Employing of the discrete Fourier transform for evaluation of crack-tip field in periodic materials

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ABSTRACT

An approach for numerical evaluation of a self-similar crack-tip field for a long (semi-infinite) crack embedded in a material with periodic microstructure is suggested. The conditions at the boundaries of a rectangular domain around the tip are formulated by the use of $K$-field for the homogeneous material possessing effective elastic properties and then the finite discrete Fourier transform is applied. This allows to replace standard analysis of a large periodic domain with many cells by the analysis of a single repetitive cell in the transform space which can be carried out by any numerical method. Consequently, the volume of calculations in comparison with the standard approach is reduced and the problem of a macrocrack embedded in a material with fine microstructure can be addressed without simplifying assumptions. The accuracy of the proposed approach is verified by a comparison with the analytical solution for a crack embedded in a homogeneous plane.

Application of the suggested method is given for a crack in a two-dimensional periodically voided material with triangular isotropic layout. The cell problem is resolved by the finite element method. The fracture toughness of the material in the framework of stress criterion for crack propagation is determined and its dependence upon the material relative density is investigated. A comparison of the fracture toughnesses of the solid and voided materials has shown for which parameter combinations voided ones will provide better crack resistance.

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1. Introduction

Investigation of new micro-architected materials with improved fracture properties is one of the current research challenges in material science (Fleck, Deshpande, & Ashby, 2010). One of the basic problems in this topic is the analysis of the crack tip field in materials with a periodic microstructure. This problem, as a rule, is a computationally expensive task due to the different scales involved in the analysis. The reason for this is that the basic stipulation for the existence of the self-similar stress field, which is of special interest, is that the crack length $a$ significantly exceeds the characteristic size of the periodic module $l$

$$a \gg l.$$  (1)

The conventional approach ($K$-field approach) to the near-tip field derivation is based on the matching of the elastic field in the near-tip non-homogeneous periodic domain with the $K$-field of semi-infinite crack in the homogeneous elastic material.
possessing effective elastic properties (e.g., Jha & Charalambides, 1998; Leguillon & Piat, 2008). The size of the domain must be large enough to meet the mentioned stipulation and, consequently, the domain includes a large number of repetitive modules. Therefore its fine meshing which is required for the adequate microstructure modeling may lead to a huge number of degrees of freedom.

The advocated technique allows to obtain without difficulty an accurate stress distribution in an arbitrarily large periodic domain surrounding the crack tip. It is applicable for any multiphase periodic composites. In the present study two-dimensional voided materials will be considered when one phase vanishes. For these materials the crack tip can be associated with a void region, the stress field is non-singular and the fracture toughness can be evaluated in terms of tensile strength of the parent solid material σ0.

In the case of cellular materials with relatively large voids, beam lattice approximation is applicable for which nodes displacements and rotations completely define the stress state. Consequently, the number of degrees of freedom in a repetitive module is small and consideration of a sufficiently large domain for the fracture toughness evaluation based on the K-field approach is not troublesome (Fleck & Qiu, 2007; Thiyagasundaram, Wang, Sankar, & Arakere, 2011). For denser voided materials when the beam model becomes invalid the direct modeling of a large domain is impractical due to the large number of degrees of freedom and, therefore, the K-field approach was not applied. Several results for these type of materials were obtained by the use of the representative cell method based on the discrete Fourier transform (Lipperman, Ryvkin, & Fuchs, 2008; Ryvkin & Aboudi, 2011). This method reduces the volume of calculations by replacing the problem for a rectangular domain with a crack by several problems for a representative periodic cell. The number of the representative cell problems is equal to the total number of cells in the domain. On the other hand, a certain disadvantage in the employed framework for the representative cell method is that, contrary to the K-field approach, a finite length crack embedded in the middle of the domain is to be considered and, consequently, the following relation between the problem length scales is to be fulfilled

$$s \gg a \gg l,$$

where $s$ is the overall size of the domain. In the present work a new method combining the positive features of the K-field and the representative cell approaches is suggested.

