LQG Control with Fixed-Rate Limited Feedback

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Caltech, Pasadena, CA, USA

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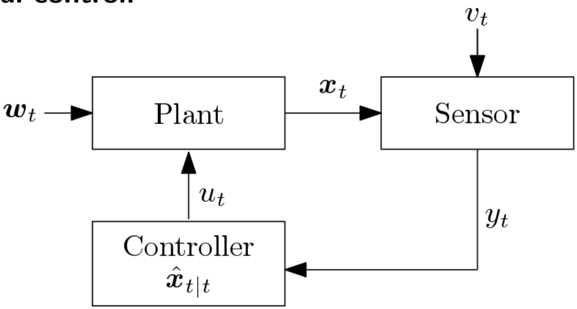
Networked Control vs. Traditional Control

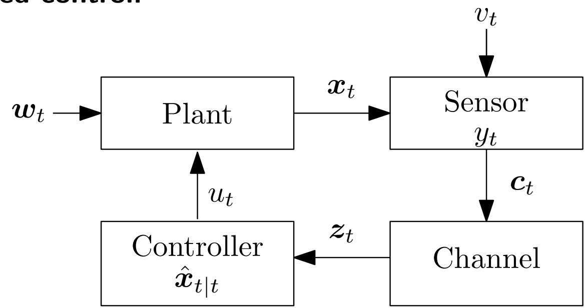
Traditional control:



- Observer and controller are co-located
- Classical systems are hardwired and well crafted

Networked Control vs. Traditional Control

Networked control:



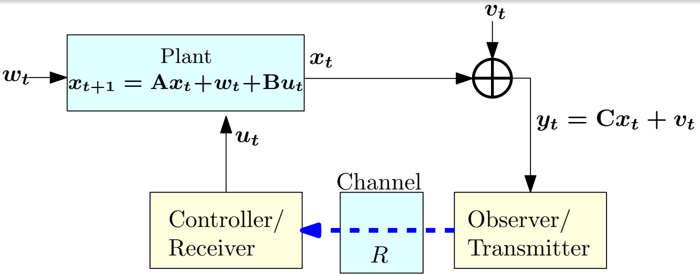
- Observer and controller are not co-located: connected through noisy link
- Suitable for new remote applications (e.g., remote surgery, self-driving cars)

Linear Quadratic Gaussian Control over Gaussian Channels

Linear quadratic Gaussian (LQG) system

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t + \mathbf{w}_t, \qquad \mathbf{w}_t \sim \text{ i.i.d. } \mathcal{N}(0, \mathbf{W})$$

$$\mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{v}_t, \qquad \mathbf{v}_t \sim \text{ i.i.d. } \mathcal{N}(0, \mathbf{V})$$



Noiseless finite-rate channel of rate R

Fixed rate: Exactly R bits are available at every time sample t

Variable rate: R bits are available **on average** at every t

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Scalar Linear quadratic Gaussian (LQG) system

$$x_{t+1} = x_t + u_t + w_t, \qquad w_t \sim \text{ i.i.d. } \mathcal{N}(0, W), \qquad |\alpha| > 1$$
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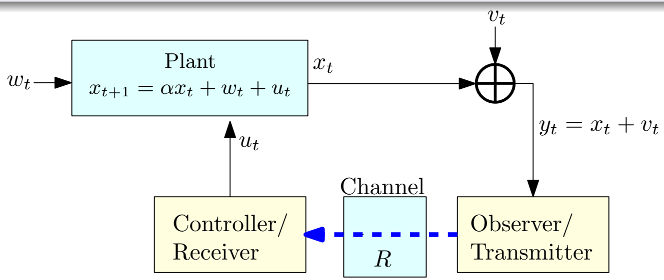

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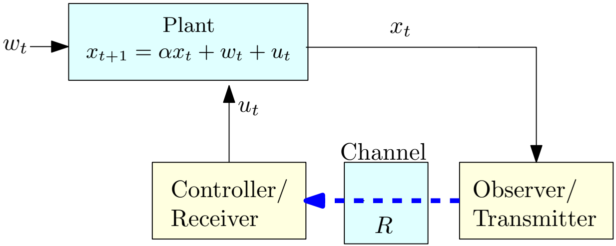

LQG cost

$$J = \mathbb{E}\left[\sum_{t=1}^{T}\left[Qx_t^2 + Ru_t^2\right] + Fx_{T+1}^2\right]$$

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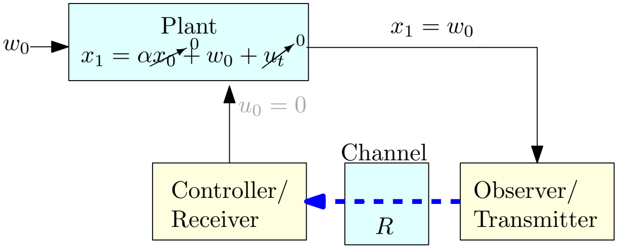

LQG cost: MMSE ($Q = F = 1, R = 0$)

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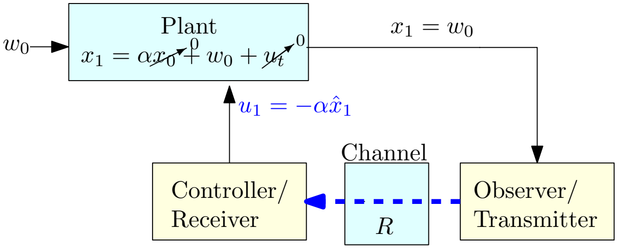

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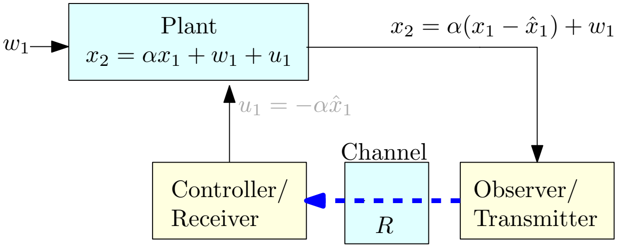

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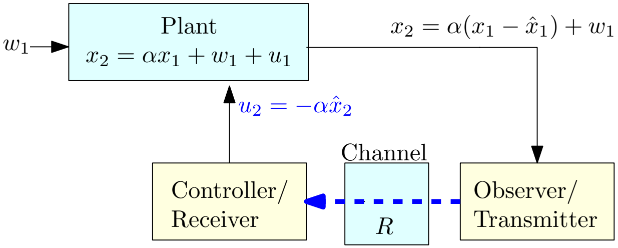

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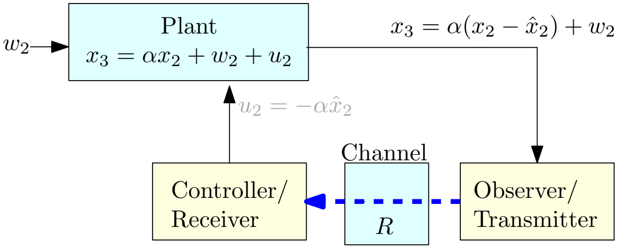

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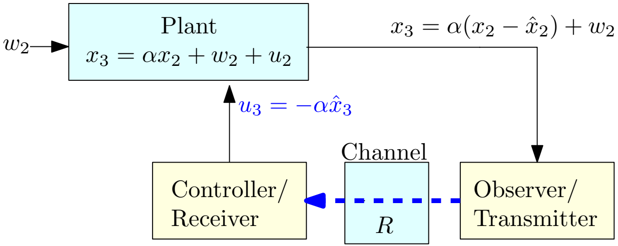

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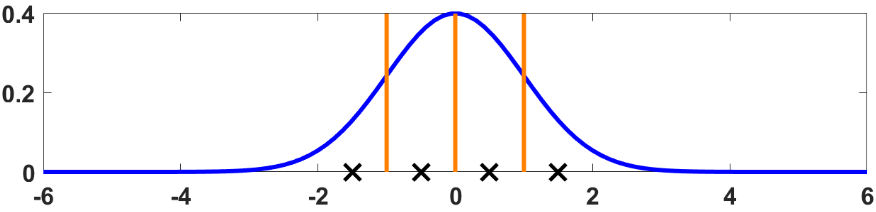
Adaptive Fixed-Rate Quantizer



- Use an adjusted quantizer to the input p.d.f.

