Transmission over Arbitrarily Permuted Parallel Gaussian Channels

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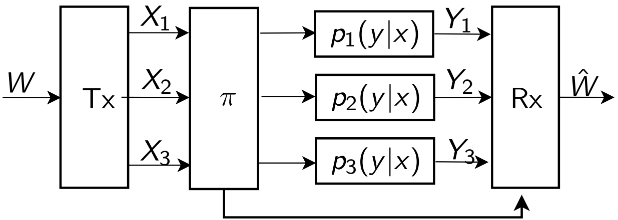
ISIT 2012
MIT, Cambridge, MA
July 6

Arbitrarily Permuted Parallel Channels



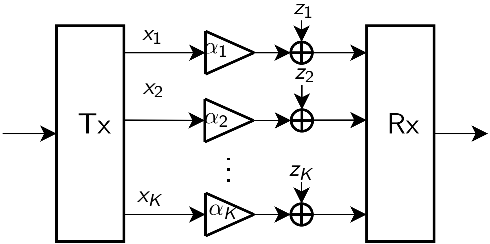
- Statistics of all channels $\{p_i(y|x)\}$ are known
- **Order** of channels is known **only to Rx** (but **not to Tx**!)
- Equivalent to **compound channel** / **multicast problem**

$$\Downarrow$$

capacity is known

- Schemes: [Willems, Gorokhov '08][Hof, Sason, Shamai '10]

Arbitrarily Permuted **Gaussian** Parallel Channels



- z_j – AWGN $\mathcal{CN}(0, 1)$
- Power constraints: $\mathbb{E}[x_j^2] \leq 1$

Motivation

Frequency bins in OFDM.

Practical Scheme

Goal

Construct a practical capacity-approaching scheme:

- Capacity-achieving
- **practical** = use only:
  - "off-the-shelf" **fixed-SNR** SISO AWGN codes
  - Standard ("black box") encoding/decoding
  - Signal processing

Equivalent Channel/Scenario Representations

Gaussian Parallel Channels

$$y_j = \alpha_j x_{\pi(j)} + z_j \,, \quad z_j \sim \mathcal{CN}(0,1)\,, \quad \mathbb{E}\left[x_j^2\right] \le 1$$

- **Equivalent MIMO multicast channel with matrices:**

$$H_i = \begin{pmatrix} \alpha_{\pi_i(1)} & 0 & \cdots & 0 \\ 0 & \alpha_{\pi_i(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{\pi_i(K)} \end{pmatrix} ; \qquad \begin{matrix} \pi_i \in S_K \\ i = 1, \ldots, K! \end{matrix}$$

- **Capacity of equivalent compound channel:**

$$C = \sum_{j=1}^{K} \log\left(1 + |\alpha_j|^2\right)$$

“Bottleneck” Problem

Channel matrices:

$$H_i = \begin{pmatrix} \alpha_{\pi_i(1)} & 0 & \cdots & 0 \\ 0 & \alpha_{\pi_i(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{\pi_i(K)} \end{pmatrix} ; \qquad \begin{matrix} \pi_i \in S_K \\ i = 1, \ldots, K! \end{matrix}$$

Naïve approach

- Use SISO coding and decoding over each SISO sub-channel
- **Rate limited to the minimum gain of all users!**

"Bottleneck" Problem

Naïve approach

- Use SISO coding and decoding over each SISO sub-channel
- **Rate limited to the minimum gain of all users!**

Example

$$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} , \; H_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

- Gains of first sub-channel to both users $= 1, 2$
- $R_1 = \log(1 + \min\{1^2, 2^2\}) = \log(1 + 1)$
- Gains of second sub-channel to both users $= 2, 1$
- $\Rightarrow R_2 = \log(1 + \min\{1^2, 2^2\}) = \log(1 + 1)$
- $C = \log(1 + 1^2) + \log(1 + 2^2) > R_1 + R_2$

Diagonal Form $\rightarrow$ Triangular Form

Idea

- Apply unitary operation on the right (@Tx)

  $\Rightarrow$ power constraints unchanged

- Apply (different) unitary operations on the left (@Rx)

  $\Rightarrow$ noise statistics unchanged

- Shape diagonals to be the **same for all users**

Problem

- Cannot be achieved for diagonal matrices...
- But... **Triangular form suffices!**

Diagonal Form → Triangular Form

Joint Equi-Diagonal Triangularization (JET)

$$U_i^{\dagger} H_i V = T_i = \begin{pmatrix} t_1 & * & \cdots & * \\ 0 & t_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_K \end{pmatrix}$$

$$\mathrm{diag}(T_1) = \mathrm{diag}(T_2) = \cdots = \mathrm{diag}(T_k) = \cdots .$$

