On Robust Dirty Paper Coding

Anatoly Khina* Uri Erez*

*Dept. EE - Systems, Tel Aviv University, Tel Aviv, Israel

May 6th, 2008
1 Background
   - DP Channel
   - Lattice Strategies for DPC / Tomlinson-Harashima Precoding
   - Compound DP Channel

2 Robust DPC - Smart Rx
   - Finite $P_S$
   - Infinite $P_S$

3 Robust DPC - Smart Tx and Rx

4 Concluding Remarks and Further Research
Dirty Paper Channel

\[ Y = X + S + N \]

\[ \frac{1}{n} \sum_{i=1}^{n} x_i^2 \leq P_x \]

- \( S \) - interference, known (causally/noncausally) at Tx with average power \( P_S \).

- \( N \) - AWGN with average power \( P_N \).

Costa: \( S \) known (noncausally) @ Tx \( \rightarrow \) as good as no \( S \)

\[ C = \frac{1}{2} \log(1 + \text{SNR}) \]
Applications of Dirty Paper Coding

Model serves as an information-theoretic framework for (known) interference cancellation in:

- ISI channels.
- Information embedding.
- MIMO broadcast channels.

Assumption: Tx knows channel gains.
### Remarks

$P_S$, for time being, is arbitrarily large.

### Transmitter

$$X = [v - \alpha S - U] \mod A_\Delta; \quad A_\Delta = \left[ -\frac{A}{2}, \frac{A}{2} \right]$$

### Receiver

$$Y' = [\alpha Y + U] \mod A_\Delta = [v + N_{\text{eff}}] \mod A_\Delta,$$

$$N_{\text{eff}} = (1 - \alpha)U + \alpha N.$$

### Performances

$$\begin{align*}
\text{SNR}_{\text{eff}} &\triangleq \frac{P_X}{P_{N_{\text{eff}}^{\beta}}} \frac{\alpha_{\text{MMSE}}}{\text{SNR+1}} \rightarrow 1 + \text{SNR}. \\
R &\geq \frac{1}{2} \log(\text{SNR}_{\text{eff}}) - \frac{1}{2} \log \frac{2\pi e}{12} = \frac{1}{2} \log(1 + \text{SNR}) - \frac{1}{2} \log \frac{2\pi e}{12}. 
\end{align*}$$

- **Causality Loss**
Compound Dirty Paper Channel

Dirty paper channel with perfect channel knowledge:

\[ Y = X + S + N \]

SNR \( \triangleq \frac{P_X}{P_N} \)

SIR \( \triangleq \frac{P_X}{P_S} \)
Compound dirty paper channel:

\[ Y = X + \frac{S}{\beta} + N \]

- \( \beta \in [1 - \delta, 1 + \delta] \) is known to Rx.
- \( \beta \) constant (non-ergodic) for whole transmission.
- Compound \( \rightarrow \) Achievable rate - worst case over all \( \beta \) values.
**Compound Channel Case**

**Remark**
\(\alpha\) is used at both ends (Tx and Rx).

**Transmitter Knows \(\beta\)**
The problem reduces to classical DPC case (\(\beta \equiv 1\)).

**Transmitter Does NOT know \(\beta\)**

**What is the best strategy?**
- To work with \(\alpha_{\text{MMSE}}\) at both ends?
- To work with \(\alpha_{\text{MMSE}}\) at Tx and \(\alpha_{\text{MMSE}}/\beta\) - at Rx?
- Other \(\alpha_T, \alpha_R\) selections?
Naïve Approach → Work as Before

Transmitter

\[ X = [v - \alpha S - U] \mod A_\Delta \]

Receiver

\[ Y = [\alpha Y + U] \mod A_\Delta = [v + N_{\text{eff}}^\beta] \mod A_\Delta, \]

\[ N_{\text{eff}}^\beta = (1 - \alpha)U + \alpha \left( \frac{\beta - 1}{\beta} \right) S + \alpha N. \]

Remark

\[ P_S \rightarrow \infty \Rightarrow R = 0 \]

What can we do?
Smart Receiver

**Distinguish between** $\alpha_R$ ($\alpha @ Rx$) and $\alpha_T$ ($\alpha @ Tx$)

**Transmitter**

$$X = [v - \alpha_T S - U] \mod A_\Delta.$$  

**Receiver**

$$Y = [\alpha_R Y + U] \mod A_\Delta = [v + N_{\text{eff}}^\beta] \mod A_\Delta,$$

$$N_{\text{eff}}^\beta = (1 - \alpha_R) U + (\alpha_T \beta - \alpha_R) \frac{S}{\beta} + \alpha_R N.$$  

**Remarks**

- $\alpha_R = \alpha_T$ - **Bad!**
- *For time being*, we optimize w.r.t. $\text{SNR}_{\text{eff}}^{\text{wc}}$.  