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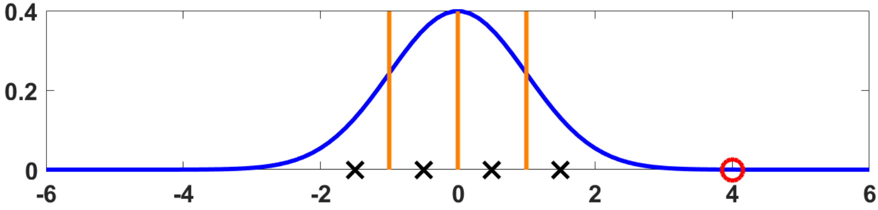

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- Input value outside effective quantization interval

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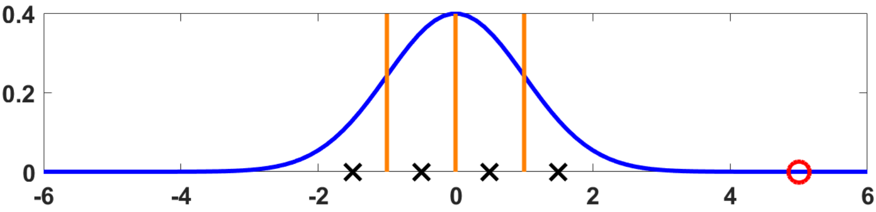

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- Next time instant: Input will be even larger!
- **Avalanche effect**

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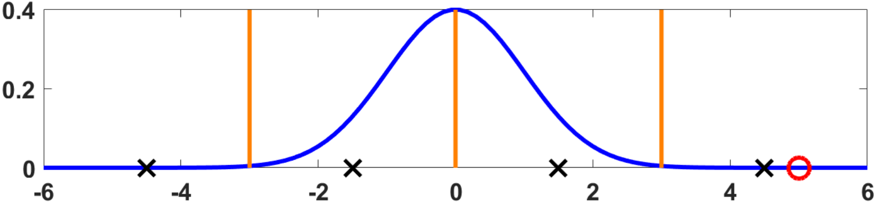

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- At some point a (rare) event will happen
- Input value outside effective quantization interval
- Next time instant: Input will be even larger!
- **Avalanche effect**
- To avoid this $\Rightarrow$ Quantizer needs to be **adaptive**

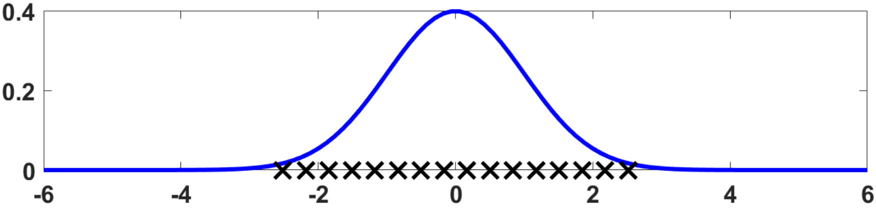
Adaptive Optimal Fixed-Rate Quantizer?

- Adaptive uniform quantizer [Yüksel AC'10]
  - Based on Jayant's adaptive quantizer [Jayant '73]
  - Similar idea in [Brockett-Liberzon AC'00]: "Zooming in/out"
- Adaptive exponential quantizer [Nair-Evans '04]
- Both results prove condition on stabilizability: $R > \log \alpha$
- But no cost optimality claims...
- Other notable contributions: [Borkar-Mitter '97] [Tatikonda-Sahai-Mitter AC'04] [Matveev-Savkin '04] [Tsumura-Maciejowski CDC'03], ...

How to optimize cost?

Optimal Quantizer for One Sample

- Let $x \sim \mathcal{N}(0,1)$
- R bits $\Rightarrow 2^R$ quantization points
- Uniform quantizer is suboptimal
- How to construct an optimal quantizer?



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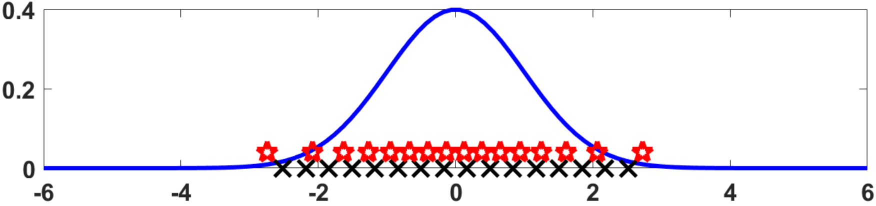


- In general a hard (NP-hard) problem
- Necessary conditions by Lloyd ['57, IT'82] and Max [IT '60]
  - Also known in machine learning as "k-means" clustering

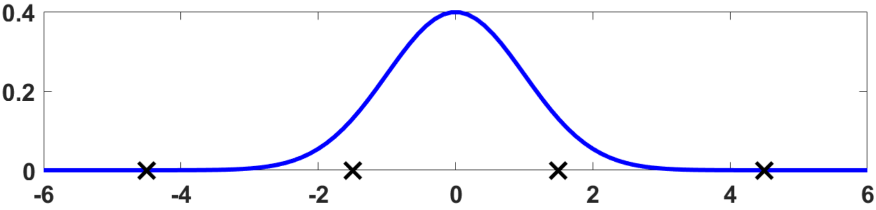
Lloyd-Max Algorithm

Nearest Neighbor: Given reconstruction points, find optimal cells

$$\text{Cell } i = \left\{ x \middle| (x - \hat{x}_i)^2 < (x - \hat{x}_j)^2, \ \forall j \neq i \right\}$$

Centroid: Given quant. cells, find optimal reconstruction points

$$\hat{x}_i = \mathbb{E}\left[x \middle| x \in \text{Cell } i \right]$$



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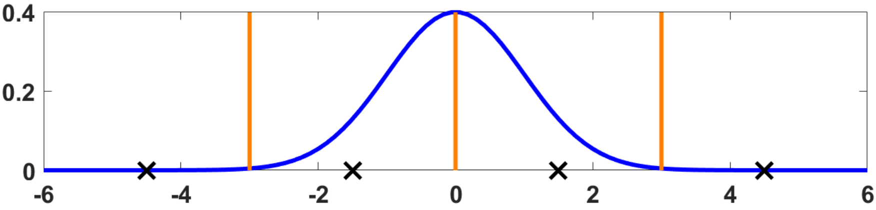

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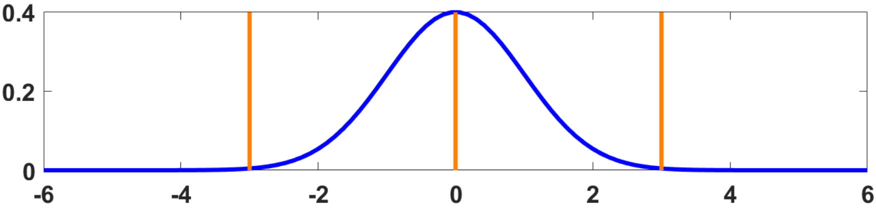

