Sequential Coding of Gauss–Markov Sources with Packet Erasures and Feedback

Anatoly Khina, Caltech

Joint work with
Victoria Kostina and Babak Hassibi, Caltech
Ashish Khisti, University of Toronto

ITW
Kaohsiung, Taiwan
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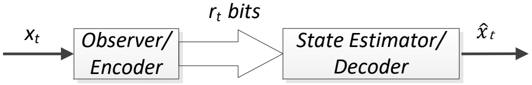
Tracking Gauss–Markov Processes over a Noiseless Channel

Tracking Gauss–Markov Processes over a Noiseless Channel

- $x_{t+1} = \alpha x_t + w_t$
- w_t – White ~~Gaussian~~Kaohsiung noise $\mathcal{N}(0, W)$



- **Goal:** Minimize the distortions $D_t \triangleq \mathbb{E}\left[(x_t - \hat{x}_t)^2\right]$
- Might be a tension betweeen optimizing D_{t_1} and D_{t_2}

Noiseless finite-rate channel of rate r

Fixed rate: Exactly r bits are available at every time sample t

Variable rate: r bits are available **on average** at every t

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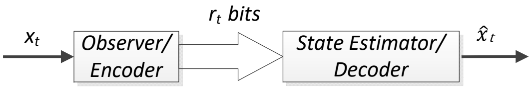

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Random-rate budget

At time t: Exactly r_t bits are given.

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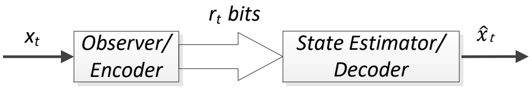

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Packet erasures with instantaneous acknowledgments (ACKs)

[Minero et al. AC'09]: Erasure + ACK $\iff$ $r_t = 0$

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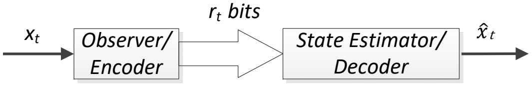

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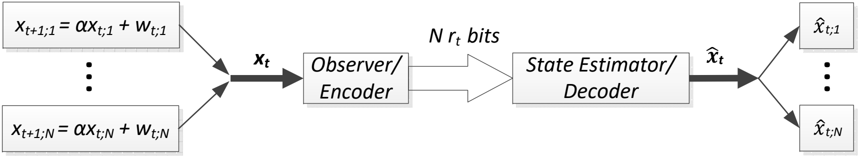
More tricky... We'll get back to it later...

Multi-Tracking Gauss–Markov Processes over a Noiseless Channel

Multi-track

Several processes are tracked and controlled over a shared channel.



- N — Number of tracked processes

(Multi-)Track: Impossibility

Lower bound

- Given rates $R_1, \ldots, R_T$
- $D_t \geq D_t^*$ where

$$D_t^* = \left(\alpha^2 D_{t-1}^* + W\right) 2^{-2R_t}$$
$$D_0^* = 0$$

Previous results

- Sum-rate explicitly proved by [Ma-Ishwar IT'11]
- Steady-state can be deduced from [Gorbunov-Pinsker PPI'74] [Tatikonda-Sahai-Mitter AC'04][Kostina-Hassibi Allerton'16]

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Inductive proof sketch

- Condition on previously received packets
- Shannon's lower bound $\Rightarrow$ Entropy-power calculations
- Entropy-power inequality $\Rightarrow$ Separates w_t from $\alpha(x_t - \hat{x}_t)$
- Jensen's inequality + simple IT inequalities

(Multi-)Track: Impossibility

Lower bound

- Random rates $r_1, \ldots, r_T$
- $D_t \geq D_t^*$ where

$$D_t^* = \left(\alpha^2 D_{t-1}^* + W\right) \mathbb{E}\left[2^{-2r_t}\right]$$

$$D_0^* = 0$$

Proof adjustment

- Condition on rates
- Take expectation at the end

(Multi-)Track: Impossibility

Lower bound

- Random rates $r_1, \ldots, r_T$
- $D_t \geq D_t^*$ where

$$D_t^* = \left(\alpha^2 D_{t-1}^* + W\right) \mathbb{E}\left[2^{-2r_t}\right]$$

$$D_0^* = 0$$

Track vs. Multi-track

- Holds for any number of tracked processes N
- Applies for single-process tracking

