

# Using a pseudo-stochastic approach for multiple-parts scheduling on an unreliable machine

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In this paper we follow previous “pseudo-stochastic” approaches that solve stochastic control problems by using deterministic optimal control methods. In a similar manner to the certainty equivalence principle, the suggested model maximizes a given profit function of the expected system outcome. However, unlike the certainty equivalence principle, we model the expected influences of all future events (including those that are *expected* beyond the planning horizon), as encapsulated by their density functions and not only by their mean values. The model is applied to the optimal scheduling of multiple part-types on a single machine that is subject to random failures and repairs. The objective of the scheduler is to maximize the profit function of the produced multiple-part mix. A numerical study is performed to evaluate the suggested pseudo-stochastic solutions under various conditions. These solutions are compared to a profit upper bound of the stochastic optimal control solutions.

## 1. Introduction

A major difficulty that has to be resolved when modeling a stochastic dynamic system is how to treat information regarding future events. Several modeling approaches have been suggested for stochastic optimal control (Boukas *et al.*, 1996; Sethi *et al.*, 1998; Bertsekas, 2000). Yet, the effects of future information on the decision-making process within a finite planning horizon are not fully understood (Neck, 1984).

Neck (1984) suggested the division of the analytical modeling of stochastic optimal control into two approaches which he called “*stochastic optimal control*” and “*pseudo-stochastic*.” The stochastic optimal control approach is concerned with models where the state variables are distributed stochastically. In these models, the dynamic equations are often approximated by Ito’s differential equations that include additive components, such as: (i) the expected outcome of random variables; (ii) the control function; and (iii) a white-noise Weiner process that represents the environmental changes. As an example of this approach, consider a Markovian process that models a dynamic system in which the state variables are random (Kimemia and Gershwin, 1983; Haurie, 1995; Farid and Davis, 1999).

The large body of literature that deals with the modeling of *stochastic optimal control* has revealed common difficul-

ties in the representation of uncertain future events. Elhafsi and Bai (1997) selected the state variable of the production system to be the random variable whose expectation is estimated. Such a selection is problematic since the distribution of the state variables strongly depends on *a priori* unknown optimal control. Kleindorfer and Glover (1973) discussed a linear system whose dynamics were described by a discrete stochastic process with a randomized component that was assumed to be additive. Such an assumption limits the consideration to approximate models of stochastic behavior. Haurie (1995) dealt with the problem in which demand and machine failures were assumed to be independent sources of uncertainty. The uncertainty sources were combined into a single, continuous-time Markov chain, where each state is a vector. The author assumed that the dynamic equation is represented by the sample mean and sample variance of a random demand. Love and Turner (1993) represented the variability of the process by a given error density function, which influenced the state variable.

The *pseudo-stochastic* approach solves stochastic problems by using deterministic optimal control methods. This approach was used, for example, in Tapiero and Venezia (1979), Kamien and Schwartz (1981) and Tapiero (1983). Perhaps the earliest and most applied pseudo-stochastic method is known as the *certainty equivalence principle*. In the context of optimal control theory, the *certainty equivalence principle* is applied to a control problem with linear dynamics and a quadratic objective function in the presence

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of a Gaussian white noise. Assuming that the state variable cannot be directly measured, it can be shown that the optimal control can be derived based on the estimate of the state variable as obtained by the Kalman filter. As indicated in Sethi and Thompson (2000), the principle is termed by different names in the literature. In Economics, it is known as the “*certainty equivalence principle*” and in Engineering and Mathematics, as the “*separation principle*”. When using the first term, one emphasizes the fact that the state variable can be replaced by its estimates and used for the purpose of optimal feedback control. In other words, one can say that the certainty equivalence holds, if the optimal policy of the stochastic control problem is unaffected when the disturbances are replaced by their means (Bertsekas, 2000). When using the term “*separation principle*,” one emphasizes that the optimal control process is broken into two steps: (i) estimating the state variable by using an optimal filter; and (ii) using that estimate in the feedback control formula for the deterministic problem (Sethi and Thompson, 2000). The *certainty equivalence control* is a related suboptimal policy that has been suggested to replace the stochastic control problem with the “deterministic equivalent” problem (Whittle, 1982; Bertsekas, 2000). It implies that a stochastic control problem is replaced by a deterministic one, where at each time period the uncertain quantities are fixed at some “typical” values: not necessarily the expected values. The remaining difficulty in all these approaches is how to handle stochastic phenomena by deterministic state variables that should capture all future realizations. Such a task seems impossible for the general case when one uses only the expected values of the unknown quantities, rather than using their (known) density functions, as we endeavor to do here.