In the next section the conventional problem for derivation of the stress field in the vicinity of a semi-infinite crack in voided material is presented. In Section 3 this problem is reformulated by the use of jump functions which allowed to apply the representative cell method for its solution as it is shown in Section 4. An application of the developed methodology for the study of fracture behavior of a plate with double periodic array of circular voids is presented in Section 5 and in the final section several conclusions are drawn.

2. Semi-infinite crack in a periodically voided material (Problem A)

The K-field approach to the fracture toughness problem, which is in common use in the analysis of cellular materials may be formulated as following in the considered case of voided material. Consider a semi-infinite crack $-\infty < X_1 < 0$, $X_2 = 0$ embedded in two-dimensional periodic voided material. The boundary problem is formulated for a rectangular domain $-L \leq X_1 \leq L$, $-L \leq X_2 \leq L$, $r = 1.2$ around the crack tip (Fig. 1a). The dashed lines in the figure should be disregarded at this stage. The displacement vector $\mathbf{u}$ and stress tensor $\sigma$ satisfy the field equations of the plane problem of elasticity defined by the differential operator $\mathcal{L}$

$$\mathcal{L}[u, \sigma] = 0,$$

the tractions $t_x$ at the voids boundaries as well as at the crack faces $t_x$ vanish

$$t_x = 0,$$

and at the outer boundaries of the rectangle the tractions defined by the K-field eigensolution for the stresses in homogeneous material possessing the effective elastic properties are applied

$$t(X_1, \pm L_2) = f^{\mathcal{K}}(X_1, \pm L_2),$$

$$t(\pm L_1, X_2) = f^{\mathcal{K}}(\pm L_1, X_2).$$

In the case when the rectangle boundaries cross the voids, as shown in the figure, a corresponding averaging procedure is to be applied. The expressions for the components of the K-field for cracks in anisotropic material can be found, for example, in Sih, Paris, and Irwin (1965).

Alternatively, the conditions at the rectangle boundaries can be formulated in terms of displacements $\mathbf{u}^K = \{u_1^K, u_2^K\}$ corresponding to the displacements K-field in the homogenized material

$$\mathbf{u}(\pm L_1, X_2) = u^{K}(\pm L_1, X_2),$$

$$\mathbf{u}(X_1, \pm L_2) = u^{K}(X_1, \pm L_2).$$
Fig. 1. Crack in periodic voided material (a), and its representative cell (b).

The stress–strain field near the rectangle boundaries in the formulated problem will differ from corresponding field in the voided material with a semi-infinite crack. But for a sufficiently large ratio \( L_2 / l \), \( r = 1.2 \) the boundary layer will not reach the near-tip region and one can evaluate the fracture toughness of the voided material by a comparison of the applied \( K \)-field and the near-tip stress distribution.

However, as mentioned, the direct solution of the formulated problem by the use of some discretization (e.g., finite element) method may be impractical due to a large number of degrees of freedom required for adequate modeling of the rectangular domain with many voids. In the following section an alternative way to reach the solution of the formulated problem (Problem A) is suggested which allows to mesh just a single periodic repetitive cell.

3. Jumps formulation of crack problems for rectangular domains

3.1. \( K \)-field problem for homogeneous domain

In order to develop an alternative way for the solution of the \( K \)-field problem for voided material, let us define functions expressing the jumps in the boundary values of the displacements and tractions of the \( K \)-field

\[
\begin{align*}
\mathbf{g}_{ uu}(X_2) &= \mathbf{u}^K(L_2, X_2) - \mathbf{u}^K(-L_2, X_2), \\
\mathbf{g}_{ tt}(X_2) &= \mathbf{t}^K(L_2, X_2) - \mathbf{t}^K(-L_2, X_2), \\
\mathbf{g}_{ uu}(X_1) &= \mathbf{u}^K(X_1, L_2) - \mathbf{u}^K(X_1, -L_2), \\
\mathbf{g}_{ tt}(X_1) &= \mathbf{t}^K(X_1, L_2) - \mathbf{t}^K(X_1, -L_2)
\end{align*}
\]  

and the crack opening displacements jump

\[
\mathbf{g}_{ tr} = \mathbf{u}^K(X_1, +0) - \mathbf{u}^K(X_1, -0).
\]  

