# (Almost) Practical Tree Codes

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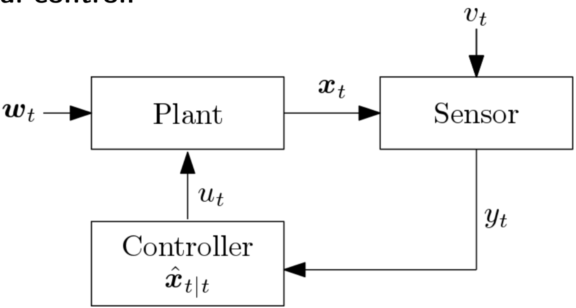
# Networked Control vs. Traditional Control

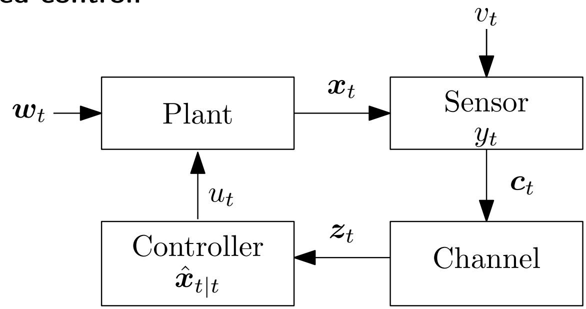
**Traditional control:**




- Observer and controller are co-located.
- Classical systems are hardwired and well crafted

# Networked Control vs. Traditional Control

**Networked control:**

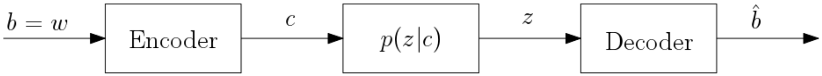



- Observer and controller are not co-located: connected through noisy link
- Suitable for new remote applications (e.g., remote surgery, self-driving cars)

# Motivating Example: Tracking a Random Walk [Sahai PhD'01]

$$x_{t+1} = \alpha x_t + w_t$$

- $\alpha > 1 \implies$ not stable!
- $w_t \in \{\pm 1\}$
- We wish to track $x_t$ with bounded expected distortion
- If tracking is possible, stability usually follows
- Allows to distill the coding problem (no quantization)




## Distortion requirement

$$\mathbb{E}\left[(x_t - \hat{x}_t)^2\right] < \infty, \qquad \forall t$$

# Motivating Example: Tracking a Random Walk [Sahai PhD'01]

- $\hat{b}_{t-d|t}$ – Estimate of $b_{t-d}$ at time $t$
- Probability of first error event at time $t-d$:
  $P_e(t,d) \triangleq \Pr\left(b_{t-d} \neq \hat{b}_{t-d|t}, \forall \delta > d, b_{t-\delta} = \hat{b}_{t-\delta|t}\right)$

$$\mathbb{E}\left[\left(x_t - \hat{x}_{t|t}\right)^2\right] \propto \sum_{d=1}^{t} P_e(t,d)\alpha^{2d} = \sum_{d=1}^{t} P_e(t,d) 2^{2\log\alpha\cdot d} < \infty$$

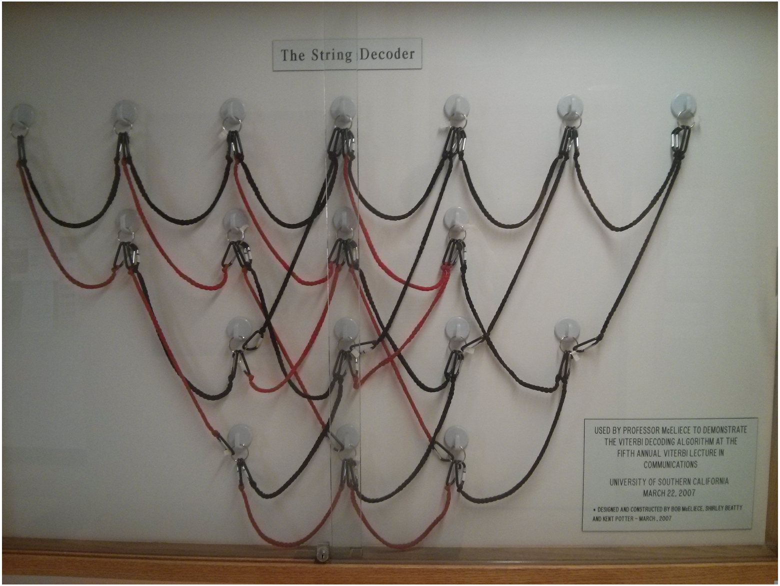
**Error probability profile: Anytime-reliable code**