Example for $K = 2$

$$\begin{aligned} \tilde{y}_1 &= [T]_{11} x_1 + \overbrace{[T]_{12} x_2}^{\text{Interference}} + \tilde{z}_1 \\ \tilde{y}_2 &= 0\, x_1 + [T]_{22} x_2 + \tilde{z}_2 \end{aligned}$$

Diagonal Form $\rightarrow$ Triangular Form

Previously known results for general matrices

- Possible for K=2 matrices [Khina, Kochman, Erez '12]
- Not possible for more...
- We have $K!$ matrices!
- But... The matrices are of **special strucure**!

Equivalence to Cholesky Decomposition

Joint Equi-Diagonal Triangularization (JET)

$$U_i^\dagger H_i V = T_i = \begin{pmatrix} t_1 & * & \cdots & * \\ 0 & t_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_K \end{pmatrix}$$

JET Revisited – Cholesky Decomposition

$$T_i^\dagger T_i = V^\dagger H_i^\dagger \cancel{U U^\dagger} H_i V = V^\dagger H_i^\dagger H_i V$$

- **Goal:** Look for V which provides Cholesky decompositions of $V^\dagger H_i^\dagger H_i V$ with equal diagonals for all users

Space Only: $K = 2$ Parallel Channels

Joint Equidiagonal Triangularization for $K = 2$

$$H_1^\dagger H_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad H_2^\dagger H_2 = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$$

$$V^\dagger H_i^\dagger H_i V = T_i^\dagger T_i \,, \quad T_i = \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix}$$

V = Hadamard Matrix

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$t_1^2 = \frac{a+b}{2}, \quad t_1^2 t_2^2 = ab$$

- Precoding does not depend on a, b (but the rates do)
- Real-valued precoding matrix suffices

Space Only: $K = 3$ Parallel Channels

Joint Equidiagonal Triangularization for $K = 3$

$$H_1^\dagger H_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$V^\dagger H_i^\dagger H_i V = T_i^\dagger T_i \,, \quad T_i = \begin{pmatrix} t_1 & * & * \\ 0 & t_2 & * \\ 0 & 0 & t_3 \end{pmatrix}$$

DFT Matrix

$$V = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{3}} & e^{-j\frac{2\pi}{3}} \\ 1 & e^{-j\frac{2\pi}{3}} & e^{j\frac{2\pi}{3}} \end{pmatrix}$$

$$t_1^2 = \frac{a+b+c}{3} \,, \quad t_1^2 t_2^2 = \frac{ab+ac+bc}{3} \,, \quad t_1^2 t_2^2 t_3^2 = abc$$

- Again precoding does not depend on a, b, c
- Complex-valued precoding matrix

Space Only for $K = 4$ Parallel Channels?

Joint Equidiagonal Triangularization for $K = 4$

$$H_1^\dagger H_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

$$V^\dagger H_i^\dagger H_i V = T_i^\dagger T_i \,, \quad T_i = \begin{pmatrix} t_1 & * & * & * \\ 0 & t_2 & * & * \\ 0 & 0 & t_3 & * \\ 0 & 0 & 0 & t_4 \end{pmatrix}$$

Problem

- FFT matrix does not work
- Hadamard matrix does not work either
- No other real/complex unitary V applies, in general

Space–Time

$K = 2$

- Hadamard Matrix
- 2×2 Real-Valued: $\mathbb{R}^2 \to \mathbb{R}^2$

$K = 3$

- FFT Matrix
- 3×3 Complex-valued: $\mathbb{C}^3 \to \mathbb{C}^3$
- Can be materialized via $\mathbb{R}^6 \to \mathbb{R}^6$:

$$(a + ib) \Longleftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Space–Time Coding

- **1 complex channel use** materialized by **2 real channel uses**

Space–Time Coding Structure

$$T_i = U_i^\dagger H_i V \qquad \text{✗}$$

- **Bunch two channel uses together:**

$$\overbrace{\begin{pmatrix} T_i & \mathbf{0} \\ \mathbf{0} & T_i \end{pmatrix}}^{\mathcal{T}_i} = \overbrace{\begin{pmatrix} U_i^\dagger & \mathbf{0} \\ \mathbf{0} & U_i^\dagger \end{pmatrix}}^{\mathcal{U}_i^\dagger} \overbrace{\begin{pmatrix} H_i & \mathbf{0} \\ \mathbf{0} & H_i \end{pmatrix}}^{\mathcal{H}_i} \overbrace{\begin{pmatrix} V & \mathbf{0} \\ \mathbf{0} & V \end{pmatrix}}^{\mathcal{V}} \qquad \text{✗}$$