Using the defined jump functions one can formulate the following boundary value problem for the homogeneous rectangular domain with a crack (see inset in Fig. 2)
\[
\frac{\sigma_{22}}{K} = 4
\]

**Fig. 2.** Comparison between the normal stress \( \sigma_{22} \) in front of the crack in a homogeneous plane as predicted by the present approach and the exact analytical solution.

\[
\mathcal{L}[u, \sigma] = 0, \\
ru(L_1, X_2) - ru(-L_1, X_2) = g_{in}(X_2), \\
t(L_1, X_2) - t(-L_1, X_2) = g_{it}(X_2), \quad -L_2 \leq X_2 \leq L_2, \\
u(X_1, L_2) - u(X_1, -L_2) = g_{in}(X_1), \\
t(X_1, L_2) - t(X_1, -L_2) = g_{it}(X_1), \quad -L_1 \leq X_1 \leq L_1, \\
t(X_1, +0) - t(X_1, -0) = 0, \\
u(X_1, +0) - u(X_1, -0) = g_{iv}(X_1), \quad -L_1 \leq X_1 \leq 0.
\]

The formulation of an elasticity problem in terms of jumps is somewhat non-standard and general issues of existence and uniqueness are to be addressed. The existence of the solution follows from the construction procedure; it is clear that the \( K \)-field (A1)-(A5) satisfies the conditions (15)-(21). The solution uniqueness can be verified by consideration of the problem for the difference \( u = u' - u'' \) of two possible solutions \( u' \) and \( u'' \). In accordance with (16)-(21) for this problem the jumps \( g_{in}, g_{it}, g_{it}, g_{iv}, g_{iv} \) vanish. Consequently, evaluating the expression for the elastic strain energy by the use of Clapeyron’s theorem one obtains

\[
\int_{-L_2}^{L_2} t(L_1, X_2) g_{in}(X_2) dX_2 + \int_{-L_1}^{L_1} t(X_1, L_2) g_{it}(X_1) dX_1 + \int_{-L_1}^{0} t(X_1, +0) g_{iv}(X_1) dX_1 = 0.
\]

Thus, the stress field \( \sigma \) vanishes and, consequently, the \( K \)-field is the unique solution of the problem.

### 3.2. \( K \)-field problem for voided domain (Problem B)

The formulation developed in the previous subsection prepared the grounds for the procedure of deriving the crack-tip field in a voided plane. Here, the jumps defined by (10)-(14) are to be evaluated from the \( K \)-field for the corresponding homogeneous material possessing effective elastic properties. Note, that for an arbitrary voids layout this material is anisotropic.

Using the above definitions let us formulate Problem B similar to Problem A (3)-(7), considered in Section 2. The field equation (3) and the equations on voids boundaries (4) remain the same as in Problem A,

\[
\mathcal{L}[u^0, \sigma^0] = 0, \\
u^0 = 0.
\]

and the conditions at the rectangle boundaries are expressed following (16)-(19) by the use of the corresponding jumps in the displacements and stress components.
\[ V^j_1(L_1, X_2) - V^j_1(-L_1, X_2) = G^j_1(X_2), \quad -L_2 \leq X_2 \leq L_2, \]  
\[ V^j_2(X_1, L_2) - V^j_2(X_1, -L_2) = G^j_2(X_1), \quad -L_1 \leq X_1 \leq L_1. \]  

(25)

(26)

Here \( V^j = (u, t)^T \) are the vectors representing displacements and tractions at the boundary \( X_r = \text{const.} \), and \( G_r = \{g_{e}, g_{r}\}, r = 1, 2 \). The conditions at the crack line are the prescribed crack opening

\[ u^j(X_1, +0) - u^j(X_1, -0) = g_{e}^j(X_1), \quad -L_1 \leq X_1 \leq L_1, \]  