In this work we follow the above-mentioned pseudo-stochastic approaches to obtain an optimal control of a dynamic system over a given planning horizon. Similar to these approaches, our model is deterministic in nature and, thus, allows us to analyze the optimal control by means of the maximum principle. The main difference with respect to these approaches, including the certainty equivalence principle, is that we try to incorporate future stochastic phenomena by using all the known information about the uncertain quantities, as captured by their density functions. In particular, we apply the suggested method to a specific optimal control problem of scheduling multiple part-types on an unreliable machine.

The rest of the paper is organized as follows. In Section 2, a general modeling formulation is presented. An application of this model is given in Section 3, where an optimal control problem of scheduling multiple part-types on an unreliable machine is analyzed. Based on the properties of the optimal solution, an optimal scheduling procedure that maximizes a given profit function of the produced parts is suggested. It is evaluated with respect to El-Ferik *et al.* (1998) that practically deals with the same application in the framework of stochastic optimal control. In Section 4,

a numerical analysis of the suggested application is conducted. In particular, the obtained pseudo-stochastic solution is compared to an upper bound of an unknown solution of the stochastic optimal control. It is found that the difference between these solutions is negligible, being in the vicinity of less than 0.8%, for all instances of the considered problem. Section 5 concludes the paper.

## 2. Pseudo-stochastic model

In this section we consider optimal control modeling of systems that operate under uncertainty. Various real-life systems fall into this category. Examples include inventory systems that are controlled by a given set of rules, heuristic scheduling systems, service systems and so on.

### 2.1. Main idea

We follow the pseudo-stochastic approach that was described in Section 1. We model future uncertain events by their probability density functions. This allows us to consider the influence of all future events (including those that are *expected* beyond the planning horizon) on the system dynamics within the finite planning horizon. The magnitude of an event’s influence on the state variables at future time  $t$  is proportional to the probability for that event to occur at time  $t$ . The model seeks an optimal control function  $u^*(t)$ , which maximizes a performance measure along the planning horizon  $T$ .

### 2.2. Modeling assumptions and formulation

We consider the following assumptions when applying the suggested pseudo-stochastic approach to a real-life system:

1. all sources of uncertainty are known;
2. different sources of uncertainty have an independent influence on the system states;
3. the dynamics of the state variables can be described by differential equations.

The model formulation now follows. Consider a dynamic system characterized by a set of state variables which define the system state at each point in time. The variables are denoted by the vector  $\mathbf{X}(t)$  and follow the dynamic equation, Equation (1), and the set of constraints of Equations (2) and (3):

$$\frac{d\mathbf{X}(t)}{dt} = \mathbf{f}(\mathbf{X}(t), \mathbf{u}(t), M(t), t), \quad \mathbf{X}(0) = \mathbf{X}^0, \quad (1)$$

$$g(\mathbf{X}(t), \mathbf{u}(t), t) \leq 0, \quad (2)$$

$$p_1(\mathbf{X}(T)) \leq 0, \quad p_2(\mathbf{X}(T)) = 0 \quad (3)$$

where  $\mathbf{f}(\mathbf{X}, \mathbf{u}, M, t)$  is a deterministic vector-valued function (differentiable with respect to (w.r.t.)  $\mathbf{X}$ ,  $\mathbf{u}$ , and integrable w.r.t.  $M$  and  $t$ ) which defines the dynamics of the state variable vector  $\mathbf{X}(t)$ ;  $\mathbf{u}(t)$  is the control vector-function which

is integrable; and  $M(t)$  is a function which represents uncertainties in the system. Particularly,  $M(t)$  encapsulates all future (discrete) events that will influence the system dynamics. Functions  $g$ ,  $p_1$  and  $p_2$  defining the constraints imposed on the system are assumed differentiable w.r.t. their arguments.