- $\mathcal{H}_i$ have a block-diagonal structure
- Use general $\mathcal{U}_i$, $\mathcal{V}$ (**not** block-diagonal):

$$\mathcal{T}_i = \left(\mathcal{U}_i\right)^\dagger \overbrace{\begin{pmatrix} H_i & \mathbf{0} \\ \mathbf{0} & H_i \end{pmatrix}}^{\mathcal{H}_i} \left(\mathcal{V}\right) \qquad \checkmark$$

- Exploit block-diagonal structure of time-extended channels $\mathcal{H}_i$

$K = 4$ Parallel Channels

Difficulty

- Search for 8×8 complex matrix becomes hard
- Instead, restrict search to special structure
- "Natural" time-extension representation of **real** $\rightarrow$ **complex**
- "Natural" time-extension of **complex** $\rightarrow$ **quaternion**

Quaternions [Hamilton 1844]

$$q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$$
$$a, b, c, d \in \mathbb{R}$$

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1\,, \quad \mathrm{ij} = \mathrm{k}\,, \quad \mathrm{jk} = \mathrm{i}\,, \quad \mathrm{ki} = \mathrm{j}$$

Quaternions over reals

$$q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$$
$$\Updownarrow$$
$$\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$$
$$a, b, c, d \in \mathbb{R}$$

Quaternions over complex

$$q = \overbrace{(a + b\mathrm{i})}^{z_1} + \mathrm{j}\,\overbrace{(c - d\mathrm{i})}^{z_2} = z_1 + \mathrm{j}z_2$$
$$\Updownarrow$$
$$\begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix}$$
$$z_1, z_2 \in \mathbb{C}$$

Quaternions [Hamilton 1844]

Why Quaternions?

- Associative:

$$q_1(q_2 q_3) = (q_1 q_2) q_3$$

- Exists an inner product:

$$(u, v) = \sum_{i=1}^{n} u_i^* v_i$$

$$(a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k})^* \triangleq a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}$$

- $\Rightarrow$ Gram-Schmidt is possible
- Also possible: QR and Cholesky decompositions
- All the desired properties of the complex
- (but not cummutative!)

Space–Time via Quaternions for $K = 4$

Equi-diagonal Triangularization over Quaternions

$$H_1^\dagger H_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

$$V^\dagger H_i^\dagger H_i V = T_i^\dagger T_i \,, \quad T_i = \begin{pmatrix} t_1 & * & * & * \\ 0 & t_2 & * & * \\ 0 & 0 & t_3 & * \\ 0 & 0 & 0 & t_4 \end{pmatrix}$$

The solution (up to degrees of freedom...)

$$V = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & x & \mathrm{i} & \mathrm{i}y \\ 1 & z & -1 & -z \\ 1 & y & -\mathrm{i} & -\mathrm{i}x \end{pmatrix}$$

$$x = \frac{1}{3}(-1-2\mathrm{i}-\sqrt{2}\mathrm{j}+\sqrt{2}\mathrm{k}) \,, \quad y = \frac{1}{3}(-1+2\mathrm{i}-\sqrt{2}\mathrm{j}-\sqrt{2}\mathrm{k}) \,, \quad z = \frac{1}{3}(-1+2\sqrt{2}\mathrm{j})$$

Space–Time via Quaternions for $K = 4$

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$$V^\dagger H_i^\dagger H_i V = T_i^\dagger T_i\,, \quad T_i = \begin{pmatrix} t_1 & * & * & * \\ 0 & t_2 & * & * \\ 0 & 0 & t_3 & * \\ 0 & 0 & 0 & t_4 \end{pmatrix}$$

The diagonal values

$$t_1^2 = \frac{A+B+C+D}{4}\,, \quad t_1^2 t_2^2 = \frac{AB+AC+AD+BC+BD+CD}{6}\,,$$

$$t_1^2 t_2^2 t_3^2 = \frac{ABC+ABD+ACD+BCD}{4}\,, \quad t_1^2 t_2^2 t_3^2 t_4^2 = ABCD$$

Space–Time for $K > 4$

$K = 5$ and $K = 6$

- There exist quaternion solutions!
- Coefficients found numerically (unlike in $K = 4$ case)

$K \geq 7$

- Problem becomes computationally hard
- Bigger structures might be needed (Clifford/cyclic-division algebras?)