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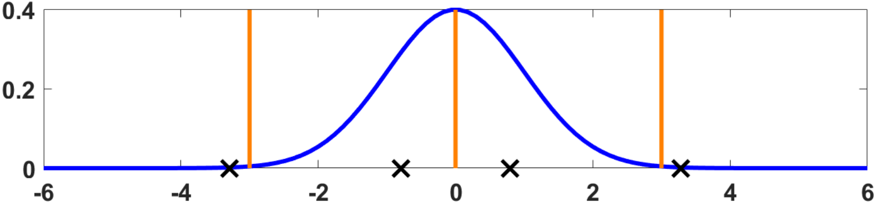

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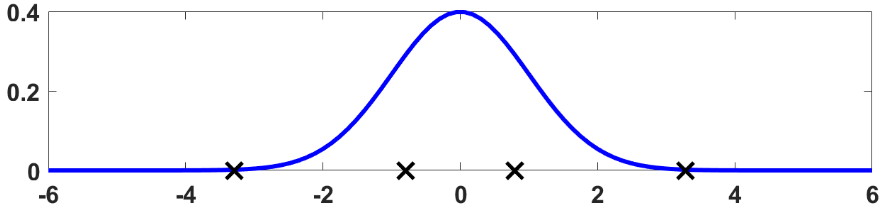

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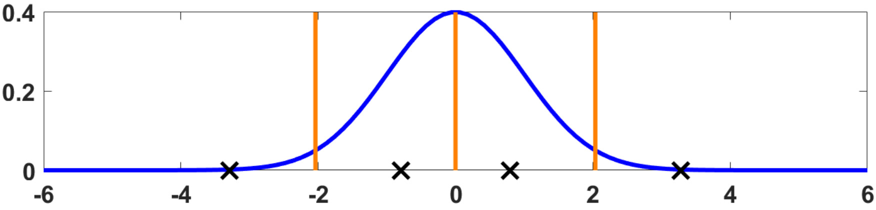

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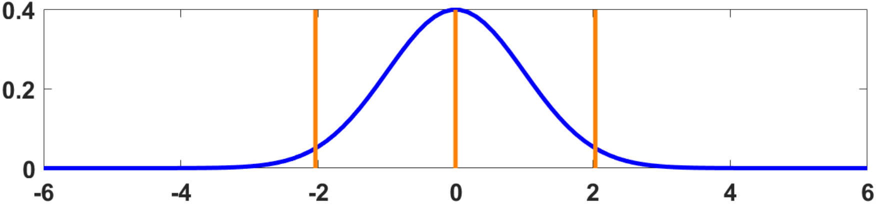

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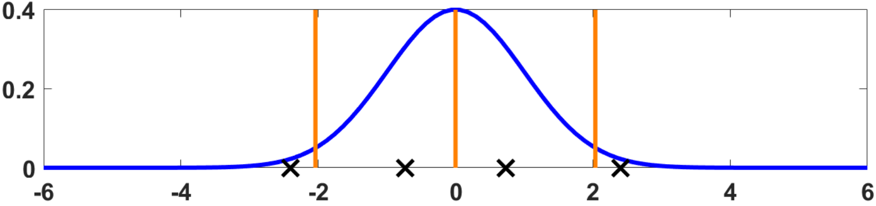

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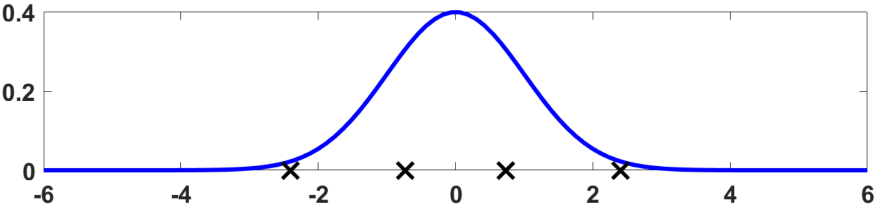

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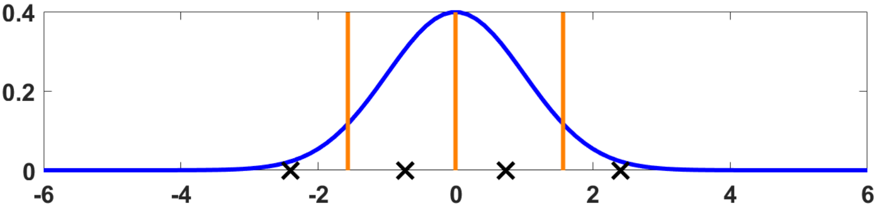

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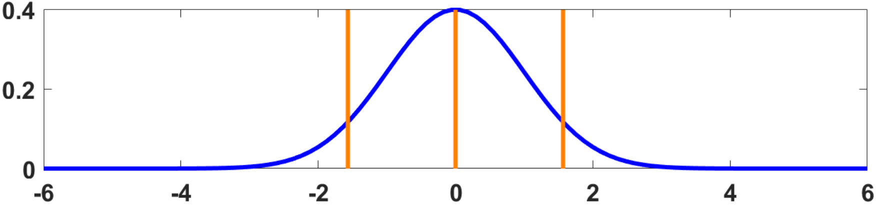

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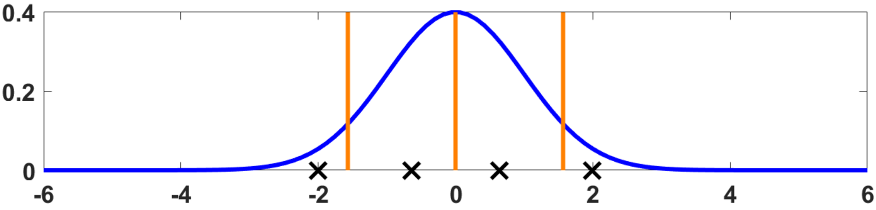

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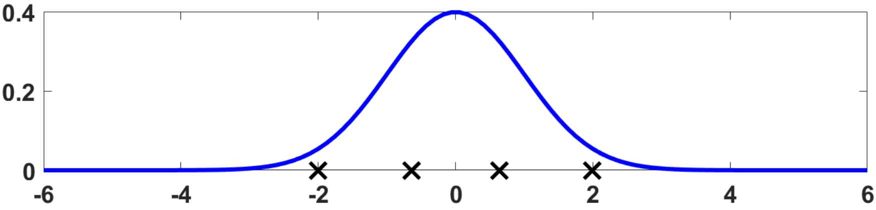

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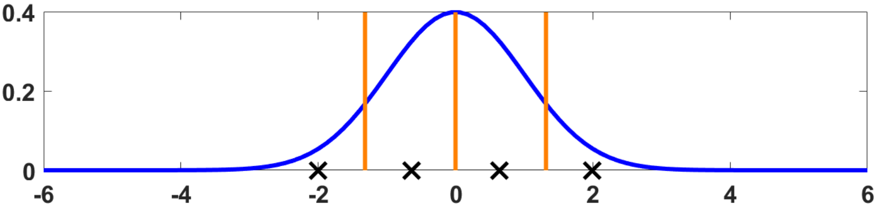

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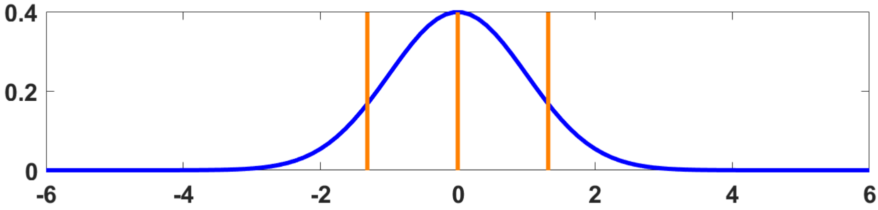