Let there be  $K$  different types of stochastic events. The inter-arrival times between successive events of type  $k$ ,  $Z^k$ ,  $k = 1, \dots, K$  are independent and identically distributed random variables with probability density  $\varphi^k(t)$ . The occurrence time  $t$ , of the  $n$ th event ( $n = 1, \dots, \infty$ ) of type  $k$ , given that the occurrence of the  $k$ th event type before  $t = 0$  happened at  $t = -s_k$ , is a stochastic variable denoted by  $Z_n^k(s_1, \dots, s_K)$  with a given density function  $\pi_n^k(t)$ . Hereafter, we omit the parameters  $s_k$  for simplicity and without loss of generality.

The function  $M(t)$  represents an expected influence of the uncertainty sources at time  $t$ . We assume that it takes an additive form with respect to  $\sum_{n=1}^{\infty} \pi_n^k(t)$ . In Appendix A we demonstrate the calculation of function  $M(t)$  for a special case. Generally, the function  $M(t)$  is assumed to be of the following form:

$$\dot{M}(t) = \sum_{k=1}^K \alpha_k \sum_{n=1}^{\infty} \pi_n^k(t), \quad (4)$$

where the multipliers  $\alpha_k$ , which may be either positive or negative, represent the effects of the  $k$ th event type on the system dynamics and normalize the units of the various density functions to a single scale.

A performance measure of the system dynamics is:

$$U(\mathbf{X}(T)) \rightarrow \max, \quad (5)$$

where  $U(\mathbf{X})$  is a profit function of outcome  $\mathbf{X}$ . Thus, the problem is to find off-line the optimal control function  $\mathbf{u}^*(t)$  that will maximize the performance measure of Equation (5) under the system dynamics of Equation (1) and the constraints of Equations (2) and (3).

The problem represented by Equations (5), (1)–(3) is the canonical form of the deterministic optimal control and can, in principle, be solved by means of the *maximum principle* and by known numerical procedures (Maimon *et al.*, 1998; Sethi and Thompson, 2000). Moreover, in certain cases such problems can be solved more easily on the basis of their specific properties, which also give a basis to the development of simple heuristics, as illustrated in the next sections.

Maximizing the profit of an expected system outcome, as we suggest here, is an entirely different problem to that of maximizing expected profit over all possible system outcomes. In fact, the famous Jensen inequality (Cover and Thomas, 1991; Hillier and Lieberman, 1995) addresses exactly that phenomenon and ensures that for a convex profit function, the outcome of the latter approach will be greater than or equal to the outcome of the former approach. The statements of stochastic problems in the framework

of stochastic optimal control are given, for example, in Akella and Kumar (1986) and in El-Ferik *et al.* (1998). It is important to note that although the stochastic optimal control tries to solve the “real problem” by the latter approach, it is not only more complicated and often computationally intractable, but also, in many cases, unfeasible. In particular, an exact control for any possible trajectory in the system is unknown in many cases, as illustrated in the next section, and, thus, obtaining a solution based on this paradigm is unfeasible. There are very rare cases (Akella and Kumar, 1986) that do allow rigorous analytical solution of a stochastic optimal control problem. At the same time, although a control for the suggested approach might be obtained easily, it should be carefully checked against some upper bound (even a numerical one) to ensure its relevance as demonstrated next.

### 3. Maximizing profit: modeling application problem

In this section we exemplify both the advantages and applicability of the suggested modeling approach by solving a stochastic scheduling problem. We consider the scheduling of multiple product-types on a single machine which can process any part type from a given set of cardinality  $I$ . The machine is subject to random failures and repairs. An off-line production planning (scheduling) has to be performed at the beginning of the planning horizon  $t = 0$ . The machine state at any future time ( $t > 0$ ), denoted by a zero for an idle state or by a one for a running state, is unknown *a priori*, yet, the time-dependent probabilities of machine failures and repairs can be calculated. In particular, we define two new random variables,  $Z_n^1 \equiv F_n(s)$  and  $Z_n^2 \equiv R_n(s)$ ,  $n = 1, \dots, \infty$ , denoting, respectively, the time of the  $n$ th failure (recovery), provided that the last change of the machine-state before  $t = 0$  happened at  $t = -s$ . The random variables  $F_n(s)$  and  $R_n(s)$  are defined under two initial conditions:

