Multi-Rate Control over AWGN Channels via Analog Coding

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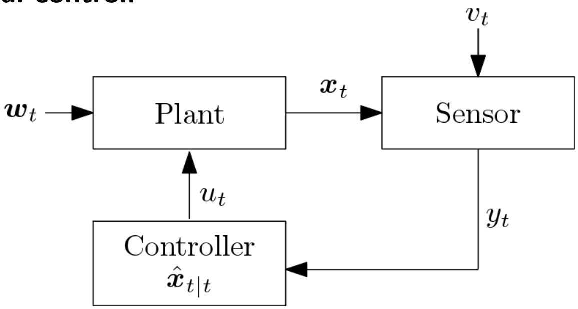
Networked Control vs. Traditional Control

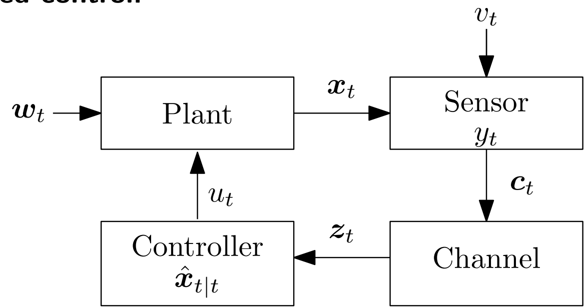
Traditional control:



- Observer and controller are co-located.
- Classical systems are hardwired and well crafted

Networked Control vs. Traditional Control

Networked control:



- Observer and controller are not co-located: connected through noisy link
- Suitable for new remote applications (e.g., remote surgery, self-driving cars)

Linear Quadratic Gaussian Control over Gaussian Channels

Linear quadratic Gaussian (LQG) system

$$\boldsymbol{x}_{t+1} = \mathbf{A}\boldsymbol{x}_t + \mathbf{B}\boldsymbol{u}_t + \boldsymbol{w}_t, \qquad \boldsymbol{w}_t \sim \text{ i.i.d. } \mathcal{N}(0, \mathbf{W})$$
$$\boldsymbol{y}_t = \mathbf{C}\boldsymbol{x}_t + \boldsymbol{v}_t, \qquad \boldsymbol{v}_t \sim \text{ i.i.d. } \mathcal{N}(0, \mathbf{V})$$

Additive white Gaussian noise (AWGN) channel

$$\boldsymbol{b}_t = \mathbf{H}\boldsymbol{a}_t + \boldsymbol{n}_t, \qquad \boldsymbol{n}_t \sim \text{ i.i.d. } \mathcal{N}(0, \mathbf{N})$$

- Power constraint: $\mathbb{E}\left[\boldsymbol{a}_t^T \boldsymbol{a}_t\right] = \sum_i \mathbb{E}\left[a_{t;i}^2\right] \leq P \cdot \text{length}(\boldsymbol{a}_t)$

LQG cost

$$J = \sum_{t=1}^{T} \left[\boldsymbol{x}_t^T \mathbf{Q}\boldsymbol{x}_t + \boldsymbol{u}_t^T \mathbf{R}\boldsymbol{u}_t\right] + \boldsymbol{x}_{T+1}^T \mathbf{F}\boldsymbol{x}_{T+1}$$

Linear Quadratic Gaussian Control over Gaussian Channels

Scalar Linear quadratic Gaussian (LQG) system

$$x_{t+1} = x_t + u_t + w_t, \qquad w_t \sim \text{ i.i.d. } \mathcal{N}(0, W)$$
$$y_t = x_t + v_t, \qquad v_t \sim \text{ i.i.d. } \mathcal{N}(0, V)$$

Scalar Additive white Gaussian noise (AWGN) channel

$$b_t = a_t + n_t, \qquad n_t \sim \text{ i.i.d. } \mathcal{N}(0, N)$$

- Power constraint: $\mathbb{E}\left[a_t^2\right] \leq P$;

LQG cost

$$J = \sum_{t=1}^{T} \left[Qx_t^2 + Ru_t^2\right] + Fx_{T+1}^2$$

Linear Quadratic Gaussian Control over Gaussian Channels

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Scalar Additive white Gaussian noise (AWGN) channel

$$b_t = a_t + n_t, \qquad n_t \sim \text{ i.i.d. } \mathcal{N}(0, N)$$

- Power constraint: $\mathbb{E}\left[a_t^2\right] \le P$; w.l.o.g. $P = 1$, $N = 1/\text{SNR}$

LQG cost

$$J = \sum_{t=1}^{T} \left[Qx_t^2 + Ru_t^2\right] + Fx_{T+1}^2$$

Linear Quadratic Gaussian Control over Gaussian Channels

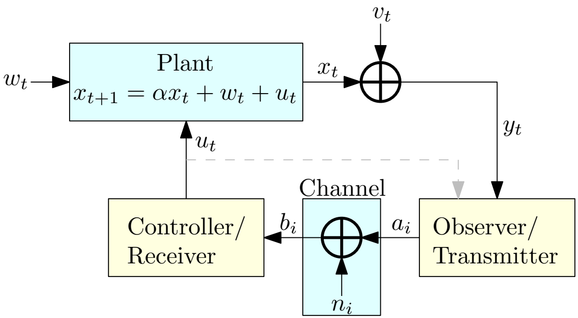
Scalar LQG system

$$x_{t+1} = x_t + u_t + w_t$$
$$y_t = x_t + v_t$$

Scalar AWGN channel

$$b_t = a_t\left(y^t, u^{t-1}\right) + n_t$$

- Power constraint: $\mathbb{E}\left[a_t^2\right] \le P$



Linear Quadratic Gaussian Control over Gaussian Channels

Control sampling rate $\neq$ Communication signaling rate!