Multi-Track: Achievability

Upper bound

- Any $\epsilon > 0$ and large enough N
- Given rates $R_1, \ldots, R_T$
- $D_t \leq D_t^* + \epsilon$ where

$$D_t^* = \left(\alpha^2 D_{t-1}^* + W\right) 2^{-2R_t}$$

$$D_0^* = 0$$

Multi-Track: Achievability

Upper bound

- Any $\epsilon > 0$ and large enough N
- Random rates $r_1, \ldots, r_T$
- $D_t \leq D_t^*$ where

$$D_t^* = \left(\alpha^2 D_{t-1}^* + W\right) \mathbb{E}\left[2^{-2r_t}\right]$$
$$D_0^* = 0$$

Multi-Track: Achievability

Optimal multi-track for large N

- The upper and lower bounds coincide for large N
- Optimal distortions:

$$D_t^* = \left(\alpha^2 D_{t-1}^* + W\right) \mathbb{E}\left[2^{-2r_t}\right]$$
$$D_0^* = 0$$

- Optimal performance achieved by greedy quantization
- No tension between optimizing D_{t_1} and D_{t_2}

Multi-Track: Achievability

Optimal multi-track for large N

- The upper and lower bounds coincide for large N
- Optimal distortions:

$$D_t^* = \left(\alpha^2 D_{t-1}^* + W\right) \mathbb{E}\left[2^{-2r_t}\right]$$

$$D_0^* = 0$$

Steady-state distortion

$$D_\infty^* \triangleq \lim_{t \to \infty} D_t^* = \frac{W \mathbb{E}\left[2^{-2r_1}\right]}{1 - \alpha^2 \mathbb{E}\left[2^{-2r_1}\right]}$$

- Recovers the data–rate theorem of [Minero et al. AC'09]:

$$\alpha^2 \mathbb{E}\left[2^{-2r_1}\right] < 1$$

- Or in the deterministic case: $R > \log|\alpha|$

Multi-Track: Achievability

DPCM scheme

Observer at time t

- Generates the prediction error
$$\tilde{\boldsymbol{x}}_t \triangleq \boldsymbol{x}_t - \alpha \hat{\boldsymbol{x}}_{t-1}$$
- Quantizes the error:
$$\hat{\tilde{\boldsymbol{x}}}_t = Q_t(\tilde{\boldsymbol{x}}_t)$$
- Sends the quantization index
- Generates next estimate:
$$\hat{\boldsymbol{x}}_t = \alpha \hat{\boldsymbol{x}}_{t-1} + \hat{\tilde{\boldsymbol{x}}}_t$$

State estimator at time t

- Receives quantization index
- Recovers $\hat{\tilde{\boldsymbol{x}}}_t$
- Generates an estimate of $\boldsymbol{x}_t$:
$$\hat{\boldsymbol{x}}_t = \alpha \hat{\boldsymbol{x}}_{t-1} + \hat{\tilde{\boldsymbol{x}}}_t$$

Random rates

Algorithm is greedy $\Rightarrow$ Does not need to know rates in advance

Packet Erasures with Instantaneous Feedback

One packet per one state frame

- I.i.d. packet erasures: $b_t = \begin{cases} 1 & \text{w.p. } \beta & (\textit{arrived}) \\ 0 & \text{w.p. } 1-\beta & (\textit{erased}) \end{cases}$
- Instantaneous feedback $\Rightarrow$ Random-rate budget scenario
- I.i.d. rates $r_t = Rb_t = \begin{cases} R, & \text{w.p. } \beta \\ 0, & \text{w.p. } 1-\beta \end{cases}$

Packet Erasures with Instantaneous Feedback

Multiple packets per one state frame

- I.i.d. $\mathscr{Ber}(\beta)$ packet erasures
- K packets each of rate R/K
- $b_t \sim \mathscr{Bin}(K, \beta)$ — Number of **successful** packet arrivals
- Instantaneous feedback $\Rightarrow$ Random-rate budget scenario
- I.i.d. rates $r_t = R\, b_t / K$