1. If at  $t = 0$  the machine is up (see Fig. 1):

$$\begin{aligned} F_n(s) &= F_1(s) + \sum_{j=1}^{n-1} R_{1j} + \sum_{j=1}^{n-1} F_{1j}; \\ R_n(s) &= F_1(s) + \sum_{j=1}^{n-1} R_{1j} + \sum_{j=1}^n F_{1j}. \end{aligned} \quad (6)$$

2. If at  $t = 0$  the machine is down:

$$\begin{aligned} F_n(s) &= R_1(s) + \sum_{j=1}^n R_{0j} + \sum_{j=1}^{n-1} F_{0j}; \\ R_n(s) &= R_1(s) + \sum_{j=1}^{n-1} R_{0j} + \sum_{j=1}^{n-1} F_{0j}, \end{aligned} \quad (7)$$

where  $R_{ij}$  is a random variable denoting the time between the  $j$ th recovery and the  $(j + i)$ th failure,  $i = 0$  and 1 and

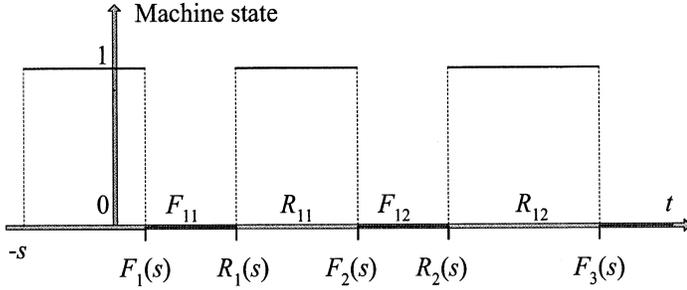


Fig. 1. Machine state dynamics.

also  $F_{ij}$  is a random variable denoting the time between the  $j$ th failure and the  $(j - i + 1)$ th recovery,  $i = 0$  and  $1$ .

The system state variable at time  $t$ , which is, in this case, the inventory level of part types, depends on the expected machine state  $M(t)$ , where:

$$0 \leq M(t) \leq 1. \tag{8}$$

The derivative of  $M(t)$  with respect to  $t$ , i.e., the rate of the expected machine state at time  $t$ , is equal to (see proof in Appendix A):

$$\dot{M}(t) = \sum_{n=1}^{\infty} [\pi_n^{R(s)}(t) - \pi_n^{F(s)}(t)], \quad M(0) = M^0, \tag{9}$$

where,  $\pi_n^{R(s)}(t)$  denotes the probability density function of  $R_n(s)$ , and  $\pi_n^{F(s)}(t)$  denotes the probability density function of  $F_n(s)$ , and  $M^0$  is the initial machine state (zero or one).

The right-hand side of Equation (9) includes an infinite sum of probability density functions, since in a stochastic environment, events which are expected beyond the planning horizon, have a positive probability to occur within the planning horizon.

If we were interested in solving a ‘‘certainty equivalence’’ approximation, i.e., a deterministic problem where the stochastic parameters in the dynamics are replaced by their expected values, we would use  $M(t)$  as shown in Fig. 2. That is, we would assume a representing ‘‘average scenario’’ where the machine is working for  $E[R_{ij}]$  time units and then it is idle for  $E[F_{ij+1}]$  time units. Our pseudo-stochastic approach goes far beyond this by making use of the entire

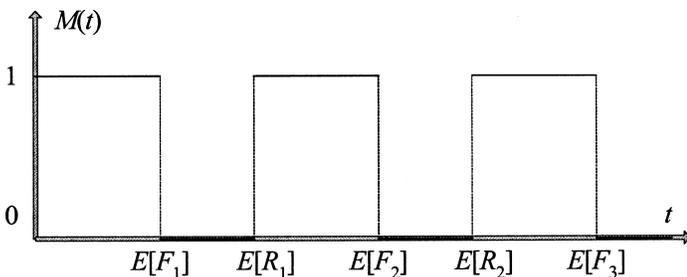


Fig. 2. Function  $M(t)$  in the ‘‘certainty equivalence’’ approximation case.

probability distribution of the events as in Equation (9), rather than only their expected values.

An expression similar to Equation (9) was used in Herbon et al. (2004) for a pseudo-stochastic description of the demand process, where the demand rate of parts arriving for processing was modeled as:

$$\sum_{n=1}^{\infty} \pi_n(t, s),$$

where  $\pi_n(t, s)$  is the probability density function of the  $n$ th part arrival at time  $t$ , provided that the last part arrival before  $t = 0$  happened at  $t = -s$ .