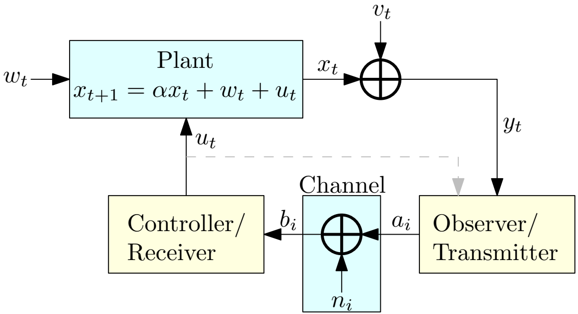
Scalar LQG system

$$x_{t+1} = x_t + u_t + w_t$$
$$y_t = x_t + v_t$$

Scalar AWGN channel

$$b_i = a_i + n_i$$

- Power constraint: $\mathbb{E}\left[a_i^2\right] \leq P$



Control Sampling Rate vs. Communication Signaling Rate

- How fast the plant dynamic is $\Rightarrow$ Control sampling rate
- Bandwidth available $\Rightarrow$ Communication signaling rate
- Communication rate can be much higher in practice

How to benefit from excess signaling rate (bandwidth)?

Networked Control Approaches: Information-Theoretic Separation

Information-theoretic separation

- Requires large blocks **(delay!)** of source samples and channel uses
- **Suboptimal for control!**
- Requires codes with strong "anytime reliability" properties [Schulman IT'96][Sahai-Mitter IT'06][Sukhavasi-Hassibi AC'16]
- Problematic in practice: Convolutional code with infinite memory [Kh.-Halbawi-Hassibi ISIT'16]

Packeting

- Assumes communication rate $\gg$ control rate, very good SNR
- Problem reduces to control-oriented quantization *[My second talk today]*
- Bad channel events are translated to packet drops / delays

Networked Control Approaches: Joint Source–Channel Coding (JSCC)

- What to do when control and communication rates are close?
- Can we do better than IT-separation?

Less familiar IT avenue

- Low-delay joint source–channel coding (JSCC)
- Control sample corresponds to source sample

1 : 1 JSCC: Rate-Matched Case

- One AWGN channel use per one control sample

1 : 1 Optimal JSCC [Goblick IT'65]

1 : 1 optimal JSCC distortion = n : n optimal JSCC distortion

- No loss of performance
- **Analog scheme is optimal:** $a_t = \sqrt{\frac{P}{P_x}} x_t$

1 : 1 JSCC: Rate-Matched Case

Scheme

- For simplicity, assume fully-observable case (can be extended to partially-observable case)

Observer/Transmitter:

- Generates the "source" signal: $s_t = x_t - \hat{x}_{t|t-1} = \tilde{x}_{t|t-1}$
- Adjusts power and transmits: $a_t = s_t / \sqrt{P_{t|t-1}}$

Controller/Receiver:

- Receives $b_t = a_t + n_t = \tilde{x}_{t|t-1} / \sqrt{P_{t|t-1}} + n_t$
- Applies Kalman filtering: $\begin{cases} \hat{x}_{t|t} = \hat{x}_{t|t-1} + \sqrt{P_{t|t-1}} \frac{\mathrm{SNR}}{1+\mathrm{SNR}} b_t \\ \hat{x}_{t|t-1} = \alpha \hat{x}_{t-1|t-1} + u_{t-1} \end{cases}$
- Generates LQG control signal: $u_t = -L_t \hat{x}_{t|t}$

1 : 1 JSCC: Rate-Matched Case

- We reduced problem to that of classical LQG control

LQR coefficients

$$L_t = \frac{\alpha S_{t+1}}{S_{t+1} + R}\,,$$

$$S_t = \frac{\alpha^2 R\, S_{t+1}}{S_{t+1} + R} + Q\,,$$

$$S_T = F\,.$$

Partially-observable case

Scheme can be extended to partially-observable case

- Generates state estimators at the transmitter $\hat{x}^t_{t|t}$ (in addition to $\hat{x}^r_{t|t}$ at the receiver)

1 : 1 JSCC: Rate-Matched Case

LQG cost

- This schemes achieves optimal LQG cost
- Formally proved by applying
  - Shannon's lower bound
  - Entropy-power inequality
  - Tightness of both in Gaussian case
  - Optimality of "1 : 1 JSCC" scheme in the Gaussian case

  in the dynamic-programming solution
  (extension of [Kostina-Hassibi Allerton'16])
- Recovers results of [Freudenberg-Middleton-Solo AC'10] as a special case

Conclusion: No coding is needed!

1 : 1 JSCC: Rate-Matched Case

Infinite-horizon steady-state average-stage LQG cost

$$\bar{J}^{\mathrm{r}} \leq \bar{J}^{\mathrm{t}} + \frac{Q + \left(\alpha^2 - 1\right) S}{1 + \mathsf{SNR} - \alpha^2} W$$
$$\bar{J}^{\mathrm{t}} = SW$$

- S is the positive solution of the DARE
$$S^2 - \left[Q + \left(\alpha^2 - 1\right) R\right] S - QR = 0$$
- System is stabilizabile if and only if $\mathsf{SNR} > \alpha^2 - 1$
- This is in stark contrast to classical LQG

Conclusion: No coding is needed!

1 : 1 JSCC: Rate-Matched Case

Infinite-horizon LQG cost: Partially-observable case

$$\bar{J}^{\mathrm{r}} \le \bar{J}^{\mathrm{t}} + \frac{Q + \left(\alpha^2 - 1\right) S}{1 + \mathrm{SNR} - \alpha^2} (P_t^t - \bar{P}_t^t)$$

$$\bar{J}^{\mathrm{t}} = S(P_t^t - \bar{P}_t^t) + Q\bar{P}_t^t$$

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- System is stabilizabile if and only if $\mathrm{SNR} > \alpha^2 - 1$
- $P^t \triangleq \lim_{t \to \infty} P^t_{t+1|t}$, $\quad \bar{P}^t \triangleq \lim_{t \to \infty} P^t_{t|t}$
- P^t is the positive solution of the DARE
$$\left(P^t\right)^2 - \left[\left(\alpha^2 - 1\right) V + W\right] P^t - VW = 0, \; \bar{P}^t = \frac{P^t V}{P^t + V}$$

Conclusion: No coding is needed!