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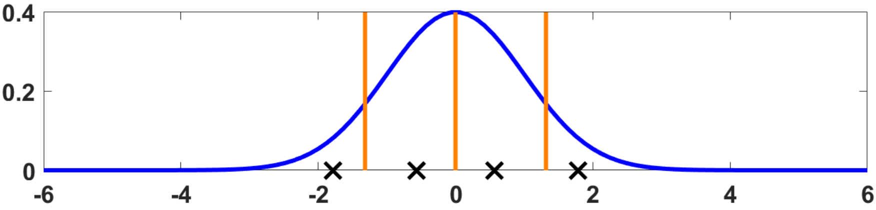

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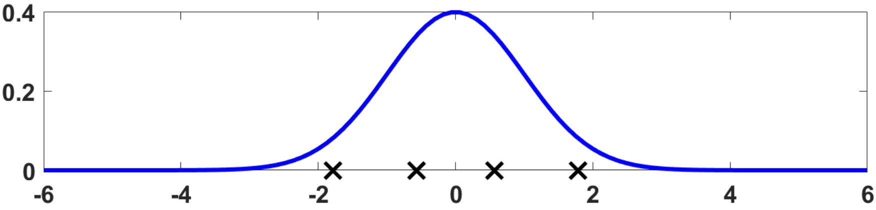

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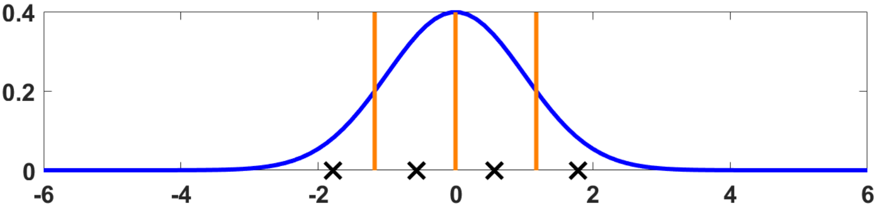

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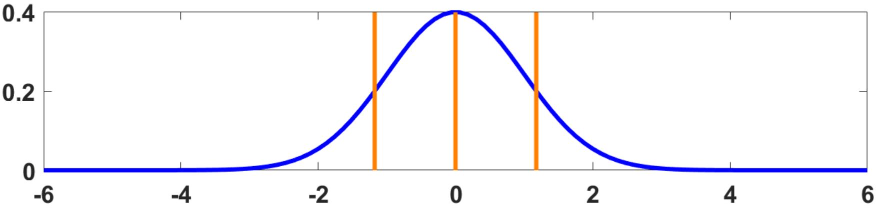

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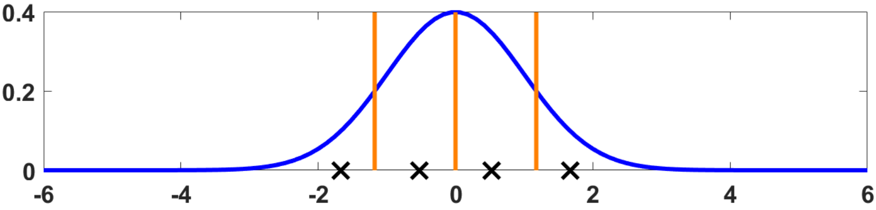

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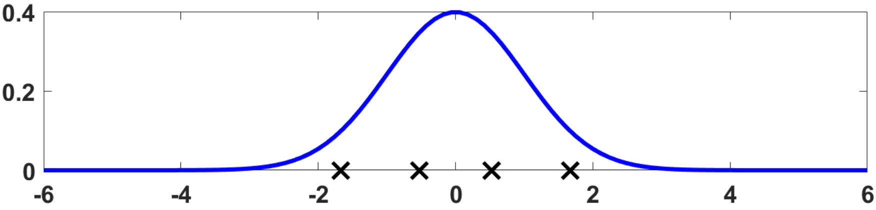

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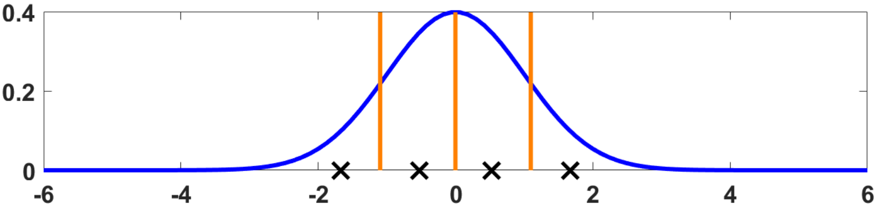

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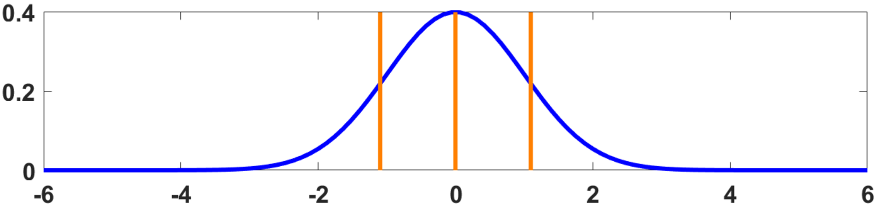

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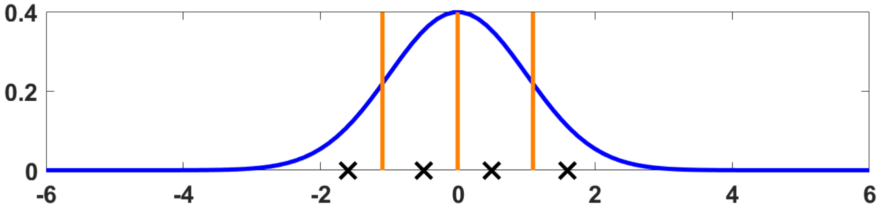

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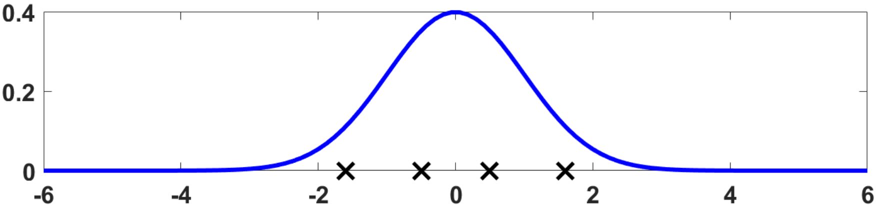

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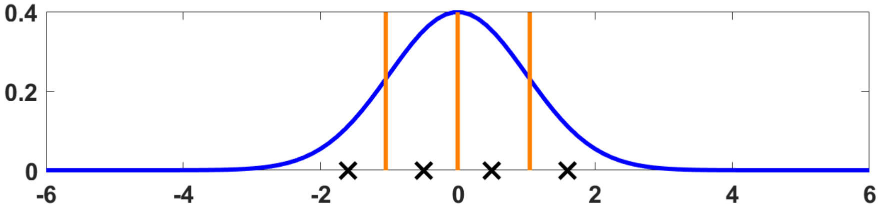