As before, we omit the parameter  $s$  when using the function  $M(t)$  for the analysis of the profit maximization problem, which is considered in the next section.

### 3.1. Optimal control model

Let  $X_i(t)$  denote the expected number of parts of type  $i = 1, \dots, I$ , accumulated by time  $t$ , and  $u_i(t)$  denote the scheduler, which controls the production process:

$$u_i(t) = \begin{cases} 1, & \text{if machine produces type } i \text{ at time } t, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

The problem is to maximize the profit function:

$$\text{Max } U(X_1(T), \dots, X_I(T)), \tag{11}$$

subject to the following constraints:

1. The change in  $X_i(t)$  is equal to the expected production rate:

$$\frac{dX_i(t)}{dt} = \frac{M(t)u_i(t)}{t_i}, \quad \forall i, t \quad X_i(0) = X_i^0, \tag{12}$$

where  $t_i$  is the processing time of part type  $i$  and  $X_i^0$  is the initial surplus of parts  $i$ .

2. The machine can process only one part at a time, i.e.,

$$\sum_{i=1}^I u_i(t) \leq 1, \quad \forall t. \tag{13}$$

As stated earlier, the problem defined by Equations (10)–(13), which is in general a very difficult problem, can be solved on the basis of its specific properties that allow us to transform the problem into an equivalent and simpler form. This is done by mapping the time variable  $t$  to a new variable,

$$t'(t) = \int_0^t M(\tau) d\tau.$$

Intuitively, such a mapping means that the ‘‘machine clock’’ slows down at a time interval where the expected machine state is less than one. The higher the probability is that the machine is down, the slower is the ‘‘machine clock’’ at that time interval. Therefore, as the expected machine state decreases, the time interval required to produce a part

increases. Following this transformation, the new planning horizon is:

$$T' = \int_0^T M(\tau)d\tau,$$

and the equivalent form of the problem can now be written as:

$$\text{Max } U(X_1(T'), \dots, X_I(T')), \quad (14)$$

subject to

$$\frac{dX_i(t')}{dt'} = \frac{u_i(t')}{t_i}, \quad \forall i, t' \in [0, T'], \quad X_i(0) = X_i^0, \quad (15)$$

$$\sum_{i=1}^I u_i(t') \leq 1, \quad u_i(t') \in \{0, 1\}. \quad (16)$$

Note that the profit function in Equation (14) is static, i.e., it depends on the machine net working time but not on the sequence (timing) of the product types along the planning horizon  $[0, T']$ . Thus, by integrating Equation (15) along  $[0, T']$ , we reduce the dynamic problem defined by Equations (14)–(16) to a static one:

$$\text{Max } U(X_1(T'), \dots, X_I(T')), \quad (17)$$

subject to

$$\sum_{i=1}^I (X_i(T') - X_i^0) t_i \leq T' \quad \text{and} \quad X_i(T') \geq X_i^0. \quad (18)$$

This is a non-linear programming problem with decision variables  $X_i(T')$  which is, in general, very hard to solve. The optimal policy for a linear profit function,  $U = c + \sum_i a_i X_i$ , where  $c$  and  $a_i$  are given constants, is to produce part type  $i = \arg \max(a_i/t_i)$ , which generates the highest increment of the profit function. The optimal solution for a convex separable function is similar. However, if  $U(\mathbf{X})$  is not linear or convex separable, the optimal solution might require the machine to switch from one part-type to another, thus producing a subset of part types. In such a case, the problem becomes hard and is solved numerically by making use of methods for maximizing the non-linear function Equation (17) on the simplex of Equation (18) (Bertsekas, 1999). Since the problem of Equations (17) and (18) is static, one may choose any sequence of the product types to obtain the optimal schedule of the machine. The switching time,  $r_i$ , at which the machine starts processing product type  $i + 1$ , is then obtained iteratively from the following expression:

$$\int_{r_{i-1}}^{r_i} M(\tau)d\tau = X_i(T')t_i, \quad \forall i = 1, \dots, I, \quad (19)$$

where  $r_0 = 0$  and  $r_I = T$ .

The suggested solution approach can now be summarized as follows.