Packet Erasures with Instantaneous Feedback

Steady-state distortion

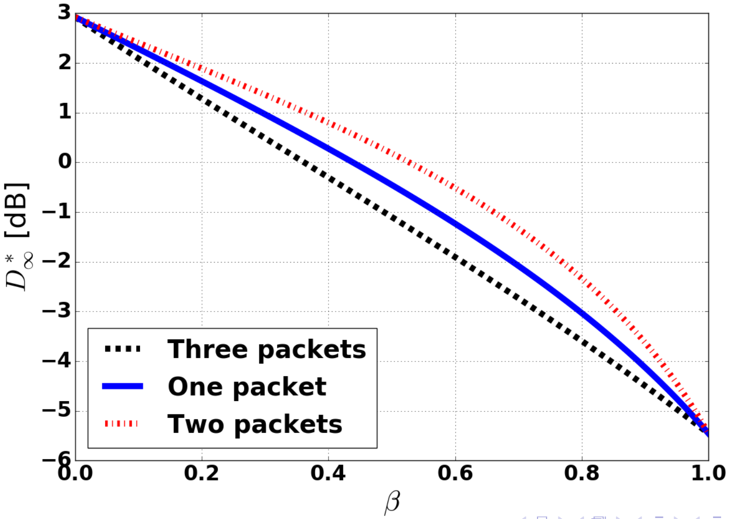
$$D_\infty^* \triangleq \lim_{t\to\infty} D_t^* = \frac{W\mathbb{E}\left[2^{-2r_1}\right]}{1-\alpha^2\mathbb{E}\left[2^{-2r_1}\right]}$$

Are more packets necessarily better?

- More packets $\Rightarrow$ Higher chance to convey with non-zero rate
- Less packets $\Rightarrow$ Higher chance to send full rate R
- How to choose the optimal K?
- Choose K that optimizes $\mathbb{E}\left[2^{-2r_t}\right]$ w.r.t. $b_t \sim \mathscr{Bin}(K,\beta)$

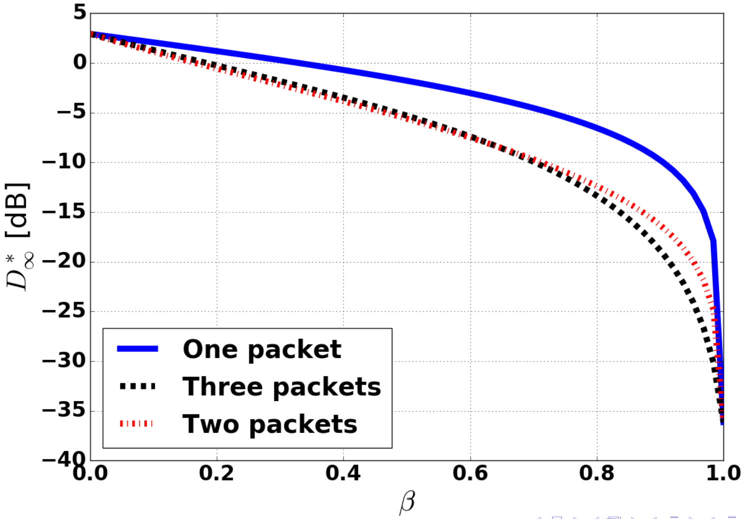
Packet Erasures with Instantaneous Feedback

- $\boldsymbol{x}_t = \alpha \boldsymbol{x}_t + \boldsymbol{w}_t$, with $\alpha = 0.7$, $R = 1$, $W = 1$



Packet Erasures with Instantaneous Feedback

- $\mathbf{x}_t = \alpha \mathbf{x}_t + \mathbf{w}_t$, with $\alpha = 0.7$, $R = 6$, $W = 1$



Packet Erasures with Delayed Feedback

Delay by one time unit

Observer at time t:

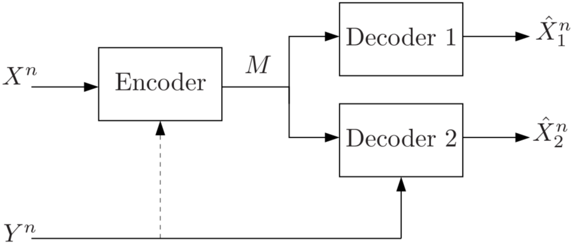
- Does not know whether the last packet arrived or not (b_{t-1})
- Knows whether all preceding packets arrived or not (b^{t-2})

Idea

At time t, treat packet of time $t-1$ as

- Side-information that may be known at the decoder
- Side-information that is known at the encoder

