Simultaneous SDR Optimality via a Joint Matrix Decomposition

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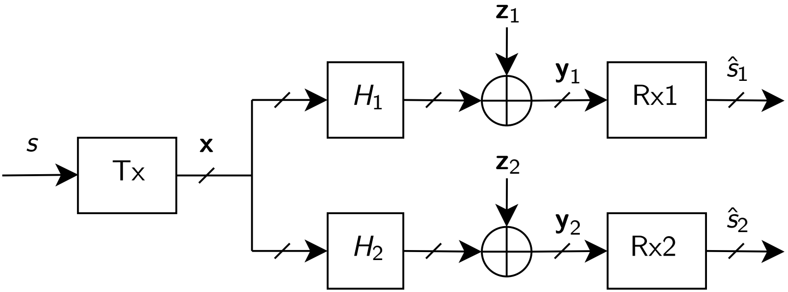
May 26, 2011

Model: Source Multicasting over MIMO Channels



- s – Scalar white Gaussian source of power P_S.
- $\mathbf{x}$ – Channel input vector of length N_t of power P_x.
- $\mathbf{y}_1$, $\mathbf{y}_2$ – Output vectors of lengths $N_r^{(1)}$, $N_r^{(2)}$.
- $\mathbf{z}_1$, $\mathbf{z}_2$ – AWGN vectors of lengths $N_r^{(i)}$ and entries of power 1.
- H_1, H_2 – Channel matrices of dimensions $N_r^{(i)} \times N_t$.

Model: Source Multicasting over MIMO Channels

- Signal-to-distortion ratio: $\mathrm{SDR}_i \triangleq \mathrm{Var}(S_i)/\mathrm{Var}(\hat{S}_i - S_i)$.

Point-to-point (single user) – "Digital scheme" (Shannon '48)

- Quantize source.
- Send index digitally using a channel code.
- **Optimal:** Minimum distortion $\Leftrightarrow$ Maximum SDR.

Goal

Find optimal SDR trade-off for two users.

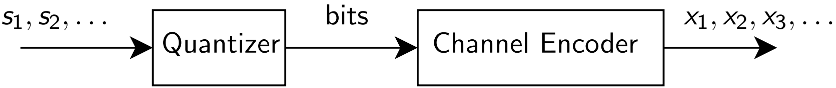
Special Case: Source Multicasting over **SISO** Channels

Scalar Case:

$$y_1 = h_1 x + z_1$$
$$y_2 = h_2 x + z_2 \,.$$

- Signal-to-noise ratio: $\mathrm{SNR}_i \triangleq |h_i|^2 P_x$.
- Optimal "individual" performance: $\mathrm{SDR}_i = 1 + \mathrm{SNR}_i$.

Special Case: Source Multicasting over **SISO** Channels



Digital Transmission

- Quantize source.
- Send (multicast) *digital* index to both users.
- Cannot be simultaneously optimal for both users!

Special Case: Source Multicasting over **SISO** Channels

Analog transmission (Goblick '61)

- Adjust power: Multiply source samples by $\sqrt{P_x/P_s}$.
- Transmit (analog) source samples.
- Perform MMSE estimation at Rx.
- Achieves **optimum** for both channels *simultaneously*:

$$\mathrm{SDR}_1 = 1 + \mathrm{SNR}_1$$
$$\mathrm{SDR}_2 = 1 + \mathrm{SNR}_2$$

Back to MIMO...

Upper Bound on Signal-to-Distortion Ratios

Upper bound

An outer bound on the achievable SDR-pairs is given by the union over all covariance matrices $C_{\mathbf{x}}$, satisfying the power constraint on:

$$\mathrm{SDR}_1 \leq \left| I + H_1 C_{\mathbf{x}} H_1^T \right| ,$$
$$\mathrm{SDR}_2 \leq \left| I + H_2 C_{\mathbf{x}} H_2^T \right| .$$

Is this bound achievable for certain cases?

Constant-Diagonal Channel Matrices

$$H_1 = \begin{pmatrix} h_1 & 0 & \cdots & 0 \\ 0 & h_1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & h_1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} h_2 & 0 & \cdots & 0 \\ 0 & h_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & h_2 \end{pmatrix}$$

Simultaneous optimality: $\mathrm{SDR}_i = (1 + \mathrm{SNR}_i)^N$, $i = 1, 2$.

Single Antenna: $N = 1$

- Optimum achieved by analog transmission.
- Digital solution is suboptimal!

Multiple Antennas: $N > 1$

- Analog solution cannot exploit all degrees of freedom.
- Simultaneous optimality impossible (Reznic et al. '06).
- Some hybrid digital-analog (HDA) schemes offer tradeoffs.

2 × 2 Diagonal Channel Matrices

For simplicity, concentrate on 2 × 2 *diagonal high-SNR* case.
(2 × 2 Diagonal Matrices / Two parallel channels)

$$H_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix}$$

- Upper bound: The union over power allocation parameter $0 \leq \gamma \leq 1$ of
$$\text{SDR}_i \leq \left(1 + \alpha_i^2 \gamma P_x\right)\left(1 + \beta_i^2 (1-\gamma) P_x\right), \quad i = 1, 2.$$
- High-SNR limit ($P_x \gg 1$):
$$\text{SDR}_i \leq \frac{\alpha_i^2 P_x}{2} \cdot \frac{\beta_i^2 P_x}{2} = \left(\frac{P_x}{2}|H_i|\right)^2, \quad i = 1, 2.$$

"Lucky" Cases

Equal capacity - $\alpha_1^2\beta_1^2 = \alpha_2^2\beta_2^2$

Digital solution:

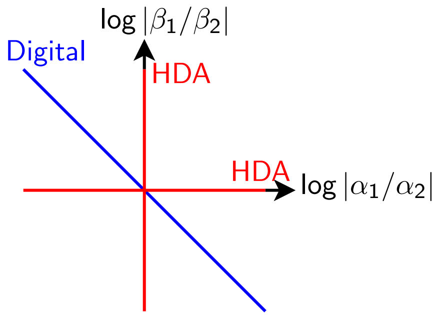
- Quantize source.
- Use channel code to convey quantization index to both users.
- Optimal since channels of same "quality" (capacity).