*Step 1.* Calculate the expected machine state,  $M(t)$ , from Equation (9) (see technical details in Appendix B)

**Table 1.** Processing times and profit function coefficients of different part types

Part types	$P_1$	$P_2$	$P_3$	$P_4$	$P_5$	$P_6$	$P_7$	$P_8$	$P_9$	$P_{10}$
$t_i$	5	10	8	12	15	9	30	3	45	25
$a_i$	0.3	0.4	0.2	0.8	0.08	0.5	1.0	0.1	0.9	0.6

*Step 2.* Calculate the transformed time horizon:

$$T' = \int_0^T M(\tau)d\tau.$$

*Step 3.* Find the optimal produced quantities of parts  $X_i(T')$ ,  $i = 1, \dots, I$ , by solving Equations (17) and (18) (a detailed algorithm for solving a case of Equations (17) and (18) is given in Appendix C).

*Step 4.* Calculate the switching times  $r_i$ ,  $i = 1, \dots, I$ , from Equation (19).

Note that the suggested pseudo-stochastic formulation admits an exact analytical solution. In contrast to the pseudo-stochastic formulation, the stochastic optimal-control formulation significantly complicates the solution, as shown in El-Ferik *et al.* (1998). In fact, in El-Ferik *et al.* (1998) only a heuristic Simple Maximal Hedging (SMH) policy was developed to control a single machine over a long time period. Numerical optimization over the class of SMH policies was then performed by solving the Hamilton-Jacobi-Bellman equations. An exact analytical solution was not obtained even for a simpler version of the problem with an additional simplifying assumption that allows a number of different part types to be produced simultaneously. Thus, using the suggested pseudo-stochastic formulations, one can solve more complex models that approximate the solutions of stochastic optimal control.

In spite of the fact that the original problem of Equations (11)–(13) seeks an “open-loop” optimal solution, it turns out that, due to the specific properties of the problem, the optimal solution of Equations (11)–(13) is a feedback control rule: this is so because the machine switches to another product when the produced quantity reaches the desirable threshold  $X_i(T')$ . In order to uniquely define the obtained feedback controller for an on-line application, and, thus, to make the controller usable (not necessarily optimal) in the stochastic optimal-control framework, we

**Table 2.** Probability density functions of work and repair times (with parameters measured in minutes)

	Case 1	Case 2	Case 3
$\varphi_F(t)$	Uniform [40,60]	Uniform [30,70]	exp(50)
$\varphi_R(t)$	Uniform [5,15]	Uniform [0,20]	exp(10)

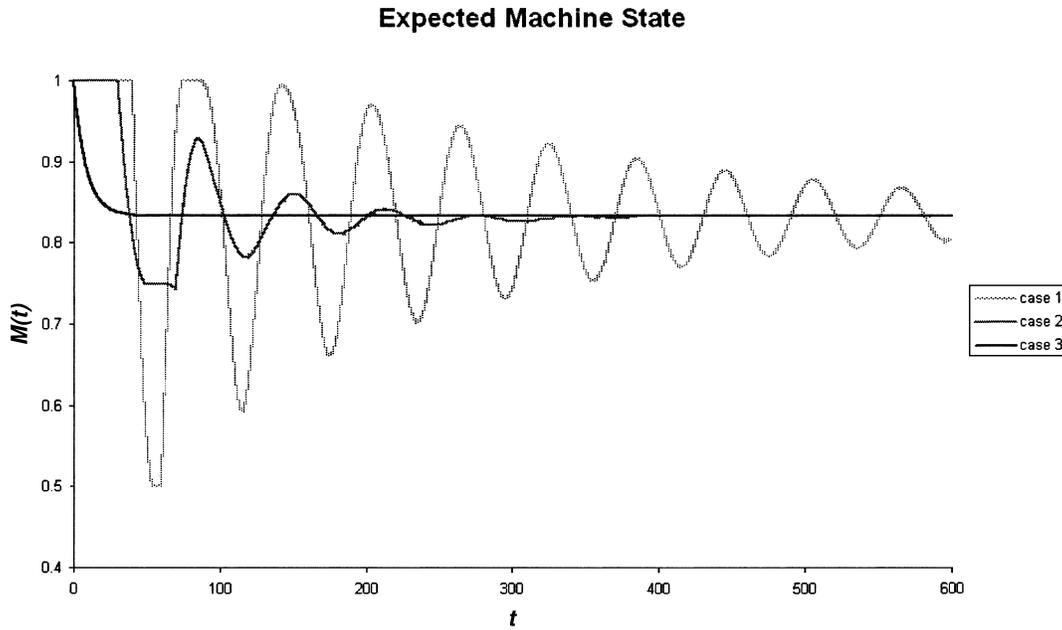


Fig. 3. Numerically obtained expected machine state,  $M(t)$ , for the three cases.