Packet Erasures with Delayed Feedback



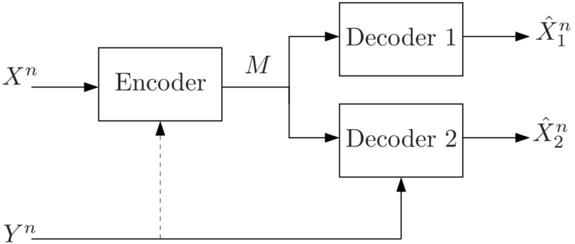
(Fig. from the lectures notes of El Gamal & Kim)

Idea

At time t, treat packet of time $t-1$ as side information

- Side-information that may be known at the decoder
- Side-information that is known at the encoder

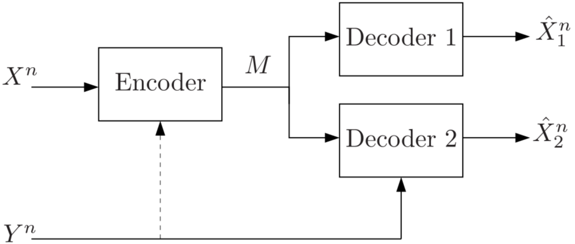
Packet Erasures with Delayed Feedback



Idea

- $X^n \to \boldsymbol{x}_t$
- $Y^n \to$ Packet sent at time $t-1$
- Decoder 1 $\to$ Previous packet did not arrive
- Decoder 2 $\to$ Previous packet arrived successfully

Packet Erasures with Delayed Feedback

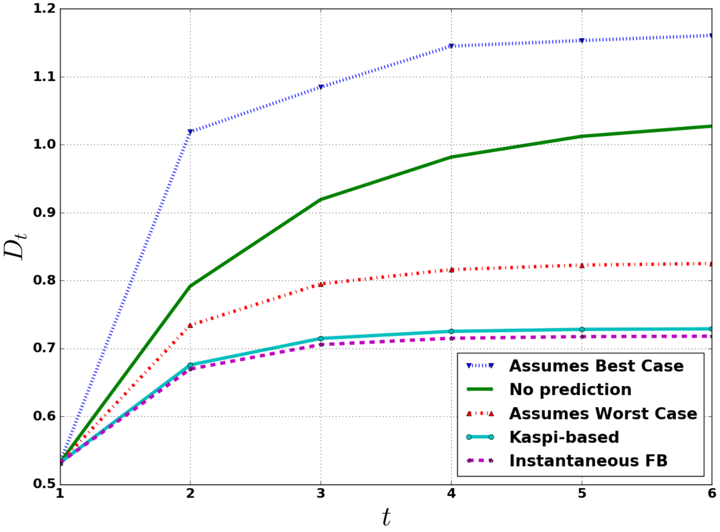


Rate–distortion region [Kaspi IT'94 ('80)]

- Gaussian case explicitly solved by [Perron et al. ISIT'06]
- Availability of SI @ Tx improves performance!
  - Does not help when SI always available @ Rx (Wyner–Ziv)
  - Does not help in when no SI available @ Rx
- SI@Tx ⇒ (anti-)correlate quantization noise and prediction error

Packet Erasures with Delayed Feedback

- $\mathbf{x}_t = \alpha \mathbf{x}_t + \mathbf{w}_t$, with $\beta = 0.5$, $\alpha = 0.7$, $R = 2$, $W = 1$



Variable-Length Coding

- Back to tracking a *single* process

Fixed-length coding

Exactly R bits are available at time t.

Variable-length coding

R bits are available *on average* at time t.

- Rx decides on bit allocation depending on the source value

Variable-Length Coding

Entropy-Coded Dithered Quantization (ECDQ) [Ziv'85][Zamir-Feder'96]

- Quantize x using an infinite grid of step size Δ
- Apply entropy-coding to resulting bits

Improvement 1: Add uniform random dither z modulo Δ:

$$[x + z] \mod \Delta$$

- Makes quantization error uniform and independent of x

Improvement 2: Linear pre- and post-processing

- Essentially applies MMSE estimation

Variable-Length Coding

ECDQ performance [Zamir-Feder IT'96]

- Source power X
- R bits on average

$$R \leq \frac{1}{2}\log\frac{X}{D} + \frac{1}{2}\log\frac{2\pi e}{12} \quad \Leftrightarrow \quad D \leq \frac{2\pi e}{12} X 2^{-2R}$$

- Applies for any distribution of x
- For higher dimensions shaping loss decreases
- For $N \to \infty$, loss goes to 0
- We assume one-to-one codes (not prefix free)