(27)

and the equality of stress components acting at the crack faces

\[ t^j(X_1, +0) - t^j(X_1, -0) = 0, \quad -L_1 \leq X_1 \leq L_1. \]  

(28)

Superscript \( j \) denotes the iteration number and the iteration procedure is arranged to obtain zero stresses at the crack faces, namely, at each iteration the opening is increased where the stress \( \sigma_{22} \) is positive and decreased where it is negative. The initial guess for the crack opening at the first iteration is taken as in the homogeneous plane with effective elastic properties \( g_{e} \) (14). This choice is motivated by the known results for the semi-infinite cracks in periodic lattices (Ryvkin, 2012; Ryvkin & Slepyan, 2010). In these works the region of validity of long-wave asymptotes is found to be surprisingly large. At the end of the iteration procedure the solution of Problems B approaches the solution of Problem A and one can determine the fracture toughness of the voided plane. The advantage of Problem B formulation is that at each iteration one can apply the representative cell method as shown in the next section.

4. Stress field evaluation by the representative cell method

In accordance with the representative cell methodology (Ryvkin & Nuller, 1997) Problem B is reformulated as following. The rectangular periodic domain \( L < X_r < L_r, \ r = 1, 2 \) is viewed as an assemblage of \( 4N_1N_2 \) identical cells and in each cell a system of local coordinates \(-l_1 \leq x_1 \leq l_1, -l_2 \leq x_2 \leq l_2 \) is defined in an identical manner. The domain division and its representative cell are illustrated in Fig. 1 for the case \( N_1 = 3 \) and \( N_2 = 2 \). The cells identity is given by indices \( k_r = -N_r, -N_r + 1, \ldots, N_r - 1; r = 1, 2 \), which are denoted by superscripts, and the relations between the local and the global coordinate systems are

\[ X_r = l_r(k_r + 1) + x_r, \quad r = 1, 2. \]  

(29)

Consequently, field equation (23) and boundary conditions (24)–(26) are presented in the following form (index \( j \) denoting the iteration number is omitted; hereafter superscripts will denote cell location)

\[ C^{k_1,k_2} u^{k_1,k_2}(x_1, x_2) = 0, \]  

\[ u^{k_1,k_2}(x_1, x_2) = 0, \quad k_r = -N_r, -N_r + 1, \ldots, N_r - 1, \quad r = 1, 2; \]  

(30)

(31)

\[ V^{k_1,k_2}(x_1, x_2) - V^{k_1,k_2}(x_1, -x_2) = G^{k_1,k_2}(x_2), \quad k_2 = -N_2, -N_2 + 1, \ldots, N_2 - 1, -l_2 \leq x_2 \leq l_2; \]  

(32)

\[ V^{k_2,k_2}(x_1, x_2) - V^{k_2,k_2}(x_1, -x_2) = G^{k_2,k_2}(x_1), \quad k_1 = -N_1, -N_1 + 1, \ldots, N_1 - 1, -l_1 \leq x_1 \leq l_1. \]  

(33)

The crack is located at the interface between the cells \( (k_1, -1) \) and \( (k_1, 0) \), \( k_1 = -N_1, -N_1 + 1, \ldots, 1 \), however, the conditions (27) and (28) may be formally extended also for \( k_1 = 0, 1, \ldots, N_1 - 1 \), if one assumes that the jumps in the right hand parts for these values of \( k_1 \) vanish. Then these conditions can be presented as following

\[ V^{k_1,0}(x_1, -x_2) - V^{k_1,-1}(x_1, -x_2) = G^{k_1,k_1}(x_1), \quad k_1 = -N_1, -N_1 + 1, \ldots, N_1 - 1, -l_1 \leq x_1 \leq l_1, \]  

(34)

where \( G^{k_1,k_1}(x_1) = \{g_{e}^{k_1,k_1}(x_1), \ldots, 0\} \). The problem formulation is to be completed by the continuity conditions at the cells interfaces, namely,

\[ V^{k_1,k_1}(x_1, x_2) - V^{k_1,k_1}(x_1, -x_2) = 0, \quad k_1 = -N_1, -N_1 + 1, \ldots, N_1 - 2; \quad k_2 = -N_2, -N_2 + 1, \ldots, N_2 - 1, \]  