$$P_e(t,d) < 2^{-(2\log\alpha+\epsilon)d}, \qquad \forall t, d_0 < d < t$$

**Larger moments**

Higher exponent $\Longrightarrow$ Cannot stabilize all moments!

# Tree / Anytime-Reliable Codes



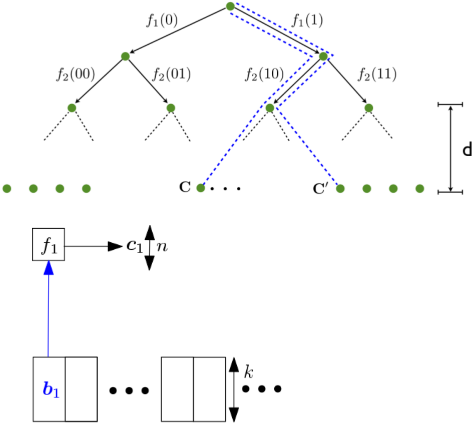
# Anytime-Reliable Codes: Basics

## Error probability profile

$$P_e(t,d) < 2^{-(2\log\alpha+\epsilon)d}\,, \qquad \forall t, d_0 < d < t$$

## How to generate such a code?

$$\begin{aligned} \boldsymbol{c}_1 &= f_1(\boldsymbol{b}_1) \\ \boldsymbol{c}_2 &= f_2(\boldsymbol{b}_1, \boldsymbol{b}_2) \\ &\vdots \\ \boldsymbol{c}_t &= f_t(\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_t) \\ &\vdots \end{aligned}$$



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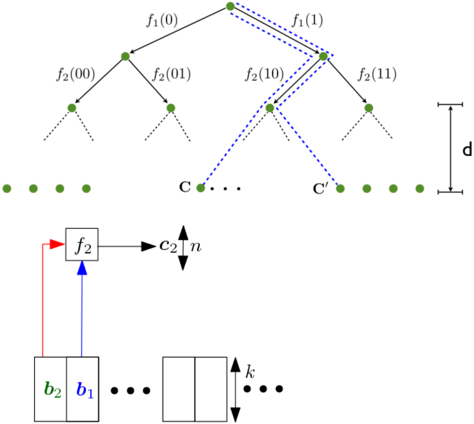

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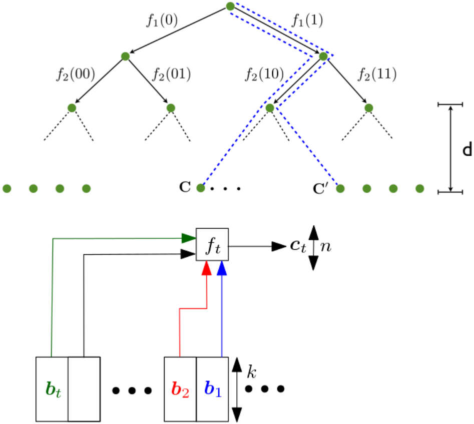


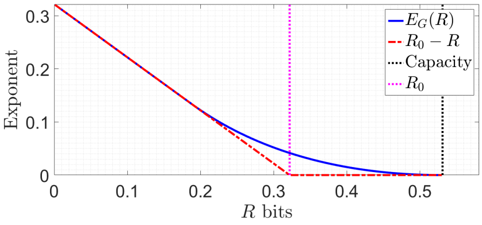
# Anytime-Reliable Codes as Convolutional Codes

