Control with Fixed-Rate Limited Feedback

Anatoly Khina

Joint work with Yorie Nakahira and Babak Hassibi

Caltech, Pasadena, CA, USA

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Networked Control vs. Traditional Control

Traditional control:

- Observer and controller are co-located.
- Classical systems are hardwired and well crafted.
Networked Control vs. Traditional Control

Observer and controller are not co-located:
- connected through noisy link
- Suitable for new remote applications
  (e.g., remote surgery, self-driving cars)
Linear Quadratic Gaussian Control over Gaussian Channels

Linear quadratic Gaussian (LQG) system

\[ x_{t+1} = Ax_t + Bu_t + w_t, \quad w_t \sim \text{i.i.d.} \; \mathcal{N}(0, W) \]
\[ y_t = Cx_t + v_t, \quad v_t \sim \text{i.i.d.} \; \mathcal{N}(0, V) \]

Noiseless finite-rate channel of rate \( R \)

**Fixed rate**: Exactly \( R \) bits are available at every time sample \( t \)

**Variable rate**: \( R \) bits are available **on average** at every \( t \)

- Transmitter decides on \( R_t \) at every \( t \) s.t. \( \frac{1}{T} \sum_{i=1}^{T} R_t \leq R \)

LQG cost

\[ J = \sum_{t=1}^{T} \left[ x_t^T Q x_t + u_t^T R u_t \right] + x_{T+1}^T F x_{T+1} \]
Linear Quadratic Gaussian Control over Gaussian Channels

Scalar Linear quadratic Gaussian (LQG) system

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Fixed rate: Exactly \( R \) bits are available at every time sample \( t \)

LQG cost

\[ J = \sum_{t=1}^{T} \left[ Qx_t^2 + Ru_t^2 \right] + Fx_{T+1}^2 \]
Linear Quadratic Gaussian Control over Gaussian Channels

Scalar LQG system

\[ x_{t+1} = x_t + u_t + w_t \]
\[ y_t = x_t + v_t \]

Finite-rate noiseless channel

Fixed-rate: \( R \) bits per time \( t \)
Adaptive Fixed-Rate Quantizer

Use an adjusted quantizer to the input p.d.f.
Adaptive Fixed-Rate Quantizer

- Use an adjusted quantizer to the input p.d.f.
- At some point a (rare) event will happen
- Input value outside effective quantization interval
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- Next time instant: Input will be even larger!
- Avalanche effect
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At some point a (rare) event will happen

Input value outside effective quantization interval

Next time instant: Input will be even larger!

Avalanche effect

To avoid this ⇒ Quantizer needs to be adaptive
Adaptive Optimal Fixed-Rate Quantizer?

- Adaptive uniform quantizer [Yüksel AC’10]
  - Based on Jayant’s adaptive quantizer [Jayant ’73]
  - Similar idea in [Brockett-Liberzon AC’00]: “Zooming in/out”

- Adaptive exponential quantizer [Nair-Evans ’04]

- Both results prove condition on stabilizability: $R > \log \alpha$

- But no cost optimality claims...

- Other notable contributions: [Borkar-Mitter ’97]
  [Tatikonda-Sahai-Mitter AC’04] [Matveev-Savkin ’04]
  [Tsumura-Maciejowski CDC’03], ...

How to optimize cost?
Optimal Quantizer for One Sample

- Let $x \sim \mathcal{N}(0, 1)$
- $R$ bits $\Rightarrow 2^R$ quantization points
- Uniform quantizer is suboptimal
- How to construct an optimal quantizer?
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- Let $x \sim \mathcal{N}(0, 1)$
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- How to construct an optimal quantizer?

- In general a hard (NP-hard) problem
- Necessary conditions by Lloyd ['57, IT'82] and Max [IT '60]
  - Also known in machine learning as “k-means” clustering
**Lloyd-Max Algorithm**

**Nearest Neighbor:** Given reconstruction points, find optimal cells

\[
\text{Cell } i = \{ x \mid (x - \hat{x}_i)^2 < (x - \hat{x}_j)^2, \forall j \neq i \}
\]

**Centroid:** Given quant. cells, find optimal reconstruction points

\[
\hat{x}_i = \mathbb{E} [x \mid x \in \text{Cell } i]
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- Optimal quantizer necessarily satisfies **Centroid** and **NN**
- But... They are not sufficient in general! 😞
- Lloyd-Max algorithm might converge to a local optimum...
Lloyd-Max Algorithm

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When does Lloyd-Max converge to global optimum?
[Fleischer ’64][Trushkin IT’82][Kieffer-Jahns-Obuljen IT’88]

- Conditions for existence of only one local optimum ⇒ Global
- Log-concave distributions satisfy these conditions
- Important special case: Gaussian distribution 😊

- One stage of LQG with finite-rate noiseless channel ✓

What about more stages?
Multi-Stage Control with Finite-Rate Feedback

- First input is Gaussian $\Rightarrow$ Log-concave pdf
- Lloyd-Max quantizer is optimal
Multi-Stage Control with Finite-Rate Feedback

- First input arrives and chooses cell
Multi-Stage Control with Finite-Rate Feedback

- Quantization noise ("error") is determined by the hit cell
Multi-Stage Control with Finite-Rate Feedback

Quantization noise pdf = truncated original pdf normalized
Multi-Stage Control with Finite-Rate Feedback

- Truncated log-concave pdf is log-concave!
Quantization noise is inflated by $\alpha$ (pdf remains log-concave)
Motivation

Multi-stage

Suc. Ref.