Variable-Length Coding

DPCM-ECDQ scheme

- Use ECDQ as the quantizer in the DPCM scheme
- Similar scheme proposed in [Silva-Derpich-Østergaard IT'11]
- Achieves the following distortions:

$$D_t = \frac{2\pi e}{12} \left(\alpha^2 D_{t-1} + W\right) \mathbb{E}\left[2^{-2r_t}\right]$$

- Steady-state: $D_\infty^* = \dfrac{\frac{2\pi e}{12} W \mathbb{E}\left[2^{-2r_1}\right]}{1 - \frac{2\pi e}{12} \alpha^2 \mathbb{E}\left[2^{-2r_1}\right]}$

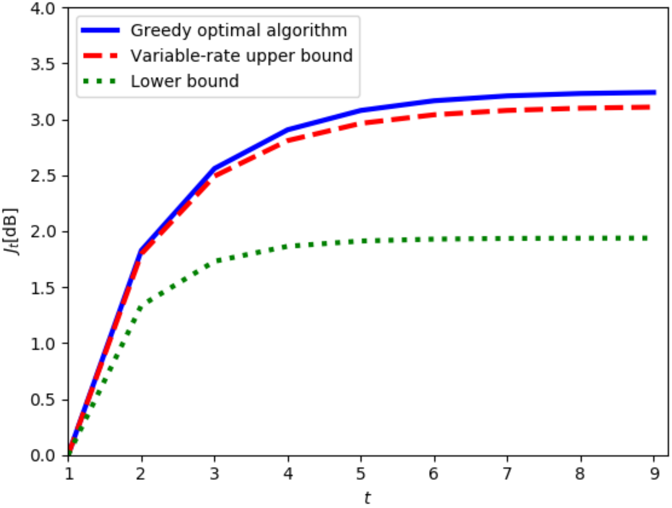
Lower bound

$$D_t = \left(\alpha^2 D_{t-1} + W\right) \mathbb{E}\left[2^{-2r_t}\right]$$

- Steady-state: $D_\infty^* = \dfrac{W \mathbb{E}\left[2^{-2r_1}\right]}{1 - \alpha^2 \mathbb{E}\left[2^{-2r_1}\right]}$

Variable-Length Coding for Control

- $A = 1.2, W = 1, \mathrm{Q}_t \equiv 1, \mathrm{R}_t = 0$, no erasures: $J_t = \alpha^2 D_t + W$



Multi-Track: Rate Allocation

Goal

Maximize average (across time) distortion: $\bar{D}_\infty = \frac{1}{T}\sum_{t=1}^{T} D_t$

- Total rate budget $\frac{1}{T}\sum_{t=1}^{T} R_t \leq R$
- How to allocate transmission rates $R_1, \ldots, R_T$?

No erasures

- Recall: $D_t = \left(\alpha^2 D_{t-1} + W\right) 2^{-2R_t}$
- Convexity arguments $\overset{T\to\infty}{\Longrightarrow}$ Uniform allocation $R_t \equiv R$ optimal

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With i.i.d. erasures

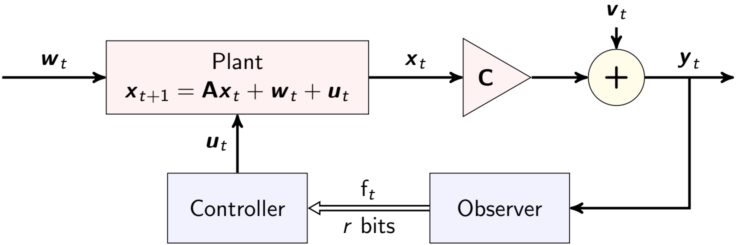
- Reminiscent of channel coding with fading that is known @ Tx
- Water-filling over current distortion level can do better(?)

Linear Quadratic Gaussian Control over Noiseless Channels

Linear quadratic Gaussian (LQG) system

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t + \mathbf{w}_t, \qquad \mathbf{w}_t \sim \text{ i.i.d. } \mathcal{N}(0, \mathbf{W})$$

$$\mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{v}_t, \qquad \mathbf{v}_t \sim \text{ i.i.d. } \mathcal{N}(0, \mathbf{V})$$



