar AWGN codes with appropriate power allocations [Erez, Trott, Wornell ’06].

- Punctured Turbo and LDPC codes [Rowitch & Milstein ’00; Ha et al. ’06].

- Raptor-based codes [Etesami & Shokrollahi ’06].

What to do for MIMO?

Can we reduce MIMO to SISO?
How to Extend to MIMO?

- Simultaneous diagonalization of both channel matrices is not possible in general.

- Even if both channels were diagonal, using scalar rateless codes over these channels would not work.

Example

\[ H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

- Different number of channel uses over each SISO sub-channel needs to be used.

⇒ Not possible to implement simultaneously for all sub-channels...
Assume, for simplicity, integer ratios: \( n_2 = m \cdot n_1 \).

Look at the \( n_2 \) channel uses as \( n_1 \) uses of the equivalent \( mN \times mN \) “augmented” channels (realized by interleaving):

\[
H_1 = \begin{bmatrix}
    H_1 & 0 & \cdots & 0 \\
    0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 0
\end{bmatrix},
\quad H_2 = \begin{bmatrix}
    H_2 & 0 & \cdots & 0 \\
    0 & H_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & H_2
\end{bmatrix},
\]

Effective covariance matrix:

\[
K = \begin{bmatrix}
    K & 0 & \cdots & 0 \\
    0 & K_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & K_2
\end{bmatrix}.
\]

For the effective matrices: \( R(H_1, K) = R(H_2, K) \).
Equivalent “Augmented” Channel Representation

- \( R(\mathcal{H}_1, \mathcal{K}) = R(\mathcal{H}_2, \mathcal{K}) \Rightarrow \) “Classical” multicasting (common-message BC) problem

- Any “good” multicasting scheme can be used over the effective channels.

How to construct a good multicasting scheme?
MIMO Multicasting Scheme

Practical schemes:
- Linear pre- and post-coding [Gohary, Davidson, Luo 2003].

Practical and capacity-achieving scheme:
- Optimal beamforming + joint unitary triangularization [Khina, Kochman, Erez 2010].
For now, assume:

- $H_1$ and $H_2$ are $N \times N$ full rank.
- $\det(H_1) = \det(H_2)$.
- High SNR.

Joint decompositions (same unitary matrix on the right)

- Joint unitary diagonalization not possible.
- Joint unitary triangularization is always possible.
- Diagonals of triangular matrices correspond to gains of sub-channels.
- We want diagonals of triangular matrices to be equal.
MIMO Multicasting Scheme

Joint equi-diagonal triangularization [Khina, Kochman, Erez 2010]

\[ H_1 = U_1 T_1 V^\dagger, \]
\[ H_2 = U_2 T_2 V^\dagger, \]

such that
- \( U_1, U_2, V \) are unitary.
- \( T_1, T_2 \) are triangular with equal diagonals.

MIMO multicast scheme

- Apply \( V \) at Tx and \( U_i \) and Rx-\( i \) ⇒ Effective channels \( T_i \)
  ⇒ \( n \) parallel sub-channels of equal gains \( \text{diag}(T_1) = \text{diag}(T_2) \).
- Use good AWGN SISO codebooks over sub-channels.
- Decode using successive interference cancellation (as in GDFE/V-BLAST).
**Problem**

In our case the channel matrices are:

$$
\mathcal{H}_1 = \begin{bmatrix}
H_1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}, \quad \mathcal{H}_2 = \begin{bmatrix}
H_2 & 0 & \cdots & 0 \\
0 & H_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & H_2
\end{bmatrix},
$$

⇒ full rank and high SNR assumptions do not hold!

**Optimal scheme for general SNR and any channel matrices**

Decompose

$$A_i \triangleq \begin{pmatrix}
H_i K^{1/2} \\
I_N
\end{pmatrix}$$

- $K$ – Optimal covariance matrix.
- Diagonals of triangular matrices $T_i$ correspond to SINRs.
Application to SISO

\[ y = \alpha x + z , \]

where \( \alpha \in \{\alpha_1, \alpha_2\} \) corresponds to SNR.

- For example, assume \( n_2 = 2 \cdot n_1 \).
  \[
  \Rightarrow \log \left( 1 + |\alpha_1|^2 \right) = 2 \log \left( 1 + |\alpha_2|^2 \right).
  \]

- Effective channel matrices:
  \[
  H_1 = \begin{bmatrix}
  \alpha_1 & 0 \\
  0 & 0
  \end{bmatrix}, \quad H_2 = \begin{bmatrix}
  \alpha_2 & 0 \\
  0 & \alpha_2
  \end{bmatrix}.
  \]

- Resulting scheme coincides with SISO rateless scheme of [Erez, Trott, Wornell 2006].

- Can be applied for any two SNRs using an appropriate number of non-zero diagonal elements.
**Half-duplex Relay**

- **Half-duplex**: Relay can receive or transmit but not both.

**Rateless relay**

- Implementation of Decode-and-Forward [Cover, El Gamal ’79].
- Proposed first for “open-loop” [Mitran, Ochiai, Tarokh ’05].
- Relay decodes the message first, before Rx does.
- Relay then transmits coherently with Tx until Rx decodes too.
Half-duplex Relay

- $n_2$ – Whole transmission duration until Rx is able to decode.
- $n_1$ – Transmission duration until relay is able to decode.
- Assume again for simplicity: $n_2 = m \cdot n_1$

**Equivalent matrix representation for SISO case:**

$$H_1 = \begin{bmatrix} \sqrt{P_1 h_{t,rel}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} \sqrt{P_1 h_{t,r}} & 0 & \cdots & 0 \\ 0 & \sqrt{P_2 h_{t,r}} + \sqrt{P_{rel}} h_{rel,r} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{P_2 h_{t,r}} + \sqrt{P_{rel}} h_{rel,r} \end{bmatrix}$$
MIMO rateless coding technique is useful even for SISO relay.

For MIMO case, replace diagonal elements in $\mathcal{H}_i$ with corresponding channel matrices.

Use additional unitary matrix at relay to achieve coherence.

Apply MIMO rateless coding technique.
Summary

- Technique for constructing practical rateless codes for SISO and MIMO channels.
- Uses “off-the-shelf” scalar AWGN codes, linear transformations and successive decoding.
- Allows designing practical transmission schemes for the half-duplex relay channel.
- Special case of “Network Modulation”:
  - Joint decomposition of channel matrices in MIMO network problems.
- Can be generalized for more than 2 users [Khina, Hitron, Erez ’2011].
Supplementary

\[ \begin{align*}
1 & = 1 + n_1 \\
2 & = 2 + n_1 \\
3 & = 3n_1 \\
1 + (m - 1)n_1 & = 2 + (m - 1)n_1
\end{align*} \]

\[ n_2 = mn_1 \]