Transmission over Arbitrarily Permutated Parallel Gaussian Channels

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Arbitrarily Permuted Parallel Channels

Statistics of all channels \( \{p_i(y|x)\} \) are known

**Order** of channels is known only to Rx (but not to Tx!)

Equivalent to **compound channel / multicast problem**

\[ \downarrow \]

capacity is known

Schemes: [Willems, Gorokhov '08][Hof, Sason, Shamai '10]
Arbitrarily Permuted **Gaussian** Parallel Channels

\[ x_1, x_2, \ldots, x_K \]

\[ z_1, z_2, \ldots, z_K \]

- \( z_j \) – AWGN \( \mathcal{CN}(0, 1) \)
- Power constraints: \( \mathbb{E}[x_j^2] \leq 1 \)

**Motivation**

Frequency bins in OFDM.
Goal

Construct a practical capacity-approaching scheme:

- Capacity-achieving

- **practical** = use only:
  - “off-the-shelf” fixed-SNR SISO AWGN codes
  - Standard ("black box") encoding/decoding
  - Signal processing
Equivalent Channel/Scenario Representations

Gaussian Parallel Channels

\[ y_j = \alpha_j x_{\pi(j)} + z_j, \quad z_j \sim \mathcal{CN}(0, 1), \quad \mathbb{E}[x_j^2] \leq 1 \]

- Equivalent MIMO multicast channel with matrices:

\[
H_i = \begin{pmatrix}
\alpha_{\pi_i(1)} & 0 & \cdots & 0 \\
0 & \alpha_{\pi_i(2)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha_{\pi_i(K)}
\end{pmatrix}; \quad \pi_i \in S_K \\
i = 1, \ldots, K!
\]

- Capacity of equivalent compound channel:

\[
C = \sum_{j=1}^{K} \log \left( 1 + |\alpha_j|^2 \right)
\]
“Bottleneck” Problem

Channel matrices:

\[ H_i = \begin{pmatrix} 
\alpha_{\pi_i(1)} & 0 & \cdots & 0 \\
0 & \alpha_{\pi_i(2)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha_{\pi_i(K)} 
\end{pmatrix} ; \quad \pi_i \in S_K 
\]

\[ i = 1, \ldots, K! \]

Naïve approach

- Use SISO coding and decoding over each SISO sub-channel
- Rate limited to the minimum gain of all users!
“Bottleneck” Problem

Naïve approach

- Use SISO coding and decoding over each SISO sub-channel
- Rate limited to the minimum gain of all users!

Example

\[
H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}
\]

- Gains of first sub-channel to both users = 1, 2
- \( R_1 = \log(1 + \min\{1^2, 2^2\}) = \log(1 + 1) \)
- Gains of second sub-channel to both users = 2, 1
- \( R_2 = \log(1 + \min\{1^2, 2^2\}) = \log(1 + 1) \)
- \( C = \log(1 + 1^2) + \log(1 + 2^2) > R_1 + R_2 \)
Diagonal Form → Triangular Form

Idea
- Apply unitary operation on the right (@Tx)
  ⇒ power constraints unchanged
- Apply (different) unitary operations on the left (@Rx)
  ⇒ noise statistics unchanged
- Shape diagonals to be the same for all users

Problem
- Cannot be achieved for diagonal matrices...
- But... Triangular form suffices!
Joint Equi-Diagonal Triangularization (JET)

\[ U_i^\dagger H_i V = T_i = \begin{pmatrix}
  t_1 & * & \cdots & * \\
  0 & t_2 & \cdots & * \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & t_K
\end{pmatrix} \]

\[ \text{diag}(T_1) = \text{diag}(T_2) = \cdots = \text{diag}(T_k) = \cdots. \]

Example for \( K = 2 \)

\[ \tilde{y}_1 = [T]_{11} x_1 + [T]_{12} x_2 + \tilde{z}_1 \]
\[ \tilde{y}_2 = 0 x_1 + [T]_{22} x_2 + \tilde{z}_2 \]
Previously known results for general matrices

- Possible for $K=2$ matrices [Khina, Kochman, Erez ’12]
- Not possible for more...
- We have $K!$ matrices!
- But... The matrices are of special structure!
Equivalence to Cholesky Decomposition

Joint Equi-Diagonal Triangularization (JET)

\[ U_i^\dagger H_i V = T_i = \begin{pmatrix} t_1 & * & \cdots & * \\ 0 & t_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_K \end{pmatrix} \]

JET Revisited – Cholesky Decomposition

\[ T_i^\dagger T_i = V^\dagger H_i^\dagger U U^\dagger H_i V = V^\dagger H_i^\dagger H_i V \]

**Goal:** Look for \( V \) which provides Cholesky decompositions of \( V^\dagger H_i^\dagger H_i V \) with equal diagonals for all users
Joint Equidiagonal Triangularization for $K = 2$

$H_1^\dagger H_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $H_2^\dagger H_2 = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$

$V^\dagger H_i^\dagger H_i V = T_i^\dagger T_i$, $T_i = \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix}$

$V =$ Hadamard Matrix

$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$t_1^2 = \frac{a + b}{2}$, $t_1^2 t_2^2 = ab$

- Precoding does not depend on $a, b$ (but the rates do)
- Real-valued precoding matrix suffices
Space Only: $K = 3$ Parallel Channels

### Joint Equidiagonal Triangularization for $K = 3$

\[
H_1^\dagger H_1 = \begin{pmatrix}
a & 0 & 0 \\ 
0 & b & 0 \\ 
0 & 0 & c
\end{pmatrix}
\]

\[
V^\dagger H_i^\dagger H_i V = T_i^\dagger T_i, \quad T_i = \begin{pmatrix}
t_1 & * & * \\ 
0 & t_2 & * \\ 
0 & 0 & t_3
\end{pmatrix}
\]

### DFT Matrix

\[
V = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\ 
e^{j\frac{2\pi}{3}} & e^{j\frac{2\pi}{3}} & e^{-j\frac{2\pi}{3}} \\ 
e^{-j\frac{2\pi}{3}} & e^{-j\frac{2\pi}{3}} & e^{j\frac{2\pi}{3}}
\end{pmatrix}
\]

\[
t_1^2 = \frac{a + b + c}{3}, \quad t_1^2 t_2^2 = \frac{ab + ac + bc}{3}, \quad t_1^2 t_2^2 t_3^2 = abc
\]

- Again precoding does not depend on $a, b, c$
- Complex-valued precoding matrix
Space Only for $K = 4$ Parallel Channels?