1 : 1 JSCC: Rate-Matched Case

Infinite-horizon LQG cost: Partially-observable case

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What about 1 : 2 case?

1 : 2 JSCC: Rate-Mismatched Case

- Two AWGN channel uses per one control sample

Naïve scheme: Repetition

Observer/Transmitter: $a_{t;1} = a_{t;2} = \tilde{x}_t / \sqrt{P_{t|t-1}}$

Controller/Receiver: $b_t^{\text{eff}} = \frac{b_{t;1}+b_{t;2}}{2}$

- Reduces to 1 : 1 JSCC with $\text{SNR}^{\text{eff}} = 2\text{SNR}$
- 3dB improvement comes from doubling total transmit power
- Same improvement is attained by
  - Using $2P$ during first channel use
  - Remaining silent during second channel use
- No real improvement due to extra degree of freedom...

Can we do better?

1 : 2 JSCC: Rate-Mismatched Case

Infinite blocklength: "n : 2n JSCC" for $n \to \infty$ [Shannon '48]

$$1 + \mathsf{SNR}^{\text{eff}} = (1 + \mathsf{SNR})^2$$

- Much better than $\mathsf{SNR}^{\text{eff}}_{\text{naïve}} = 2\mathsf{SNR}$ at high SNR

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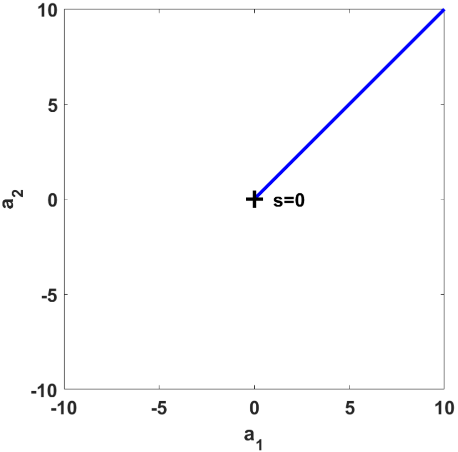
What about 1 : 2 JSCC?

Non-linear mappings can do better! [Kotel'nikov '47][Shannon '49]

1 : 2 JSCC: Rate-Mismatched Case

$$\begin{cases} a_1(s) = s \\ a_2(s) = s \end{cases}$$



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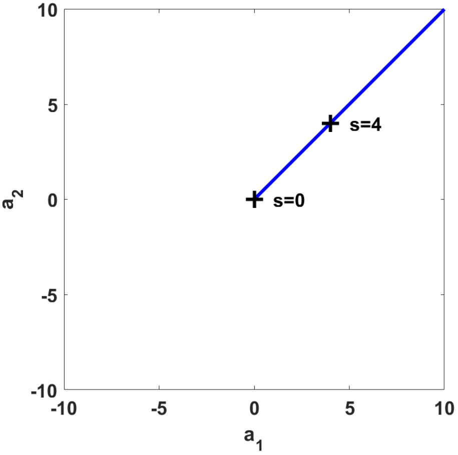

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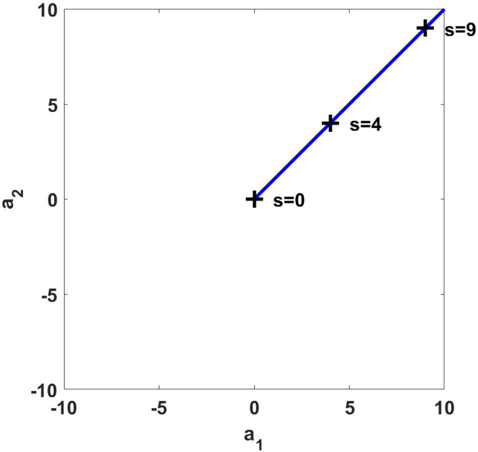

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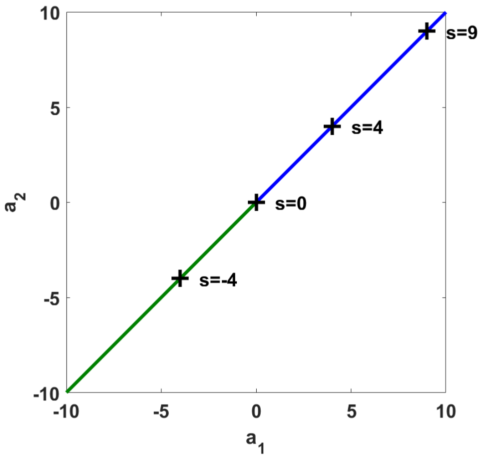

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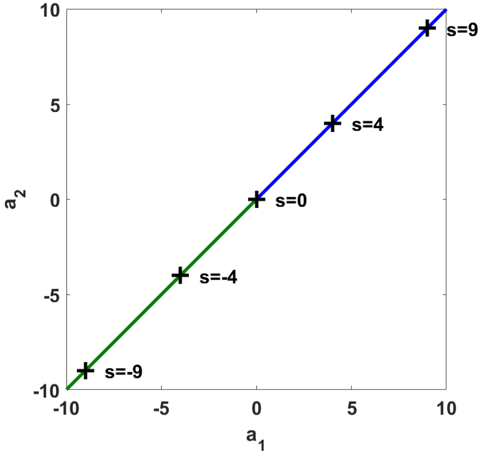

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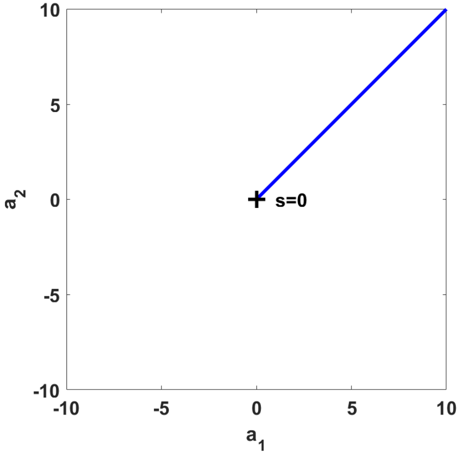