assume the following: (i) if in a specific realization the net machine working time is smaller than  $T'$ , then only a portion of the required quantities  $X_i(T')$  are produced; and (ii) if in a specific realization the net machine working time is greater than  $T'$ , then all the required quantities  $X_i(T')$  are produced, and in the remaining time the machine produces one of the part types, which is chosen arbitrarily. Since this “pseudo-stochastic” policy is implementable, when applying it on-line it serves as a lower bound for the stochastic optimal control, which is unknown as discussed above. In the next section we also develop a numerical upper bound for the stochastic optimal control. Thus, the stochastic optimal control lies between the suggested “pseudo-stochastic” solution and the numerical upper bound.

#### 4. Numerical analysis

In this section, we present a numerical example to demonstrate the applicability of the proposed method and to compare its result with an upper bound for the stochastic optimal control solution.

#### 4.1. System description

Consider a single machine subjected to random failures and repairs. The machine can process 10 part types, each requiring a single operation. The processing times of the parts,  $t_i, i = 1, \dots, 10$  are given in Table 1.

Three cases are considered and presented in Table 2. Each case consists of different density functions of the work time,  $\varphi_F(t)$ , and repair time,  $\varphi_R(t)$ . The density functions are chosen to have identical mean values, in order to emphasize that the optimal solution (scheduler) takes into account higher distribution moments. In terms of uncertainty, case 3 has a much higher variance for both work and repair times than the other two cases, whereas case 1 is the closest to the deterministic case.

The planning horizon over which the production system operates is  $T = 600$  minutes. The initial quantity of parts  $X_i^0, i = 1, \dots, 10$  is equal to zero. Next, the profit function is to be chosen.

Several methods (Korhonen and Wallenius, 1988; Brans and Mareschal, 1994) have been suggested over the years to assist users to define appropriate profit functions. In this example, we choose a simple concave separable

Table 3. Parameters of the optimal solution

		$P_1$	$P_2$	$P_3$	$P_4$	$P_5$	$P_6$	$P_7$	$P_8$	$P_9$	$P_{10}$
Case 1	$X_i$	7.14	4.43	2.39	8.05	0.0	6.54	3.52	3.52	1.71	2.26
	$r_i$	109.2	223.0	417.3	0.0	600.0	152.5	276.6	403.8	508.0	440.6
Case 2	$X_i$	7.13	4.42	2.38	8.03	0.0	6.53	3.5	3.51	1.71	2.25
	$r_i$	110.0	224.3	417.9	0.0	600.0	153.5	277.8	405.1	509.2	441.0
Case 3	$X_i$	7.10	4.40	2.37	8.00	0.0	6.50	3.50	3.50	1.70	2.24
	$r_i$	113.5	226.5	418.2	0.0	600.0	156.2	279.4	405.5	508.3	441.0

**Table 4.** Profit function values

	“Worst-sequence” pseudo-stochastic solution	“Best-sequence” pseudo-stochastic solution	Numerical upper bound
Case 1	7.578	7.588	7.591
Case 2	7.558	7.583	7.590
Case 3	7.453	7.523	7.563

function:

$$U(\mathbf{X}) = \sum_i a_i \ln(X_i + 1), \tag{20}$$

where the weights  $a_i$  (given in Table 1) represent subjective profit function coefficients obtained from producing part type  $i$ .

**4.2. Results**

Figure 3 presents the expected machine state,  $M(t)$ , obtained by integrating Equation (9) for all three cases. A procedure for computing  $M(t)$  is detailed in Appendix B. The resulting quantities of the produced parts are presented in Table 3. They are obtained analytically by solving the problem defined by Equations (17) and (18) with the Lagrange multipliers method (a solution algorithm for a case of the problem of Equations (17) and (18) can be found in Appendix C).