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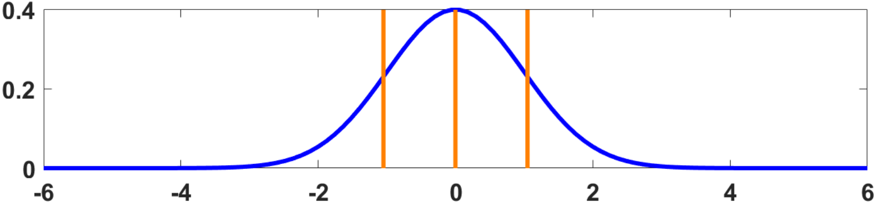

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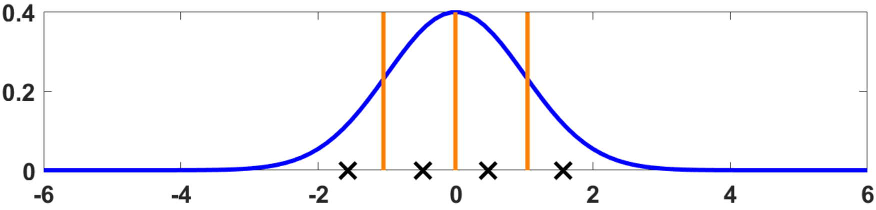

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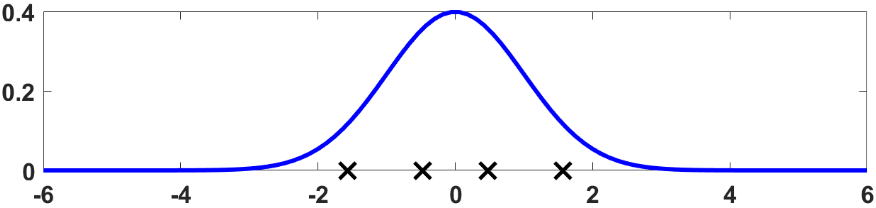

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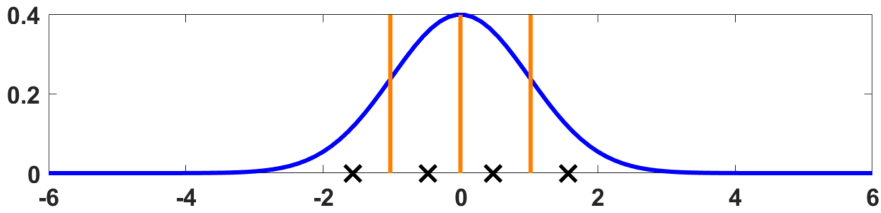

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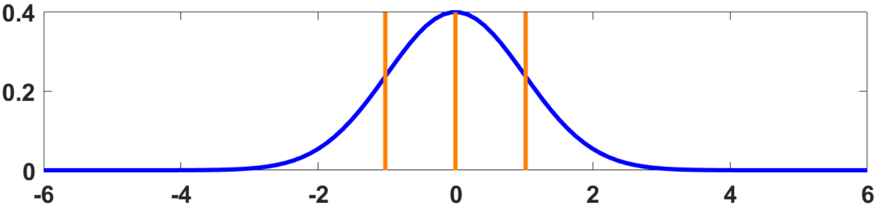

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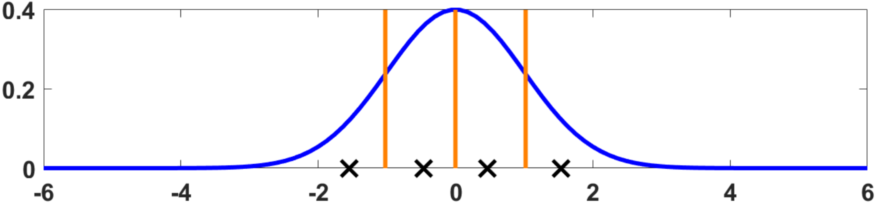

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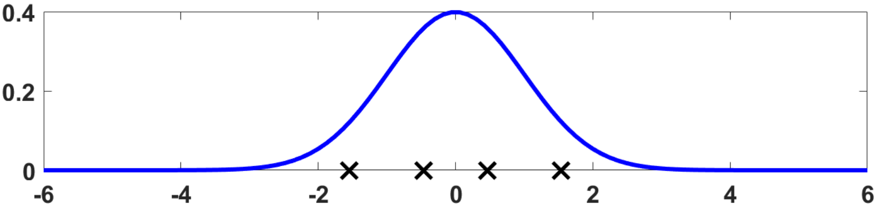

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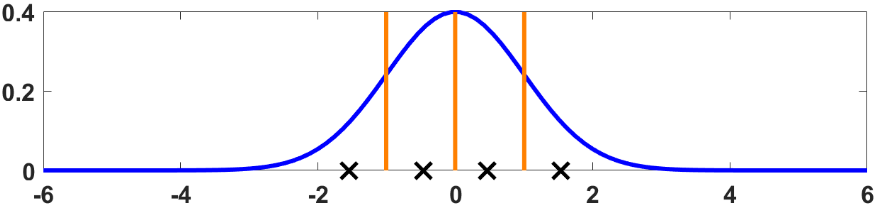

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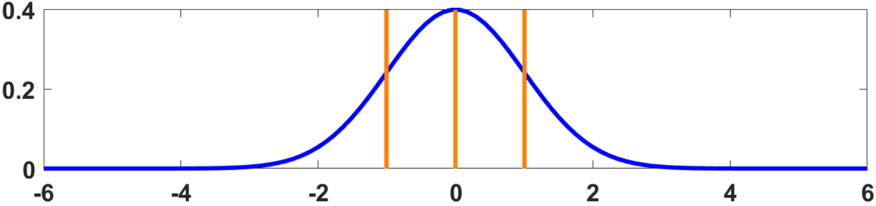

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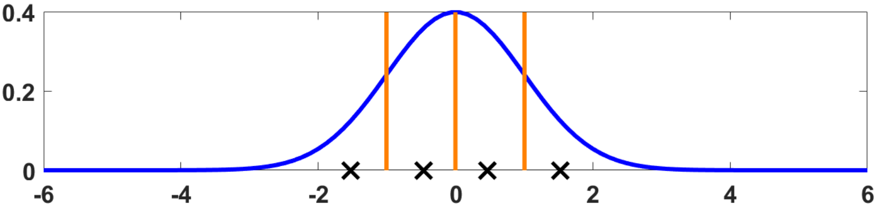

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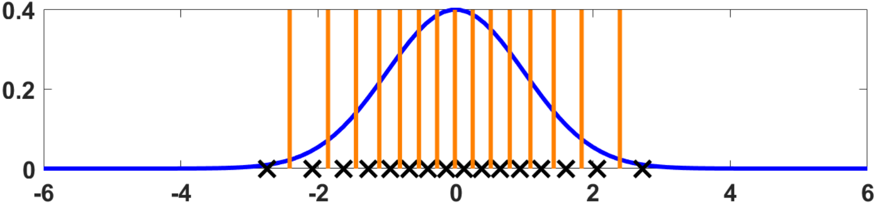
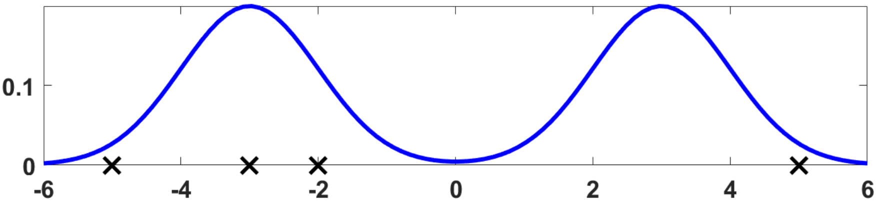


Lloyd-Max Algorithm

- Optimal quantizer necessarily satisfies *Centroid* and *NN*
- But... They are not sufficient in general! ☹
- Lloyd-Max algorithm might converge to a local optimum...