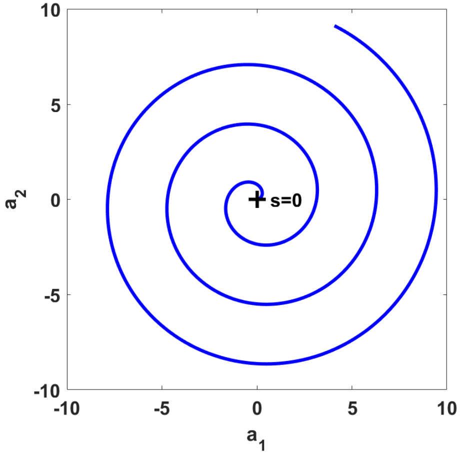

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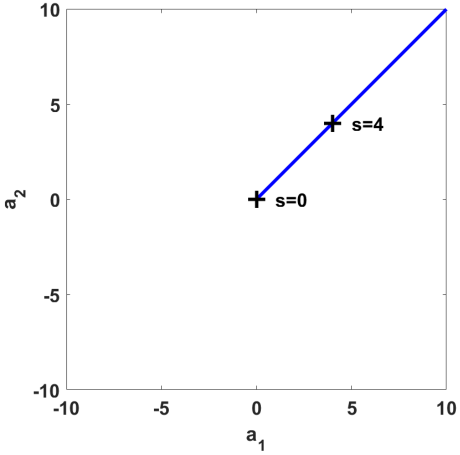

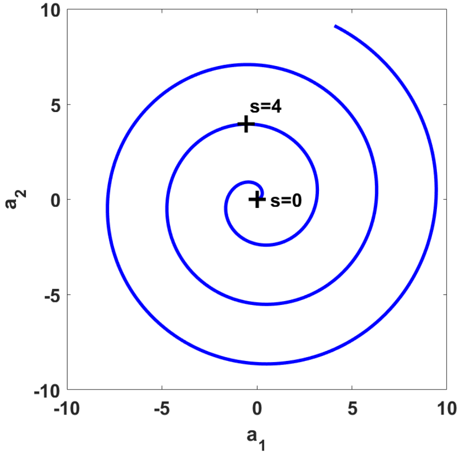

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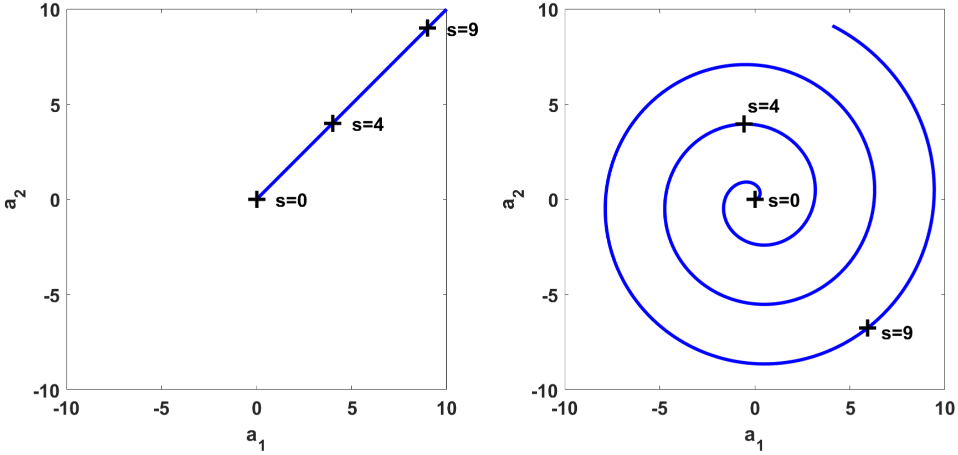

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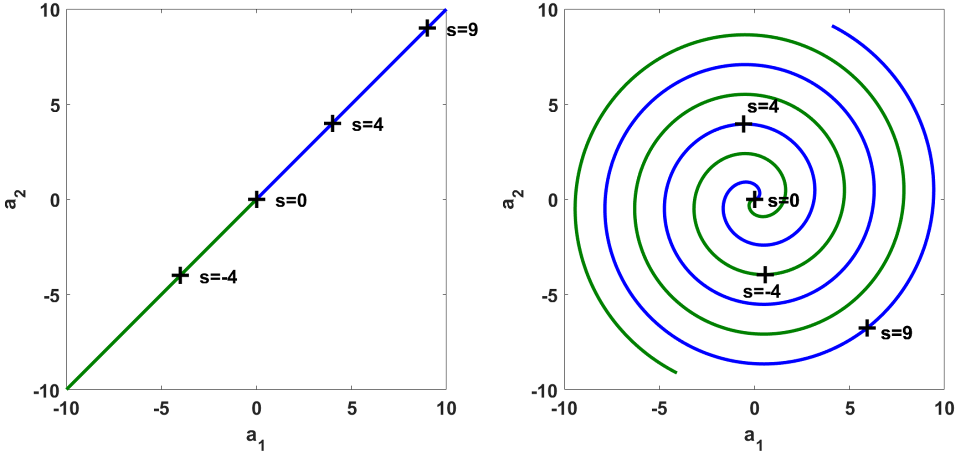

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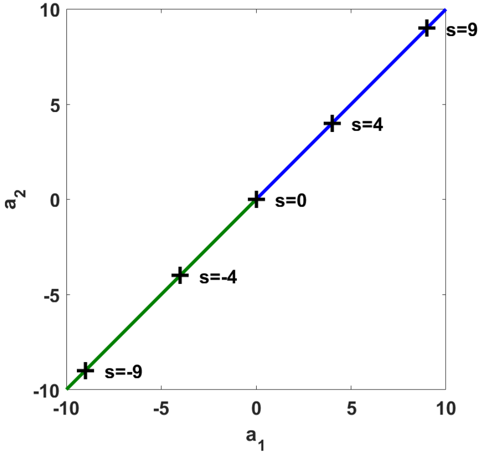