## Random **time-varying** convolutional-code ensemble [Viterbi, Yudkin, Zigangirov, Shulman–Feder, ...]

- Most results assume infinite stream ($\gg$ delay-line length)
- We wish to recover a bit using subsequent $nd$ output symbols
- The random time-varying CC ensemble achieves:

$$\mathbb{E}[P_e(t,d)] \leq 2^{-E_G(R)nd}$$

- $E_G(R) > 0$ for $R < C$ – Gallager's error exponent



# Anytime-Reliable Codes as Convolutional Codes

**Good ensemble performance $\Rightarrow$ Good specific code performance?**

- Block codes: Yes, with high probability!
- Anytime reliable-code?
- Such a code exists [Schulman IT'96], but **not w.h.p.** ☹
  (Proof requires min-distance $\propto$ delay)
- **How to construct a good anytime-reliable code?**

# Ensemble Performance $\Rightarrow$ Specific Code Performance?

## Ensemble performance

$$\mathbb{E}[P_e(t,d)] \leq 2^{-E_G(R)nd}$$

## Specific $d$ and $t$

Using Markov's inequality:

$$\Pr\left(P_e(t,d) \geq 2^{-[E_G(R)-\epsilon]nd}\right) \leq \frac{\mathbb{E}[P_e(t,d)]}{2^{-[E_G(R)-\epsilon]nd}} = 2^{-\epsilon nd}$$

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Using the union bound:

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# Linear Time-Invariant Codes [Sukhavasi–Hassibi ISIT'11]

## Linear time-variant code

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{1,1} & \mathbf{0} & \cdots & \cdots & \cdots \\ \mathbf{G}_{2,1} & \mathbf{G}_{2,2} & \mathbf{0} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ \mathbf{G}_{t,1} & \mathbf{G}_{t,2} & \cdots & \mathbf{G}_{t,t} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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$$= \frac{2^{-\epsilon n d_0}}{1 - 2^{-\epsilon n}}$$

# Linear Time-Invariant Codes

- Random LTI convolutional codes are anytime-reliable w.h.p. ✓
- But the exponent result was valid for time-variant codes
- Valid also for LTI codes [S~~c~~hulman–Feder IT'00] ✓
  - (Proved independently in [Sukhavasi–Hassibi ISIT'11])
- **No gain** for general codes over LTI codes in this regime!
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**What about decoding?**

# Decoding of Linear Time-Invariant Codes

- All results assumed maximum-likelihood (ML) decoding
- ML complexity rises exponentially with $t$

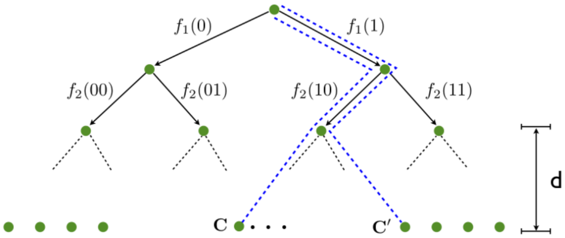
**Binary Erasure Channel (BEC)**

- For LTI codes: ML = Solving linear equations

- What about other channels?

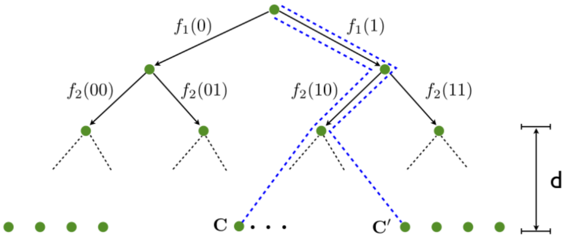
# Sequential Decoding

- Before Viterbi algo.: Sequential decoding *de facto* standard
- Sequential decoding = class of algorithms
- Introduced originally in [Wozencraft '57] for **tree codes**
- Common to all: Explore only subset of (likely) codewords
- Most prominent variants: Stack and Fano's algorithms
- Proposed for general tree ensembles in [Sahai-Palaiyanur '05]



# Sequential Decoding

- Fano's metric: $M(\boldsymbol{c}_1, \cdots, \boldsymbol{c}_t) = \sum_{i=1}^{nt} \left[ \log \frac{p(z_t|c_t)}{p(z_t)} - \overbrace{B}^{\text{bias}} \right]$
- For ML decoding: $\arg\max_{\{\boldsymbol{c}_i\}} p(z_t|c_t) = \arg\max_{\{\boldsymbol{c}_i\}} \left[ \log \frac{p(z_t|c_t)}{p(z_t)} - B \right]$
- For partial tree exploration: Fano's metric penalizes longer incorrect paths via bias $B$