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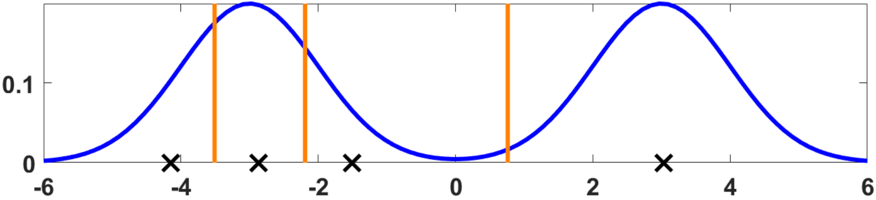


Lloyd-Max Algorithm

When does Lloyd-Max converge to global optimum?
[Fleischer '64][Trushkin IT'82][Kieffer-Jahns-Obuljen IT'88]

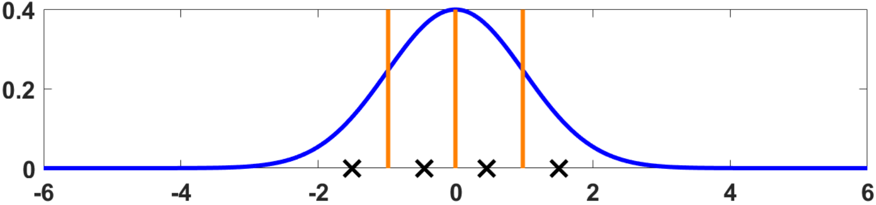
- Conditions for existence of only one local optimum ⇒ **Global**
- **Log-concave** distributions satisfy these conditions
- Important special case: **Gaussian distribution** ☺

- One stage of LQG with finite-rate noiseless channel ✓

What about more stages?

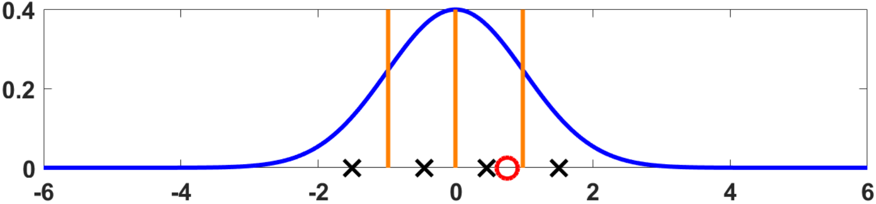
Multi-Stage Control with Finite-Rate Feedback

- First input $x_1 = w_0$ is Gaussian $\Rightarrow$ Log-concave pdf
- Lloyd-Max quantizer is optimal



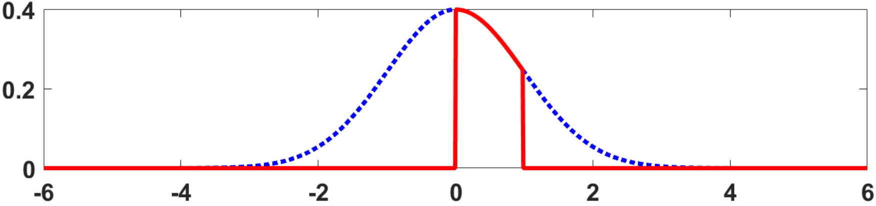
Multi-Stage Control with Finite-Rate Feedback

- First input x_1 arrives
- Chooses cell: cell i
- Chooses reconstruction point: $\hat{x}_i$



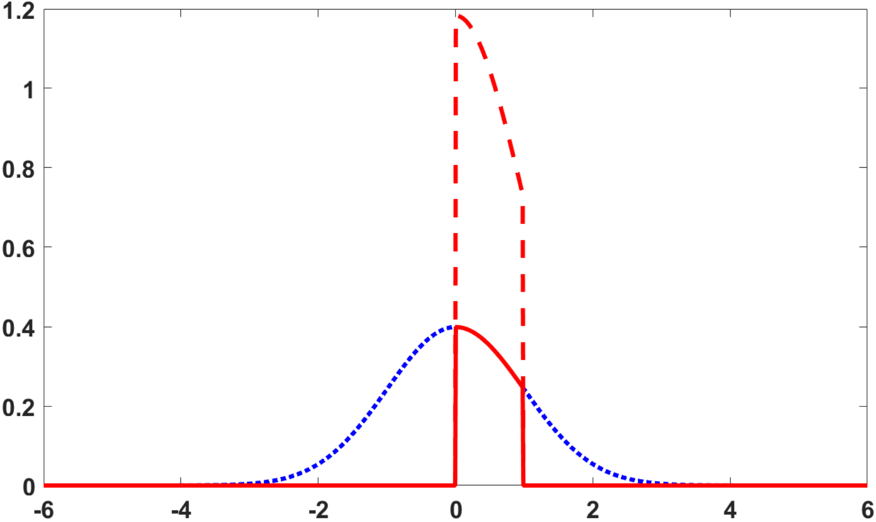
Multi-Stage Control with Finite-Rate Feedback

- $$\underset{p(x_1 | x_1 \in \text{cell } i)}{\text{pdf given hit cell } i} = \underset{p(x_1)}{\text{truncated original pdf}}$$



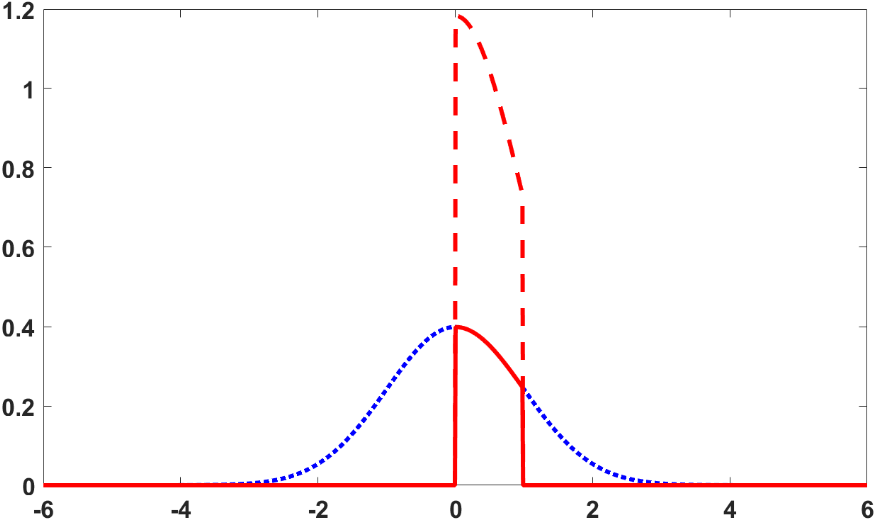
Multi-Stage Control with Finite-Rate Feedback

- Up to scaling...