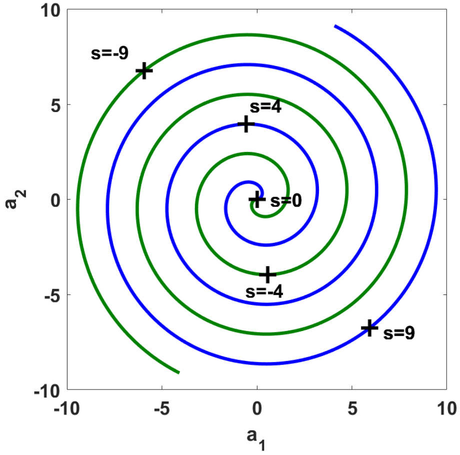

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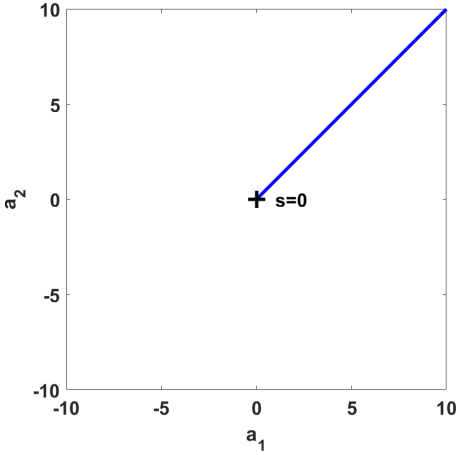

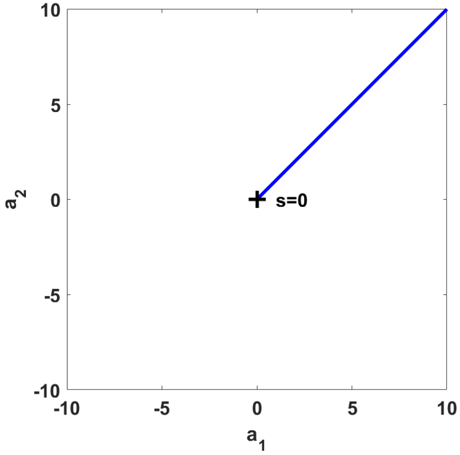

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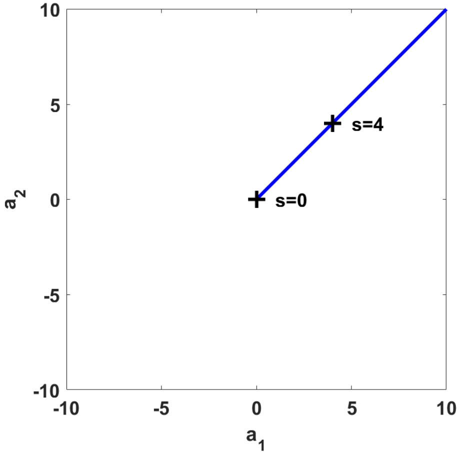

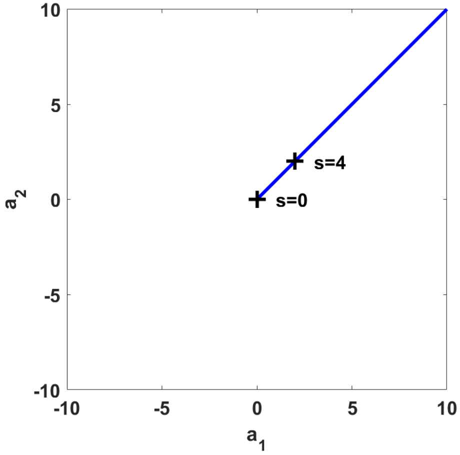

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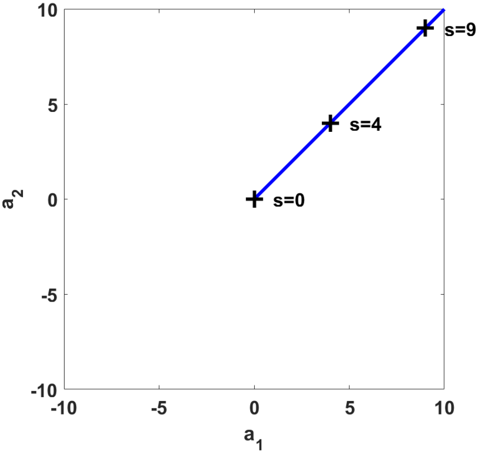

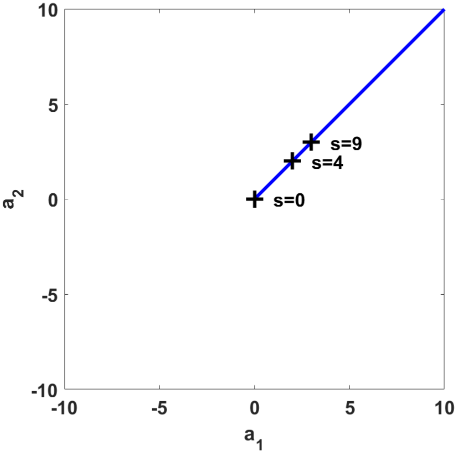

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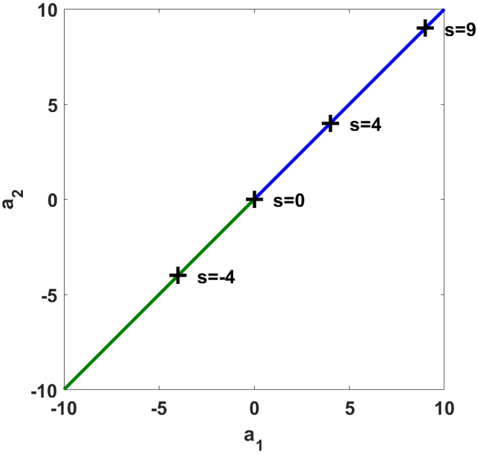