(35)

\[ V^{k_2,k_2}(x_1, x_2) - V^{k_2,k_2}(x_1, -x_2) = 0, \quad k_2 = -N_2, \ldots, -2, 0, 1, \ldots, N_2 - 2; \quad k_1 = -N_1, -N_1 + 1, \ldots, N_1 - 1. \]  

(36)

The Finite Discrete Fourier Transform (FDFT) in the direction \( r = 1, 2 \) is defined by the formula

\[ f^{k_r} = \sum_{k_r = -N_r}^{N_r - 1} \int f_r \exp i k_r q_r \]  

(37)

where \( \phi_r = \frac{\pi q_r}{N_r} \), \( q_r = -N_r, -N_r + 1, \ldots, N_r - 1. \)  

(38)

Consequently the double Finite Discrete Fourier Transform reads as following

\[ f^T = \{f^T, f^T\} \]  

(39)

Applying this transform to problem (30)–(36) yields after some manipulation the problem for the representative cell with respect to the complex valued displacement transforms \( u^j(x_1, x_2, q_1, q_2) \)
\[ \mathcal{L}[\mathbf{u}, \sigma^r] = 0, \tag{40} \]
\[ \mathbf{t}^r = 0, \tag{41} \]
\[ \gamma_1^{-1} \mathbf{V}^r_1(-l_1, x_2) - \mathbf{V}^r_1(l_1, x_2) = -\gamma_1^{N_1-1} \mathbf{G}^r_1(x_2), \tag{42} \]
\[ \gamma_2^{-1} \mathbf{V}^r_2(x_1, -l_2) - \mathbf{V}^r_2(x_1, l_2) = -\gamma_2^{N_2-1} \mathbf{G}^r_2(x_1) + \gamma_2^{-1} \mathbf{G}^r_2(x_1). \tag{43} \]

where \( \gamma_r = \exp(iq_r) \), \( r = 1, 2. \)

After resolving this problem for all values of \( q_r \) defined in (38) one can determine actual displacements and stresses in any location of the rectangular domain by the inverse transform formula. For example,

\[ \sigma^{k_1, k_2}_{22}(x_1, x_2) = \frac{1}{4N_1N_2} \sum_{q_1=-N_1}^{N_1-1} \sum_{q_2=-N_2}^{N_2-1} \sigma_D^{k_1, k_2}(x_1, x_2, q_1, q_2) \exp[-i(q_1k_1 + q_2k_2)]. \tag{44} \]

The representative cell problem (40)-(43) can be solved by any discretization method and in the present study the finite element method (FEM) is employed. An interested reader can find the details of the application of the FEM in the representative cell environment in previous works (Lipperman et al., 2008; Moses, Ryvkin, & Fuchs, 2001; Ryvkin & Nuller, 1997). The representative cell problem in the present work was resolved with PDE Toolbox of MATLAB software.

The verification of the suggested method accuracy was carried out by means of the Mode I crack problem in homogeneous isotropic elastic domain (see inset in Fig. 2). The jumps \( \mathbf{g}^{k_1, k_2}_{ua}(x_i), \mathbf{g}^{k_1, k_2}_{ut}(x_i), \mathbf{g}^{k_1, k_2}_{tt}(x_1) \) at the rectangular boundaries entering the boundary conditions (32)-(34) are calculated from formulas (10)-(14) and (A1)-(A5) in Appendix A. For the boundaries \( X_1 = \pm l_1; \)

\[ \mathbf{g}^{k_1}_{ua}(x_2) = \frac{K}{2E} \begin{pmatrix} r^{1/2} \\ 2 \pi \end{pmatrix} \{ D_1(\theta, r), D_2(\theta, r) \}, \tag{45} \]
\[ \mathbf{g}^{k_1}_{ut}(x_2) = \frac{K}{(2\pi r)^{1/2}} \{ D_3(\theta), D_4(\theta) \}, \tag{46} \]