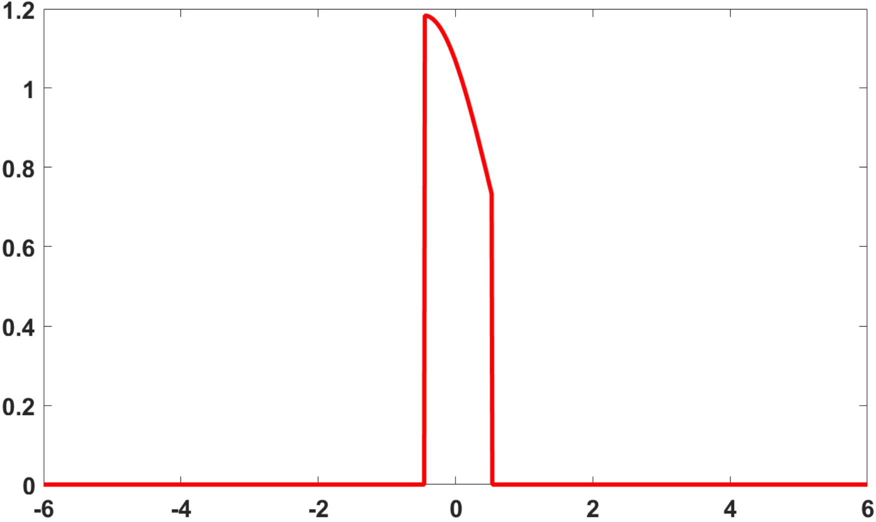
Multi-Stage Control with Finite-Rate Feedback

- Truncated log-concave pdf is **log-concave**!



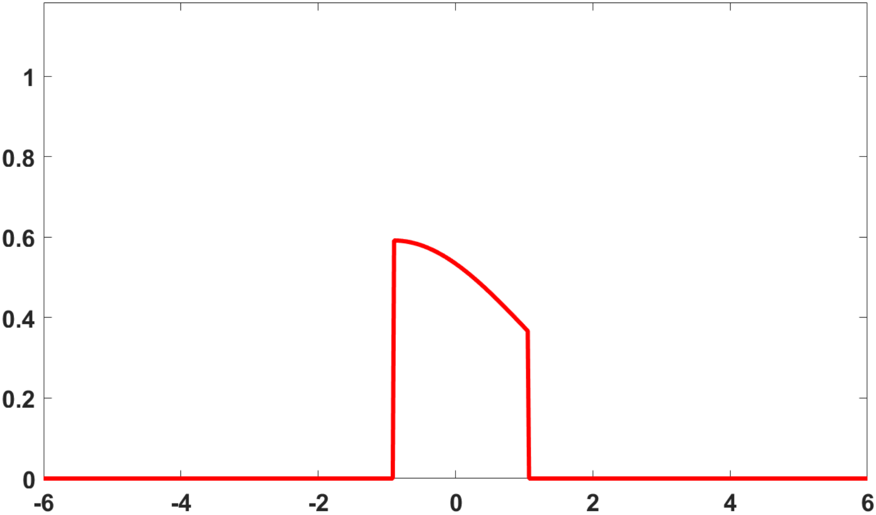
Multi-Stage Control with Finite-Rate Feedback

- pdf of quantization noise $p(x_1 - \hat{x}_1 | x_1 \in \text{ cell } i)$



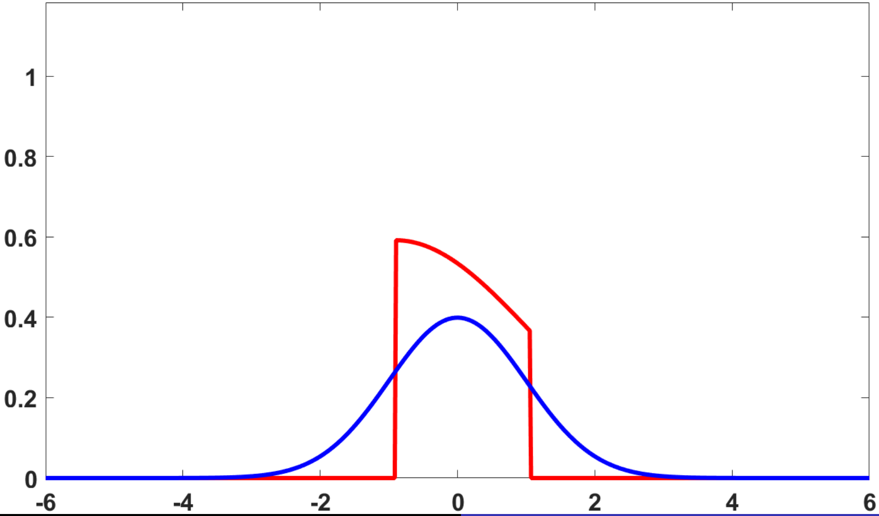
Multi-Stage Control with Finite-Rate Feedback

- Quantization noise inflated by α: $\alpha(x_1 - \hat{x}_1)$



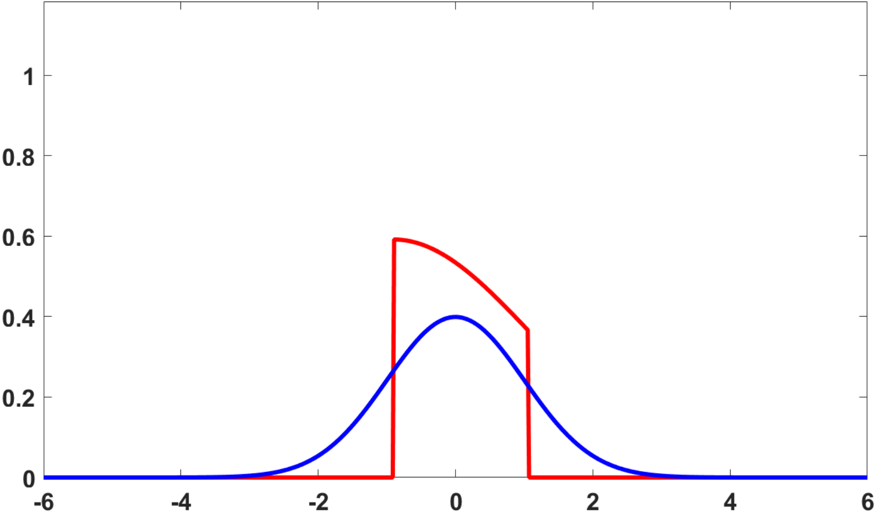
Multi-Stage Control with Finite-Rate Feedback

- New w_t added: $\alpha(x_1 - \hat{x}_1) + w_1 \Rightarrow$ Convolution of pdfs



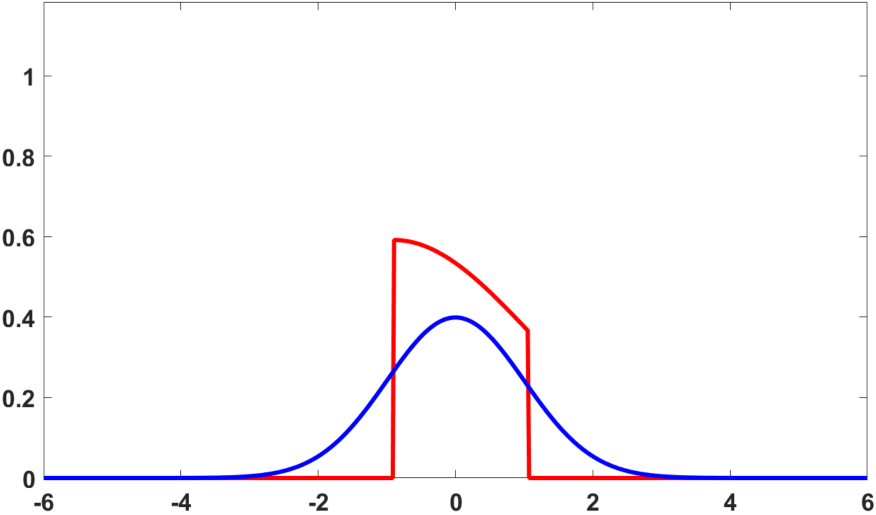
Multi-Stage Control with Finite-Rate Feedback

- $w_t \sim \mathcal{N}(0, W) * $ log-concave quantization error



Multi-Stage Control with Finite-Rate Feedback

- Convolution of log-concave functions is also **log-concave**!