Joint Equidiagonal Triangularization for $K = 4$

$$H_1^\dagger H_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

$$V^\dagger H_i^\dagger H_i V = T_i^\dagger T_i, \quad T_i = \begin{pmatrix} t_1 & * & * & * \\ 0 & t_2 & * & * \\ 0 & 0 & t_3 & * \\ 0 & 0 & 0 & t_4 \end{pmatrix}$$

Problem

- FFT matrix does not work
- Hadamard matrix does not work either
- No other real/complex unitary $V$ applies, in general
Space–Time Coding

- **1 complex channel use** materialized by **2 real channel uses**
Space–Time Coding Structure

\[ T_i = U_i^\dagger H_i V \]

- Bunch two channel uses together:

\[
\begin{bmatrix}
  T_i \\
  0 & T_i
\end{bmatrix} = \begin{bmatrix}
  U_i^\dagger \\
  0 & U_i^\dagger
\end{bmatrix} \begin{bmatrix}
  H_i \\
  0 & H_i
\end{bmatrix} \begin{bmatrix}
  V \\
  0 & V
\end{bmatrix}
\]

- \( H_i \) have a block-diagonal structure
- Use general \( U_i, V \) (not block-diagonal):

\[
T_i = (U_i)^\dagger \begin{bmatrix}
  \mathcal{H}_i \\
  0 & \mathcal{H}_i
\end{bmatrix} V
\]

- Exploit block-diagonal structure of time-extended channels \( \mathcal{H}_i \)
$K = 4$ Parallel Channels

**Difficulty**

- Search for $8 \times 8$ complex matrix becomes hard
- Instead, restrict search to special structure
- “Natural” time-extension representation of $\text{real} \rightarrow \text{complex}$
- “Natural” time-extension of $\text{complex} \rightarrow \text{quaternion}$
Quaternions [Hamilton 1844]

\[ q = a + bi + cj + dk \]
\[ a, b, c, d \in \mathbb{R} \]

\[ i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = j \]

### Quaternions over reals

\[ q = a + bi + cj + dk \]
\[ \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \]
\[ a, b, c, d \in \mathbb{R} \]

### Quaternions over complex

\[ q = (a + bi) + j(c - di) = z_1 + jz_2 \]
\[ \begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix} \]
\[ z_1, z_2 \in \mathbb{C} \]
Quaternions [Hamilton 1844]

Why Quaternions?

- Associative:
  \[ q_1(q_2q_3) = (q_1q_2)q_3 \]

- Exists an inner product:
  \[ (u, v) = \sum_{i=1}^{n} u_i^* v_i \]

- \((a + bi + cj + dk)^* \triangleq a - bi - cj - dk\)

- \(\Rightarrow\) Gram-Schmidt is possible

- Also possible: QR and Cholesky decompositions

- All the desired properties of the complex

- (but not commutative!)
Space–Time via Quaternions for $K = 4$

**Equi-diagonal Triangularization over Quaternions**

$$H_1^\dagger H_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

$$V^\dagger H_i^\dagger H_i V = T_i^\dagger T_i, \quad T_i = \begin{pmatrix} t_1 & * & * & * \\ 0 & t_2 & * & * \\ 0 & 0 & t_3 & * \\ 0 & 0 & 0 & t_4 \end{pmatrix}$$

**The solution (up to degrees of freedom...)**

$$V = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & x & i & iy \\ 1 & z & -1 & -z \\ 1 & y & -i & -ix \end{pmatrix}$$

$$x = \frac{1}{3}(-1 - 2i - \sqrt{2}j + \sqrt{2}k), \quad y = \frac{1}{3}(-1 + 2i - \sqrt{2}j - \sqrt{2}k), \quad z = \frac{1}{3}(-1 + 2\sqrt{2}j)$$
### Equi-diagonal Triangularization over Quaternions

\[ H_1^\dagger H_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \]

\[ V^\dagger H_i^\dagger H_i V = T_i^\dagger T_i \], \quad T_i = \begin{pmatrix} t_1 & * & * & * \\ 0 & t_2 & * & * \\ 0 & 0 & t_3 & * \\ 0 & 0 & 0 & t_4 \end{pmatrix} \]

### The diagonal values

\[ t_1^2 = \frac{A + B + C + D}{4} \], \quad t_1^2 t_2^2 = \frac{AB + AC + AD + BC + BD + CD}{6} \]

\[ t_1^2 t_2^2 t_3^2 = \frac{ABC + ABD + ACD + BCD}{4} \], \quad t_1^2 t_2^2 t_3^2 t_4^2 = ABCD \]
Space–Time for $K > 4$

$K = 5$ and $K = 6$
- There exist quaternion solutions!
- Coefficients found numerically (unlike in $K = 4$ case)

$K \geq 7$
- Problem becomes computationally hard
- Bigger structures might be needed (Clifford/cyclic-division algebras?)