where \( r = \sqrt{l_1^2 + x_2^2}, \; \theta = \arccos(x_2/l_1), \) and for the boundaries \( X_2 = \pm l_2 \)

\[ \mathbf{g}^{k_2}_{ua}(x_2) = \frac{K}{2E} \begin{pmatrix} r^{1/2} \\ 2 \pi \end{pmatrix} \{ 0, D_5(\theta, r) \}, \tag{47} \]
\[ \mathbf{g}^{k_2}_{ut}(x_2) = \frac{K}{(2\pi r)^{1/2}} \{ 0, D_6(\theta) \}, \tag{48} \]

where \( r = \sqrt{x_1^2 + l_2^2}, \; \theta = \arccos(l_2/x_1). \) The functions \( D_{s}, \; s = 1, 2, \ldots 6 \) are presented in Appendix B and passing from the global coordinates \( X_1, X_2 \) to the local ones \( k_1, k_2, x_1, x_2 \) is carried out in accordance with (29).

For the considered problem the applied crack opening jump corresponds to the exact solution and, consequently, there is no need in iteration procedure. The number of degrees of freedom in the square representative cell is 4700 and the rectangular domain composed of 36 cells with \( N_1 = N_2 = 3 \) is considered. The comparison of the obtained results for normal stress in front of the crack with the analytical solution is shown in Fig. 2. It is readily observed that the present approach predicts results which are in excellent agreement with the analytical expression.

### 5. Application: does perforation of a plate increase its crack resistance?

As an example of application of the developed technique the Mode I fracture toughness of voided material with a fully triangulated circular voids layout is examined (Fig. 3a). The voids have radius \( R \), the distance between their centers is \( l \), and the relative density \( \rho \) of the material for the considered layout is given by

\[ \rho = 1 - \frac{2\pi}{\sqrt{3}} \frac{(R/l)^2}{l}. \tag{49} \]

Note, that the maximal possible voids radius for the considered layout is bounded \( R < l/2 \), and, therefore, \( \rho > 0.093 \). The representative cell with \( l_1 = l \) and \( l_2 = \sqrt{3}l \) which is chosen for the analysis is depicted in Fig. 3b and the rectangular material domain shown in the figure includes 196 cells. This number corresponds to the values \( N_1 = N_2 = 7 \) which were established by numerical experiments from the condition that their increase does not affect the stress field in the crack-tip vicinity. The mesh for the representative cell (Fig. 3b) has 4700 degrees of freedom. This corresponds to 921,200 degrees in the overall domain to be considered in the framework of conventional K-field approach.

The in-plane stiffness of the 2D orthotropic solid corresponding to the considered layout of circular voids is described by two effective elastic moduli: Young modulus \( E \) and Poisson ratio \( \nu \). Their values were determined by the same program code which was developed for the advocated approach by application of uniform jumps at the opposite sides of periodic rectangular domain without crack. The Poisson ratio \( \nu \) of a solid material has an impact on the effective value \( \nu' \) of a voided one and, consequently, on the jumps in boundary displacements and the crack opening which enter the boundary conditions (25)-(27). In spite of this, stress distribution in the considered plane problem is independent of the parent solid material
elastic properties. Therefore, the conducted numerical experiments which confirmed this statement can be considered as an additional verification of the suggested methodology.

The distribution of maximal principal stress in the crack-tip vicinity for $R/l = 0.29$ and $\rho = 0.7$ is shown in Fig. 4. The stress concentration at the intersection of the crack line and crack tip void contour, where the local tensile stress approaches its maximum, is clearly visible. As in the case of cellular materials (Gibson & Ashby, 1997; Romijn & Fleck, 2007; Lipperman, Ryvkin & Fuchs, 2007) this value is used for the prediction of macroscopic fracture behavior of the considered voided one. Its fracture toughness $K_{IC}$ is the value of remote $K$ at which the maximal local stress attains the tensile strength of the parent brittle material $\sigma_T$. The toughness for the relative density $\rho = 0.5$ was compared with the result obtained in Lipperman et al. (2008) where simplifying assumptions regarding the stress distribution at the intevent ligaments were adopted. The observed 5% discrepancy is reasonable in view of the difference in the employed approaches.