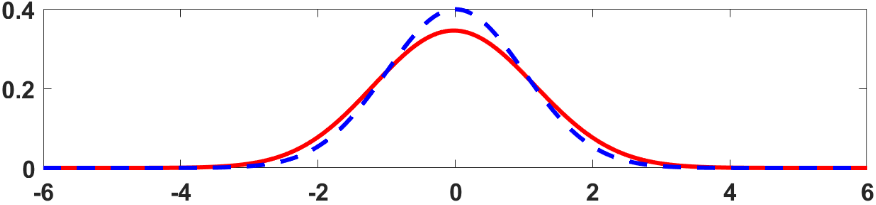
Multi-Stage Control with Finite-Rate Feedback

Resulting pdf (in red)

- Depends on cell index chosen in previous stage(s)
- Log-concave

Applying Lloyd-Max quantization in second stage is optimal!

- First-stage pdf (in blue) for comparison



Optimal Greedy Algorithm

- Lloyd-Max quantization minimizes squared error of that stage
- Lloyd-Max quantization = Optimal **greedy** algorithm
- But... It is not necessarily globally optimal...
- Quantizer used affects pdf of future stages
- Quantizer should be chosen according to the dynamic program (take into account the "cost-to-go")

Linear Quadratic Regulator (LQR) Example

- LQR setting with $x_0 \sim \mathcal{N}(0, X)$ and $\alpha = 1$:

$$\begin{cases} x_{t+1} &= x_t + u_t \\ y_t &= x_t \end{cases}$$

- Assume for simplicity we are interested in accumulated MMSE:

$$J = \sum_{t=1}^{T} \mathbb{E}\left[x_t^2\right] \triangleq \sum_{t=1}^{T} J_t$$
$$J_t \triangleq \mathbb{E}\left[x_t^2\right]$$

- In this case, clearly $u_1 = -\hat{x}_0$, $u_2 = -\left(\widehat{x_0 - \hat{x}_0}\right)$
- $\{u_t\}$ sequence refines the reconstruction of x_0 at every stage
- Equivalent to the *successive refinement* problem

Successive Refinement

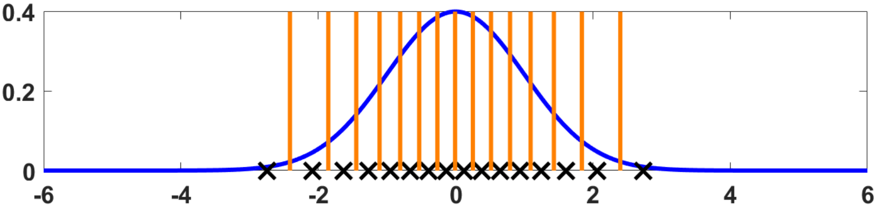
Successive refinement with encoding/decoding of long blocks [Equitz-Cover IT'91][Rimoldi IT'94]

- Optimal trade-off $(R_1, R_2) \leftrightarrow (J_1, J_2)$ is known
- J_2 is the same as if $R_1 + R_2$ was given to begin with (no J_1)

- But... **Optimal scalar quantizer for J_1 is not optimal for J_2**
- Tension between optimizing J_1 and J_2
- **Suboptimality of greedy algorithm** in LQR example [Fu AC'12]

Successive Refinement: Example with $R = 2$

Lloyd-Max algorithm with $2^{2R} = 16$ quantization points:



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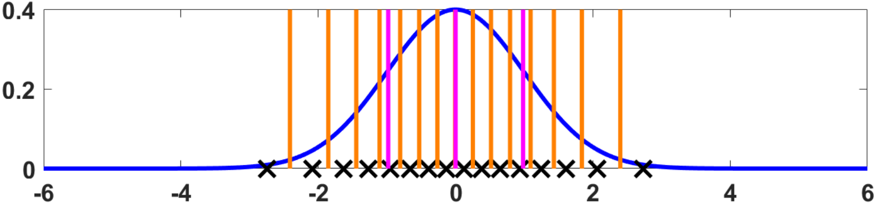
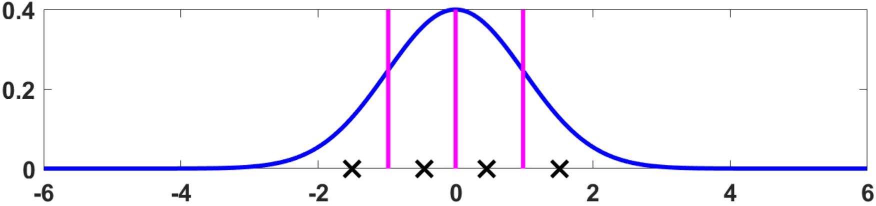


Lloyd-Max algorithm with $2^{R} = 4$ quantization points:



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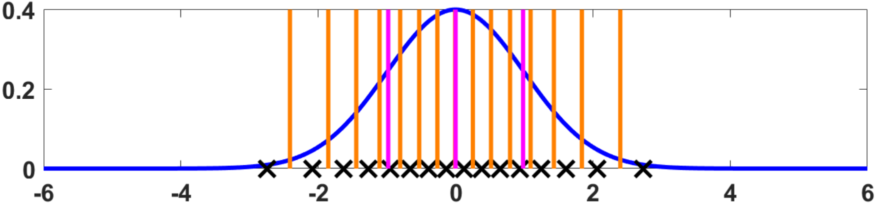
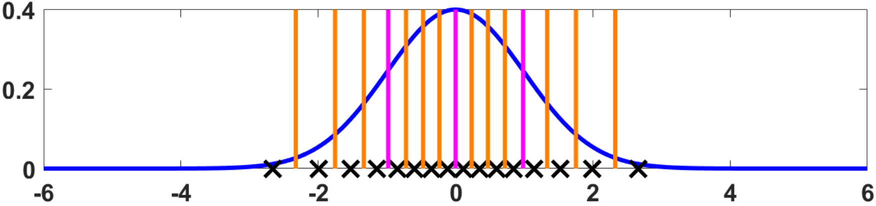


Lloyd-Max algorithm ran for each cell with $2^{R} = 4$ points:



Optimal Scalar Successive Refinement

Optimal average-stage MMSE of scalar successive refinement [Dumitrescu-Wu IT'09]

- Generalized Lloyd-Max can be constructed [Brunk-Farvardin '96]
- Converges to **optimal average-stage MMSE**
- Extends Trushkin's conditions to successive refinement setting
- Conditions are satisfied for log-concave pdfs

LQR for $\alpha \neq 1$

- $J = J_1 + \alpha^2 J_2 + \cdots + \alpha^{2(T-1)} J_t$
- Adequate generalized LM algorithm can be constructed
- Converges to optimal weighted MSE J for log-concave pdfs

Optimal Quantization for LQG

- We saw how to construct optimal quantizers for LQR
- How to construct optimal quantizers for LQG control?
- Input pdf at every stage is log-concave
- Variant of generalized Lloyd-Max quantization will be optimal
- What variant to use would be dictated by dynamic program
- How to construct a good low-complexity scheme?

Complementary Results

High Resolution: Bennett's (Approx.) Optimal Quantizer

- Assume a large number of points
- Overload noise (noise outside dynamic range) is negligible
- Quantization points "can" be approximated by continuous pdf
- Optimal quantization points distribution $\propto f_X^{1/3}$
- Optimal distortion $= \frac{1}{12N^2} \|f_X\|_{1/3}$
- Under these assumptions, successively refinable
  (no tensions between J_1 and J_2)