Discuss.

Greedy LQG

Multi-Stage Control with Finite-Rate Feedback

Inflated error added to new $w_t \Rightarrow$ Convolution of pdfs
$w_t \sim \mathcal{N}(0, W) \ast \text{log-concave quantization error}$
Convolution of log-concave functions is also log-concave!
Multi-Stage Control with Finite-Rate Feedback

Resulting pdf (in red)
- Depends on cell index chosen in previous stage(s)
- Log-concave

Applying Lloyd-Max quantization in second stage is optimal!

- First-stage pdf (in blue) for comparison
Optimal Greedy Algorithm

- Lloyd-Max quantization minimizes squared error of that stage
- Lloyd-Max quantization = Optimal greedy algorithm
- But... It is not necessarily globally optimal...
- Quantizer used affects pdf of future stages
- Quantizer should be chosen according to the dynamic program
Linear Quadratic Regulator (LQR) Example

- LQR setting with $x_0 \sim \mathcal{N}(0, X)$ and $\alpha = 1$:
  \[
  \begin{cases}
  x_{t+1} = x_t + u_t \\
  y_t = x_t
  \end{cases}
  \]

- Assume for simplicity we are interested in accumulated MMSE:
  \[
  J = \sum_{t=1}^{T} x_t^2 \triangleq \sum_{t=1}^{T} J_t
  \]
  \[
  J_t \triangleq x_t^2
  \]

- In this case, clearly $u_1 = -\hat{x}_0$, $u_2 = -\left(\hat{x}_0 - \hat{x}_0\right)$

- $\{u_t\}$ sequence refines the reconstruction of $x_0$ at every stage

- This problem is known in IT as successive refinement
Successive Refinement

- Two descriptions of the source $x$:
  - Description of rate $R_1$
  - First description of rate $R_1$ and another description of rate $R_2$

Successive refinement with encoding/decoding of long blocks
[Equitz-Cover IT’91][Rimoldi IT’94]

- Optimal trade-off $(R_1, R_2) \leftrightarrow (J_1, J_2)$ is known
- $J_2$ is the same as if $R_1 + R_2$ was given to begin with (no $J_1$)

- But... Optimal scalar quantizer for $J_1$ is not optimal for $J_2$
- Tension between optimizing $J_1$ and $J_2$
  $\Rightarrow$ Suboptimality of Lloyd-Max in LQR example [Fu AC’12]
Successive Refinement

Lloyd-Max algorithm with $2^R = 16$ quantization points:
Successive Refinement

Lloyd-Max algorithm with $2^R = 16$ quantization points:

![](chart1.png)

Lloyd-Max algorithm with $2^{R_1} = 4$ quantization points:

![](chart2.png)
Successive Refinement

Lloyd-Max algorithm with $2^R = 16$ quantization points:

Lloyd-Max algorithm ran for each cell with $2^{R_2} = 4$ points:
Optimal Scalar Successive Refinement

Optimal average-stage MMSE of scalar successive refinement
[Dumitrescu-Wu IT’09]

- Generalized Lloyd-Max can be constructed [Brunk-Farvardin ’96]
- Converges to **optimal average-stage MMSE**
- Extends Trushkin’s conditions to successive refinement setting
- Conditions are satisfied for log-concave pdfs

LQR for $\alpha \neq 1$

- $J = J_1 + \alpha^2 J_2 + \cdots + \alpha^{2(T-1)} J_t$
- Adequate generalized LM algorithm can be constructed
- Converges to optimal weighted MSE $J$ for log-concave pdfs
Future Research: Optimal Quantization for LQG

- We saw how to construct optimal quantizers for LQR
- How to construct optimal quantizers for LQG control?
- Input pdf at every stage is log-concave
- Variant of generalized Lloyd-Max quantization will be optimal
- What variant to use would be dictated by dynamic program
- How to construct a good low-complexity scheme?
Complementary Results
Assume a large number of points

Overload noise (noise outside dynamic range) is negligible

Quantization points “can” be approximated by continuous pdf

Optimal quantization points distribution $\propto f_X^{1/3}$

Optimal distortion $= \frac{1}{12N^2} \| f_X \|_{1/3}$

Under these assumptions, successively refinable (no tensions between $J_1$ and $J_2$)