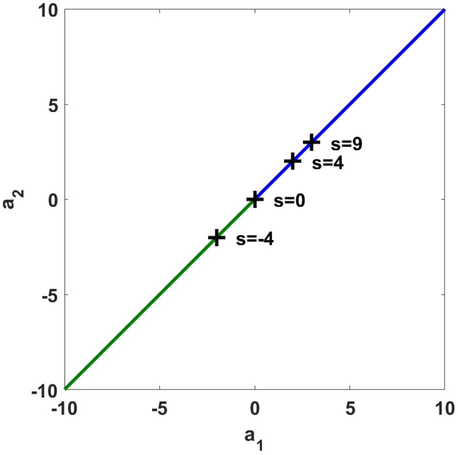

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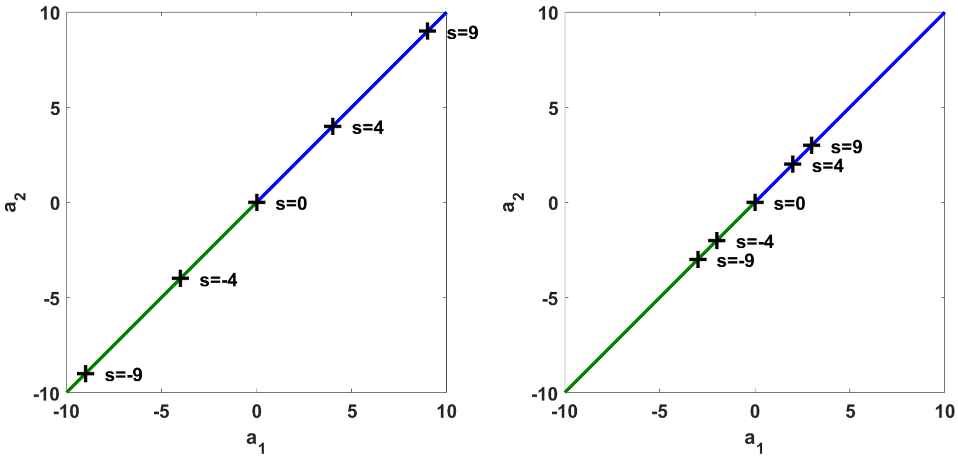

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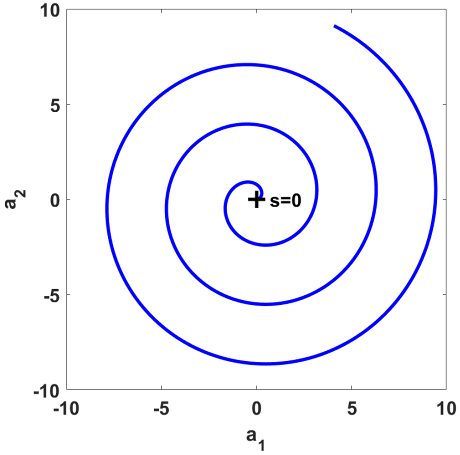

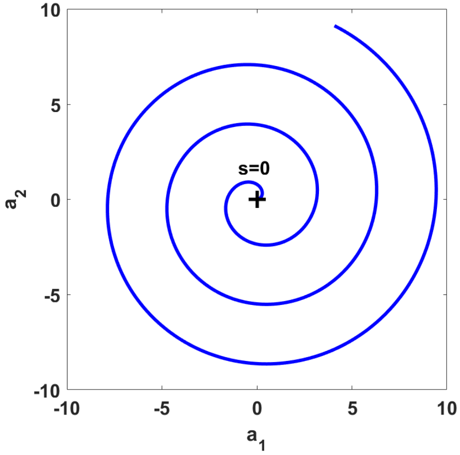

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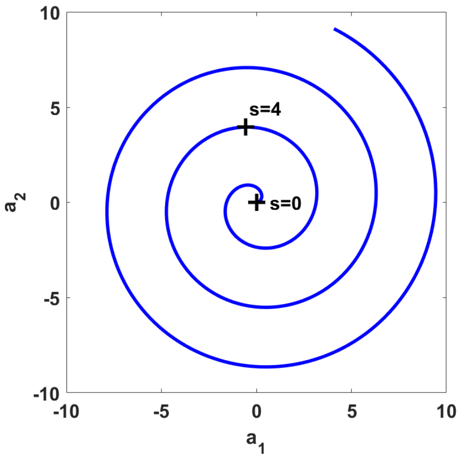
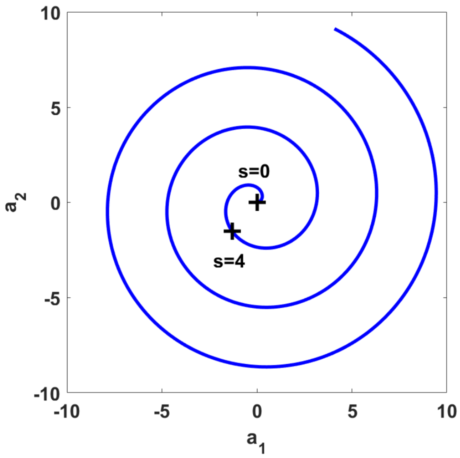
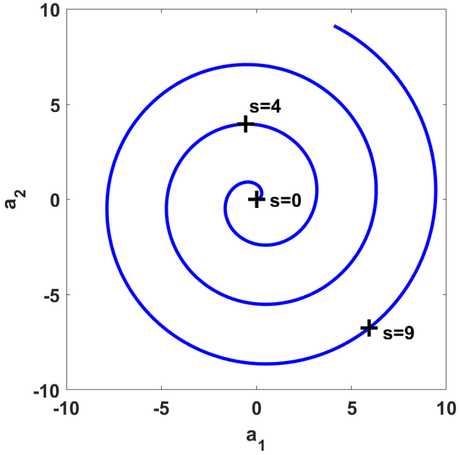