The dependence of the normalized fracture toughness upon the material relative density is presented in Fig. 5. It is seen, that for low and moderate densities $0.15 < \rho < 0.7$ the fracture toughness increases proportionally to the relative density

![Fig. 4. (color online) Stress field in the crack tip vicinity for $\rho = 0.7$; maximal principal stress $\sigma_1$ is shown.](image-url)
increment. Interestingly, this type of behavior was reported by Fleck and Qiu (2007) also for honeycombs with the same triangular layout as in the present example. However, for high densities, close to unity, the behavior changes and steep toughness decrease takes place. This non-monotonic dependence of the fracture toughness evaluated by the strength criterion was obtained also in a similar problem for two-dimensional material with square voids (Ryvkin & Aboudi, 2011). It follows from the competition between two effects related to the density increase, i.e., decrease of the voids radii. On the one hand the radius decrease extends the thickness of intervoid ligament which reduces the level of near tip stresses, and on the other hand a too small void becomes a stress concentrator which leads to increase of the maximal stress value. The interested reader can find a comprehensive discussion on the influence of pore size on the fracture toughness of porous materials in Leguillon and Piat (2008). It is readily seen that for a certain value of the relative density \( \rho = 0.85 \) the normalized fracture toughness approaches its maximum value. This result, which can be also expressed by the use of (49) in terms of voids radius \( R/l = 0.203 \) is of practical importance and can be used in the design of crack resistant perforated plates.

It is of interest to compare the fracture behavior of solid and voided plates. The voids clearly increase the value of maximal stress in a plate subjected to a remote tensile loading and therefore raise the danger of crack nucleation. On the other hand, they can play a positive role in arresting an already existing crack thanks to the crack blunting phenomenon when the crack terminates in a void. This phenomenon of cracks arrest by a dilute system of non-interacting voids was investigated recently by Liu and Chen (2008). Concentrating on the analysis of long cracks it is worthwhile to make a note regarding the criteria for crack propagation. The fracture toughness of voided material is evaluated above by the use of the strength criterion in the framework of which the crack propagation takes place when a new microcrack nucleates at the crack tip void's boundary. Consequently, as in the case of cellular materials, the fracture toughness \( K_{IC} \) is calculated in terms of the parent material tensile strength \( \sigma_f \). On the contrary, addressing the crack propagation in a homogeneous material without voids it is natural to use the energy criterion, expressed by the fracture toughness of solid material \( K_{IC} \) which is determined experimentally and represents an independent material parameter. Thus, the influence of voids on the resistance to crack propagation in a plate can be estimated by a comparison of the values \( K_{IC} \) and \( K_{IC} \). To this end one has to choose the distance between the voids \( l \) and calculate the non-dimensional value \( K_{IC}/(\sigma_f \sqrt{l}) \) for a specific material. For example, the dashed line in Fig. 5 indicates this value for the Alumina 99.9% with \( l = 1.5 \) mm. The relative densities for which the calculated curve located above this line correspond to parameter combinations when a voided plate provides better crack arresting ability than a solid one.

The only variable parameters defining the fracture toughness of the specific voided material are the distance between the voids \( l \) and their radius \( R \). Therefore, it is worthwhile to present the obtained results in \( R-l \) space. Such presentation may also be useful for the design purposes. In view of (49) the dependence depicted in Fig. 5 can be symbolically presented in the form

\[
\frac{K_{IC}}{\sigma_f \sqrt{l}} = f(R,l).
\]

Replacing now \( K_{IC} \) by the actual value \( K_{IC} \) for a given material one can consider the latter equation as a definition of the implicit function \( l = l(R) \). The corresponding curve in the \( l-R \) plane will separate two regions corresponding to voided/unvoided
design preference. The results obtained for several brittle materials are presented in Fig. 6. The values of $R$ and $l$ for which a voided plate is a better design solution from the fracture toughness point of view is defined by the domain above the curve for a specific material. Recall that for the considered layout there is a geometric stipulation $R < l/2$ and, therefore, only points above the dashed line, referring this constraint, should be addressed. Finally, it should be noted, that the shape of the obtained dependencies allows to determine for each material lower bounds for the voids radius and the distance between their centers for the cases where a voided plate has a higher crack arresting ability than a solid one.