**ITW 2008**
MMSE Estimation

\[ \alpha_{R}^{\text{MMSE}} = \frac{1 + \frac{\alpha_{T}^{\text{MMSE}} \beta}{\text{SIR}}}{1 + \frac{1}{\text{SIR}} + \frac{1}{\text{SNR}}} \]

\[ \Rightarrow \text{SNR}_{\text{eff}}^{\text{wc}} = (1 + \text{SNR}) \left( 1 + \frac{1}{\text{SNR}} + \frac{1}{\text{SIR}} \right) \frac{1 + \frac{1}{\text{SNR}} + \frac{1}{\text{SIR}}}{\left( 1 + \frac{1}{\text{SIR}} + \frac{1}{\text{SNR}} \right) + \frac{\text{SNR}}{\text{SIR}} (1 - \beta)^2} \]

High SNR Limit

- \( \alpha_{T} \rightarrow 1 \).

- \( \alpha_{R}^{\text{MMSE}} = \frac{1 + \frac{\beta}{\text{SIR}}}{1 + \frac{1}{\text{SIR}}} = \frac{\text{SIR} + \beta}{\text{SIR} + 1} \).

- Optimized \( \alpha_{R} \Rightarrow \text{SNR}_{\text{eff}}^{\text{wc}} = \frac{1 + \text{SIR}}{(1 - \beta)^2} \).

- Non-Optimized \( \alpha_{R} \Rightarrow \text{SNR}_{\text{eff}}^{\text{wc}} = \frac{\text{SIR}}{(1 - \beta)^2} \).
Example: \( \text{SIR} = 1 \)

Remark

Even for \( \text{SIR} = 1 \), the gain is 3dB, in the limit of high SNR.
Conclusions

- *In the high SNR regime, the scheme is interference limited due to $\beta$.*

- *Ignoring $\beta$ at Rx is suboptimal and could lead to significant rate losses.*

- *For large $S$, Rx partially compensates for ignorance of Tx.*

\[
\max SNR^{wc}_{eff} \text{ criterion } \neq \max MI \text{ criterion,} \\
\text{since the effective noise is not Gaussian.}
\]
Random Dirty Paper Strategies

- So far - “Smart Rx”.
- Tx does not know $\beta$, but is aware of its ignorance...
- Can Tx do better?

How about Guessing $\beta$?

- $\alpha_T$ varies from symbol to symbol.
- Common Randomness: for any $\alpha_T$, Rx uses optimal $\alpha_R$ as previously.
- Increases (worsens) the MSE.
- Improves the mutual-information.
Motivation

Remark
Concentrate on the case of $P_N = 0$.

Example ($B = 2$)

- $\beta \in \{1 \pm \delta\}$
- $\alpha_T \equiv 1$ minimizes the MSE but achieves a finite rate.
- $P(\alpha_T = 1 - \delta) = P(\alpha_T = 1 + \delta) = 0.5$
  obtains a larger MSE, but achieves infinite rate.
Back to our case of interest...

\[ \beta \in [1 - \delta, 1 + \delta] \]

- Optimal distribution of \( \alpha_T \rightarrow \) numerical optimization.

- Lower bounds by specific choices of distribution of \( \alpha \).

- Derived upper bound

\[
\log(1 + \delta) - \log(\delta) + 1.
\]
Robust DPC Performances

\[ f(\alpha) \]

\[ R \text{ [nats]} = 1 + \delta f(\alpha) \]

\[ P(\alpha=1) = 1 \]
Robust DPC Performances

\[ f(\alpha) \]

\[ R \text{ [nats]} \]

\[ \alpha \sim \text{Unif}[\delta/2, \delta/2) \]

\[ \Pr(\alpha = 1) = 1 \]
Robust DPC Performances

\[ f(\alpha) \]

\[ R \text{ [nats]} \]

\[ \delta \]

\[ V\text{-like} \]

\[ \alpha \sim \text{Unif}[-\delta/2, \delta/2) \]

\[ P(\alpha=1)=1 \]

ITW 2008
Open Question: How tight is the Upper Bound?
Further Research

- Multidimensional Lattices attain smaller gain than in classical DP channel.

- Finding the Optimal $\alpha_T$ distribution for the robust DPC scheme.

- Quantifying gain at finite-SNR DPC.

- Finding the capacity of the compound DP channel.

- Capacity of the “Fast-Fading” DP channel.
Noisy Performances for SNR = 17dB

\[ R \text{ [nats]} = \begin{cases} \alpha & \text{if } \alpha = 1 \\ \alpha = \alpha_{\text{MMSE}} & \text{else} \end{cases} \]

\[ \text{Unif}[1-\delta,1+\delta] \]

\[ \text{Unif}[\alpha_{\text{MMSE}}(1-\delta),\alpha_{\text{MMSE}}(1+\delta)] \]