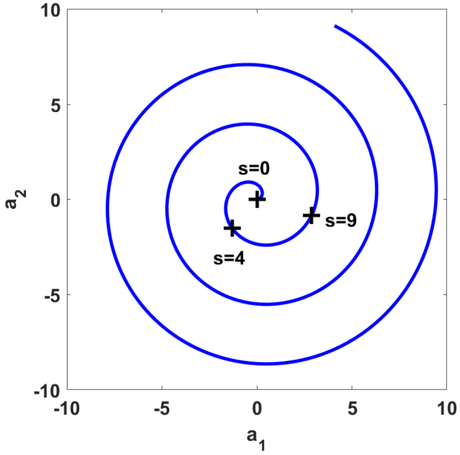


1 : 2 JSCC: Rate-Mismatched Case

$$\begin{cases} a_1(s) = s\cos(2s) \\ a_2(s) = s\sin(2s) \end{cases}$$

$$\begin{cases} a_1(s) = \sqrt{s}\cos(2\sqrt{s}) \\ a_2(s) = \sqrt{s}\sin(2\sqrt{s}) \end{cases}$$

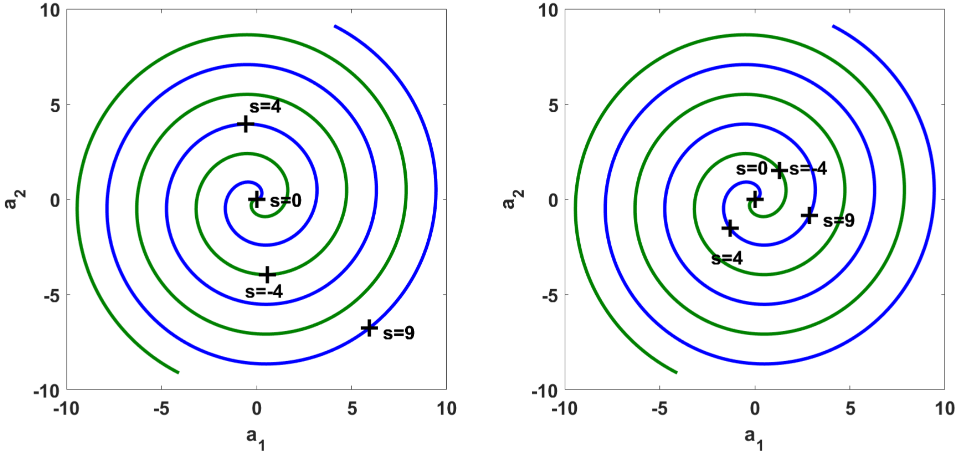




1 : 2 JSCC: Rate-Mismatched Case

$$\begin{cases} a_1(s) = s\cos(2s) \\ a_2(s) = s\sin(2s)\,\mathrm{sign}(s) \end{cases}$$

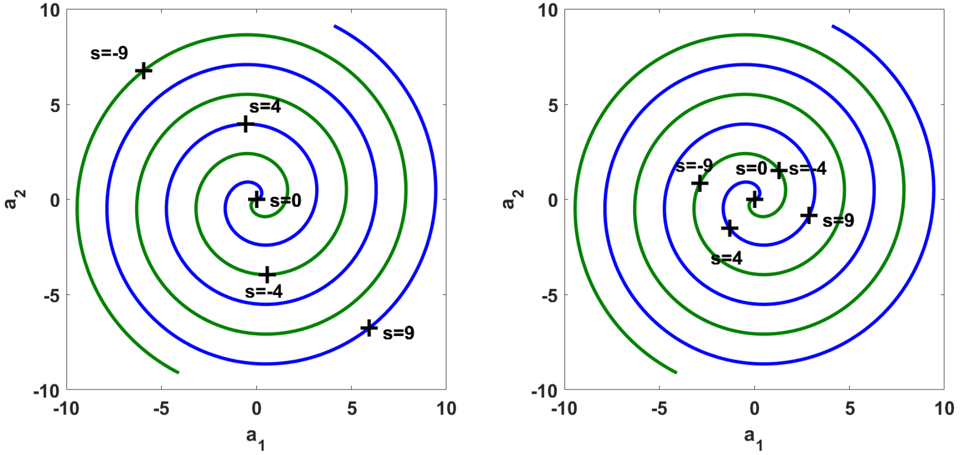
$$\begin{cases} a_1(s) = \sqrt{|s|}\cos(2\sqrt{|s|})\,\mathrm{sign}(s) \\ a_2(s) = \sqrt{|s|}\sin(2\sqrt{|s|})\,\mathrm{sign}(s) \end{cases}$$



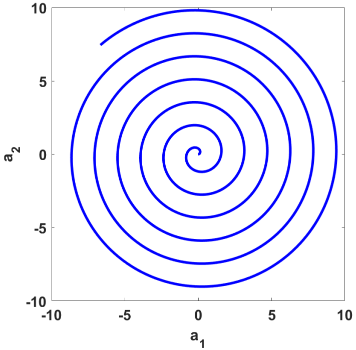
1 : 2 JSCC: Rate-Mismatched Case

$$\begin{cases} a_1(s) = |s| \cos(2|s|) \operatorname{sign}(s) \\ a_2(s) = |s| \sin(2|s|) \operatorname{sign}(s) \end{cases}$$

$$\begin{cases} a_1(s) = \sqrt{|s|} \cos(2\sqrt{|s|}) \operatorname{sign}(s) \\ a_2(s) = \sqrt{|s|} \sin(2\sqrt{|s|}) \operatorname{sign}(s) \end{cases}$$