6. Concluding remarks

Modeling of fracture behavior of materials with periodic microstructure often presents a serious computational problem due to a large number degrees of freedom involved into analysis. The representative cell method based on the discrete Fourier transform was successfully employed previously for reducing the calculations volume in the cases when the damaged region is localized, and, consequently, stress perturbation does not reach the remote boundaries. The present work extends the application of the representative cell approach for problems where fracture is caused by a long crack and where it is therefore natural to consider a semi-infinite crack model. It appears that in this case it is also possible to reduce the analysis of a periodic domain around the crack tip to a multiple analysis of a single repetitive module in the transform space.

The suggested technique is applicable for a semi-infinite crack problem in any 2D periodic composite. It was verified by a comparison with analytical solution for the degenerate case of a homogeneous plane. For the sake of brevity, however, all the general considerations and the example of application address the case of voided material.

In the considered application a plane with triangulated circular voids isotropic layout is examined. The numerical efficiency of the suggested method allowed to calculate the fracture toughness of voided material by the use of stress criterion for crack propagation. For low and moderate densities the dependence of the fracture toughness upon the material relative density is found to be linear. For higher densities, however, a non-monotonic behavior with a single maximum is observed. Such a behavior results from the concurrence between two phenomena caused by the voids: crack tip blunting and stress concentration around the voids. A parametric study has shown for which voids radii and distances between their centers the fracture toughnesses of voided material is higher than that of a homogeneous one.

The results show that the suggested approach is a convenient tool for the investigation and design of new materials with improved fracture properties. The essential next step for its application is the analysis of cracks in three-dimensional micro-architected materials.

Appendix A

Eigensolution ($K$-field) for Mode I crack in isotropic material with Young modulus $E$ and Poisson ratio $\nu$
\[ \sigma_{11} = \frac{K}{2\pi r^{1/2}} \cos \frac{\theta}{2} \left[ 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right], \]  
\[ \sigma_{22} = \frac{K}{2\pi r^{1/2}} \cos \frac{\theta}{2} \left[ 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right], \]  
\[ \sigma_{12} = \frac{K}{2\pi r^{1/2}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2}, \]  
\[ u_1 = \frac{K}{2E} \left( \frac{r}{2\pi} \right)^{1/2} (1 + v) \left[ (2\kappa - 1) \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \right], \]  
\[ u_2 = \frac{K}{2E} \left( \frac{r}{2\pi} \right)^{1/2} (1 + v) \left[ (2\kappa + 1) \cos \frac{\theta}{2} - \sin \frac{3\theta}{2} \right], \]  
where \( \kappa = \frac{3 - v}{1 + v} \) for plane strain and \( \kappa = 3 - 4v \) for plane strain.

Appendix B

\[ D_n(\theta, \nu) = (1 + \nu) \left\{ (1 - (-1)^n) \pi \frac{\sin \theta}{2} \cos \frac{\theta}{2} - \sin \frac{3\theta}{2} \cos \frac{3\theta}{2} \right\}, \quad n = 1, 2 \]
\[ D_3(\theta) = \cos \frac{\theta}{2} \sin \frac{\theta}{2} + \frac{1}{2} \sin \theta \left( \cos \frac{3\theta}{2} - \sin \frac{3\theta}{2} \right), \]
\[ D_4(\theta) = \frac{1}{2} \sin \theta \left( \cos \frac{3\theta}{2} + \sin \frac{3\theta}{2} \right), \]
\[ D_5(\theta, \nu) = 2(1 + \nu) \left[ (1 + 2\kappa) \sin \frac{\theta}{2} - \cos \frac{3\theta}{2} \right], \]
\[ D_6(\theta) = \sin \theta \cos \frac{3\theta}{2}, \]

References