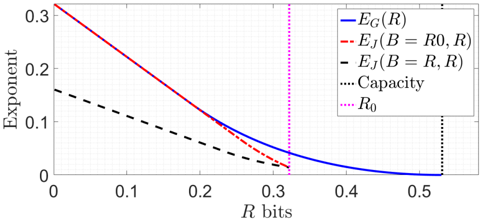


# Sequential Decoding: Error Probability

## Error probability of **general** conv. ensemble [Jelinek's Book '68]

$$\mathbb{E}[P_e(t,d)] \le A\, 2^{-E_J(B,R)nd}$$

- $A$ is finite for $B < R_0$
- $E_J(B,R)$ properties:
  - $\frac{1}{2}E_G(R) \le E_J(B,R) < E_G(R)$
  - $E_J(B,R) \xrightarrow{B \to R_0} E_G(R)$, for $R < R_{\text{crit}}$
- Does not guarantee a good **specific code** w.h.p.



# Sequential Decoding: Error Probability

Proof for general codes requires:

- **Pairwise independence:** Every two paths are independent (from divergence point)
- **Individual codeword distribution:** Entries within each codeword are i.i.d.

# Sequential Decoding: Error Probability

Use the following affine time-invariant ensemble:

$$\begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \vdots \\ \boldsymbol{c}_t \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{0} & \cdots & \cdots & \cdots \\ \mathbf{G}_2 & \mathbf{G}_1 & \mathbf{0} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ \mathbf{G}_t & \mathbf{G}_{t-1} & \cdots & \mathbf{G}_1 & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_t \\ \vdots \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_t \\ \vdots \end{bmatrix}$$

- Entries of $\{\mathbf{G}_t\}$, $\{\boldsymbol{b}_t\}$ and $\{\boldsymbol{v}_t\}$ are i.i.d. uniform
- $\{\boldsymbol{v}_t\}$ random translation vectors

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- i.i.d. uniformity of $\{\mathsf{G}_t\}$ $\Rightarrow$ pairwise independence $\checkmark$

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- $\{\boldsymbol{v}_t\}$ random translation vectors
- i.i.d. uniformity of $\{\mathsf{G}_t\}$ $\Rightarrow$ pairwise independence $\checkmark$
- i.i.d. uniformity of $\{\boldsymbol{v}_t\}$ $\Rightarrow$ i.i.d. uniformity of each codword $\checkmark$

# Sequential Decoding: Complexity

- $W_t$ – Number of branch computations at time $t$
- $W_t$ is a random variable

## Cutoff rate [Arıkan IT'88]

For any "good" code (general or LTI), $\mathbb{E}[W_t]$ is unbounded for $R > R_0$.

## Pareto distribution of $W_t$ [Gallager, Zigangirov, Viterbi–Omura, ...]

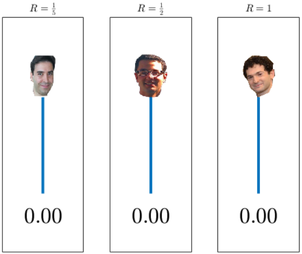
$$\Pr\left(W_t \geq m\right) \leq Am^{-\rho}$$

- For $B, R < R_0$ and $R < \frac{B+R_0}{2\rho}$, $\rho \in (0,1]$:
  Tight for **general** and **LTI** codes $\Rightarrow \mathbb{E}[W_t] < \infty$ for $R < R_0$
- For $\rho > 1$, $R = E_0(\rho)/\rho$:
  - Tight for **general** codes
  - Widely conjectured to be true for **LTI** codes
- Heavy tailed even if expectation is finite!

# Simulation: Cart–Stick over BSC(0.01)

$n = 20$

| | | |
|---|---|---|
| $k = 4$ | $10$ | $20$ |
| $R = \frac{1}{5}$ | $\frac{1}{2}$ | $1$ |
| $E = 0.5382$ | $0.2382$ | $0$ |




- Cart–stick system model [Franklin–Powell–Emami-Naeini Book]
- BSC(0.01)
- For this setting [Sukhavasi–Hassibi ISIT'11]: $k_{\min} = 3$, $E_{\min} = 0.2052$