Noiseless finite-rate channel of rate r

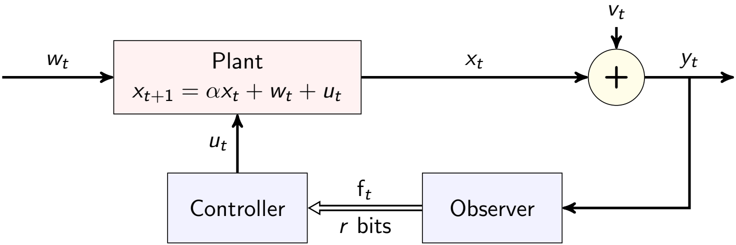
Fixed rate: Exactly r bits are available at every time sample t

Variable rate: r bits are available **on average** at every t

Linear Quadratic Gaussian Control over Noiseless Channels

Scalar linear quadratic Gaussian (LQG) system

$$x_{t+1} = \alpha x_t + u_t + w_t, \qquad w_t \sim \text{ i.i.d. } \mathcal{N}(0, W), \qquad |\alpha| > 1$$
$$y_t = x_t + v_t, \qquad v_t \sim \text{ i.i.d. } \mathcal{N}(0, V)$$



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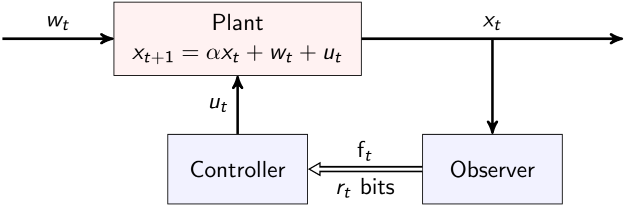

Random-rate budget

At time t: Exactly r_t bits are given.

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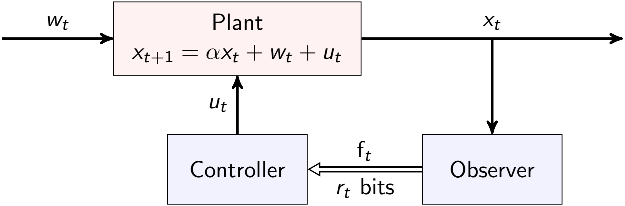

Packet erasures with instantaneous acknowledgments (ACKs)

[Minero et al. AC'09]: Erasure + ACK $\Longleftrightarrow$ $r_t = 0$

Linear Quadratic Gaussian Control over Noiseless Channels

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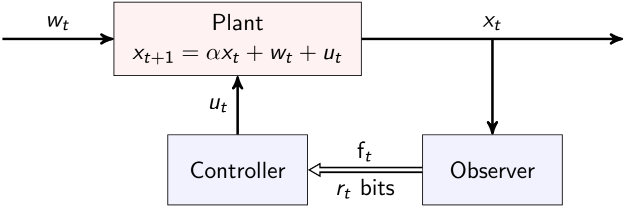

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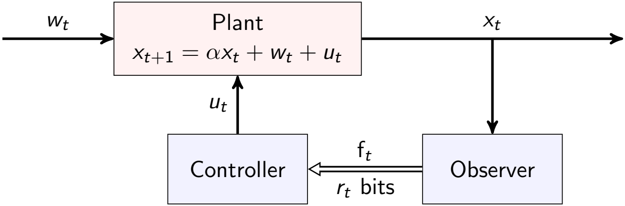

LQG cost

$$\bar{J}_T = \mathbb{E}\left[\sum_{t=1}^{T-1}\left(Q_t x_t^2 + R_t u_t^2\right) + Q_T x_T^2\right]$$

The Control–Estimation Separation Principle

Control–estimation separation for networked control systems [Fischer AC'82][Tatikonda-Sahai-Mitter AC'04]

- Optimal control action: $u_t = -K_t \hat{x}_t$
- LQR coefficients: $\begin{cases} K_t = \frac{\alpha L_{t+1}}{R_t + L_{t+1}}, & K_T = 0, \\ L_t = Q_t + \alpha R_t K_t, & L_{T+1} = 0 \end{cases}$
- MMSE estimate: $\hat{x}_t = \mathbb{E}\left[x_t | f^t\right]$
- Optimal cost: $\bar{J}_T^* = \frac{1}{T} \sum_{t=1}^{T} \left(W L_t + \alpha K_t L_{t+1} D_t^* \right)$
- $D_t = \mathbb{E}\left[(x_t - \hat{x}_t)^2 \right]$

- Past u_t known to all $\Rightarrow$ Same $\{D_t\}$ for all control actions $\{u_t\}$
- Control–estimation separation extends to networked control