One gain equal (e.g. $\beta_1 = \beta_2$)

Hybrid Digital-Analog (HDA) solution (Mittal & Phamdo):

- Quantize source.
- Use channel code to convey quantization index over channel of same gain.
- Transmit quantization error over other band in an analog manner.

"Lucky" Cases



Solution by Transformation (1)

Equivalent parallel channels representation:

$$\underbrace{\begin{bmatrix} y_{i;1} \\ y_{i;2} \end{bmatrix}}_{\mathbf{y}_i} = \underbrace{\begin{bmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{bmatrix}}_{H_i} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} z_{i;1} \\ z_{i;2} \end{bmatrix}}_{\mathbf{z}_i}, \ i = 1, 2$$

,
where x_1 and x_2 are i.i.d. of power $P/2$ each.

Orthogonal transformation

- Apply an orthogonal matrix V at Tx: $\tilde{\mathbf{x}} = V\mathbf{x}$.
- Apply orthogonal matrices U_i at Rx-i: $\tilde{\mathbf{y}}_i = U_i^T \mathbf{y}_i$.
- upper bound and power stay the same!

Solution by Transformation (2)

Equivalent channel adequate for HDA transmission:

- Need one of the diagonal gains equal for the digital element.
- Do **not** need a diagonal matrix: Triangular suffices.

Theorem: existence of transformation

Iff $\alpha_1^2 \geq \alpha_2^2$ and $\beta_1^2 \leq \beta_2^2$, or vice versa, then:

$$\begin{bmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{bmatrix} = U_1 \begin{bmatrix} a_1 & c_1 \\ 0 & b \end{bmatrix} V^T \triangleq U_1 R_1 V^T$$

$$\begin{bmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{bmatrix} = U_2 \begin{bmatrix} a_2 & c_2 \\ 0 & b \end{bmatrix} V^T \triangleq U_2 R_2 V^T$$

where U_1, U_2, V are orthogonal matrices.

Solution by Transformation (3)

By applying V at Tx and U_i^T at Rx, equivalent BC channel:

$$\tilde{\mathbf{y}}_1 = \begin{bmatrix} \tilde{y}_{1;1} \\ \tilde{y}_{1;2} \end{bmatrix} = \begin{bmatrix} a_1 & c_1 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \tilde{z}_{1;1} \\ \tilde{z}_{1;2} \end{bmatrix}$$

$$\tilde{\mathbf{y}}_2 = \begin{bmatrix} \tilde{y}_{2;1} \\ \tilde{y}_{2;2} \end{bmatrix} = \begin{bmatrix} a_2 & c_2 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \tilde{z}_{2;1} \\ \tilde{z}_{2;2} \end{bmatrix}$$

Due to orthogonality:

- $a_i^2 b^2 = \alpha_i^2 \beta_i^2$.
- $\tilde{\mathbf{z}}_i$ have the same statistics as $\mathbf{z}_i$.

Solution by Transformation (4)

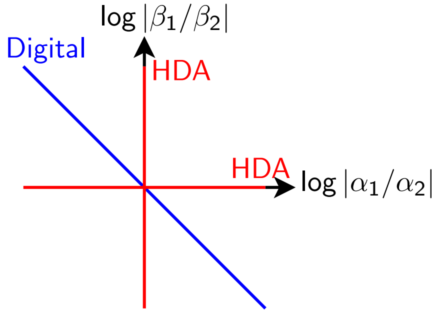
$$\tilde{y}_{i;1} = a_i \, x_1 + \overbrace{c_i \, x_2}^{\text{Interference}} + \tilde{z}_{i;1}$$

$$\tilde{y}_{i;2} = 0 \cdot x_1 + b \, x_2 + \tilde{z}_{i;2}$$

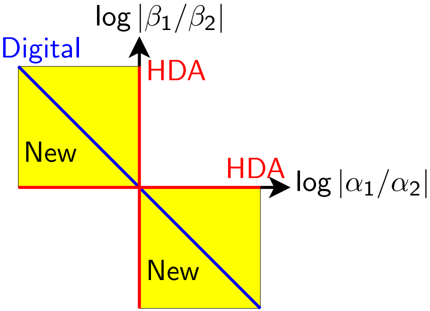
Optimal transmission over equivalent channel

- Quantize and send digital channel code over second channel.
- Send analog quantization error over first channel.
- Interference cancelation: $\tilde{y}_{i;1} - c_i x_2 = a_i x_1 + \tilde{z}_{i;1}$.
- Optimal reconstruction as in HDA for parallel channels with one equal band.

Optimality Region



Optimality Region



Generalizations

- Optimality for general SNR under the same condition.
- MIMO (non-diagonal) channel: optimality under the same condition, applied to the generalized singular values.

Higher dimension

- $(N-1)$ digital sub-channels, one analog.
- Explicit condition on singular values - a generalized form of Weyl's condition used in the GTD (Jiang et al.).

- Results carry over to colored/ISI channels (diagonal matrix $\Leftrightarrow$ filter in frequency domain)
- Special case of "Network Modulation": Joint decomposition of channel matrices for MIMO network problems.