1 : 2 JSCC: Rate-Mismatched Case

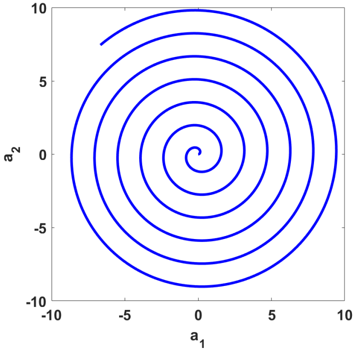


- Small distance between branches
  $\Rightarrow$ better for "weak noise"
- Large distance between branches
  $\Rightarrow$ better for "strong noise"

Standard spiral

$$\begin{cases} a_1(s) \propto s\,\cos(\omega s) & = |s|\,\cos(\omega |s|)\,\text{sign}(s) \\ a_2(s) \propto s\,\sin(\omega s)\,\text{sign}(s) & = |s|\,\sin(\omega |s|)\,\text{sign}(s) \end{cases}$$

1 : 2 JSCC: Rate-Mismatched Case



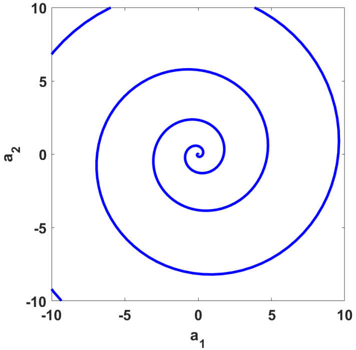
- Small distance between branches
  $\Rightarrow$ better for "weak noise"
- Large distance between branches
  $\Rightarrow$ better for "strong noise"

Stretched-source spiral

Stretch input before mapping to spiral: $s \to |s|^\lambda \operatorname{sign}(s)$

$$\begin{cases} a_1(s) \propto |s|^\lambda \cos\left(\omega |s|^\lambda\right) \operatorname{sign}(s) \\ a_2(s) \propto |s|^\lambda \sin\left(\omega |s|^\lambda\right) \operatorname{sign}(s) \end{cases}$$

1 : 2 JSCC: Rate-Mismatched Case



$\Longleftarrow$ Control requirements

- Small distance between branches
  $\Rightarrow$ better for "weak noise"
- Large distance between branches
  $\Rightarrow$ better for "strong noise"

Bounded average distortion given any input

Avoid increase in distortion with $|s|$ $\Rightarrow$ Slower rotation with $|s|$

$$\begin{cases} a_1(s) \propto |s|^{\lambda\beta} \cos\left(\omega|s|^{\lambda}\right) \operatorname{sign}(s) \\ a_2(s) \propto |s|^{\lambda\beta} \sin\left(\omega|s|^{\lambda}\right) \operatorname{sign}(s) \end{cases}$$

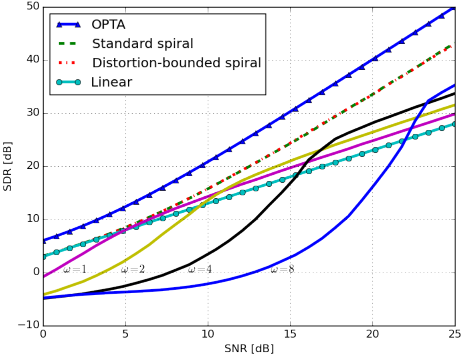
1 : 2 JSCC: Rate-Mismatched Case

⟸ Control requirements

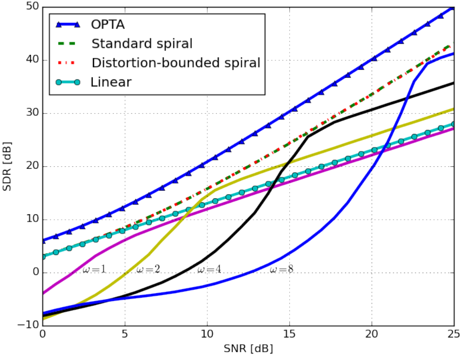
- Small distance between branches
  ⇒ better for "weak noise"
- Large distance between branches
  ⇒ better for "strong noise"

1 : 2 JSCC: Rate-Mismatched Case

- Average distortion given (almost) any s needs to be small!
- E.g., transmitters that truncate the signal do not perform well (avalanche effect)



(a) $\lambda = 1.$



(b) $\lambda = 0.5.$

1 : 2 JSCC: Rate-Mismatched Case

Inner bound: Black-box approach

Assume a JSCC scheme with bounded distortion $D = \frac{1}{\mathrm{SNR}^{\mathrm{eff}}}, \forall s$. Then,

$$\bar{J}^{\mathrm{r}} \leq \bar{J}^{\mathrm{t}} + \frac{Q + (\alpha^2 - 1)\, S}{1 + \mathrm{SNR}^{\mathrm{eff}} - \alpha^2} (P_t^t - \bar{P}_t^t)$$

Outer bound: Extension of [Kostina-Hassibi Allerton'16]

$$\bar{J}^{\mathrm{r}} \leq \bar{J}^{\mathrm{t}} + \frac{Q + (\alpha^2 - 1)\, S}{1 + \mathrm{SNR}^{\mathrm{eff}}_{n\to\infty} - \alpha^2} (P_t^t - \bar{P}_t^t)$$

- $1 + \mathrm{SNR}^{\mathrm{eff}}_{n\to\infty} = (1 + \mathrm{SNR})^2$

- Difference between bounds is only due to effective SNR

Further Results and Future Research

- Inner bound can be improved: Optimization over curves, e.g. [Akyol-Vishwanatha-Rose-Ramstad IT'14]
- Outer bound for low-delay JSCC can be improved [Ziv-Zakai IT'73]
- High dimensional curves
- Other low-delay JSCC techniques: e.g., repetitive quantization [Kleiner-Rimoldi GLOBECOM'09]
  - Easy to generalize to higher dimensions
- Vector $\boldsymbol{x}$, vector $\boldsymbol{u}$, scalar y: *Simple extension of scalar setting!*
- Rate-matched case with vector $\boldsymbol{y}$: "n : 1 JSCC" is needed
  - Switch roles between Transmitter and Receiver
  - Improves over [Freudenberg-Middleton-Solo AC'10]