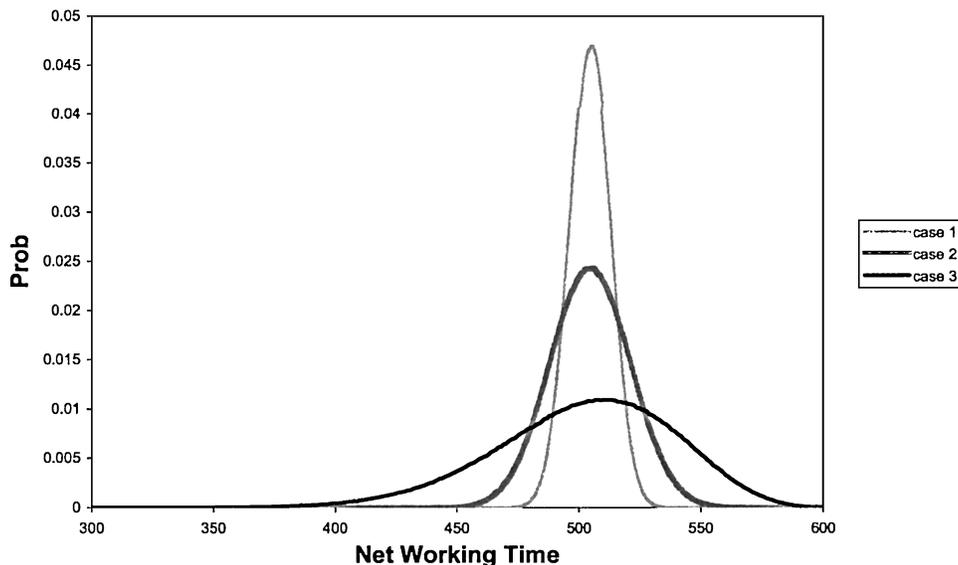
Now, in order to apply the obtained controller on-line as described in the previous section, we have to choose a production sequence. Generally, the sequence of part types should not affect the value of the profit function significantly, since their relative contributions are already considered by the quantities to be produced. Thus, even if a

non-valuable part type is produced in the beginning of the time horizon, its low quantity will result in a short production time slot. For the purpose of illustration, we considered two extreme types of sequences for production. The first sequence,  $\{4,1,6,2,7,8,3,10,9,5\}$ , which is supposed to be the most effective one (termed as “best sequence” in Table 4), is ordered with respect to a decreasing ratio  $a_i/t_i$ . The second, least effective sequence,  $\{5,9,10,3,8,7,2,6,1,4\}$ , is simply in reverse order of the best sequence (this sequence is termed as “worst sequence” in Table 4).

The obtained profit function values for the pseudo-stochastic solution are given in Table 4. Note that for all cases, as expected, the difference between the “worst sequence” and “best sequence” is negligible. Although the distributions of all cases have the same mean, the optimal values are different, emphasizing that the suggested scheduler depends on higher moments of the probability distributions.

Since the solution for the stochastic optimal control is unknown, we compare our pseudo-stochastic solution to a numerical upper bound for the stochastic optimal control problem, which is given in the last column of Table 4. The upper bound is computed by the following procedure:

- Step 1. Generate 5000 000 arbitrary realizations of complete sequences of machine breakdowns and repairs in the planning horizon of 600 time units. Use the given distributions to generate machine work times and repair times. For each realization calculate the machine net working time.
- Step 2. Calculate the empirical distribution of the machine net working time with a resolution of one time unit,  $p_j, j = 1, \dots, 600$ . Figure 4 depicts the empirical distributions of the machine net working time



**Fig. 4.** The empirical distributions of the machine net working time for the three cases.

**Table 5.** Profit function values

	“Worst-sequence” pseudo-stochastic solution	“Best-sequence” pseudo-stochastic solution	Numerical upper bound
Case 1	37.79	37.96	38.00
Case 2	37.50	37.90	37.99
Case 3	36.65	37.62	37.91

(rounded up) for all three cases. Note that for all cases, the empirical distributions resemble a Gaussian distribution with its variance being proportional to the initial variance of the machine work and repair times.

*Step 3.* Solve analytically the problem defined by Equations (17) and (18), for each value of the machine net working time, and obtain the optimal profit function values,  $U_j, j = 1, \dots, 600$ .

*Step 4.* Obtain the numerical upper bound by the empirical expectation of the optimal profit function, i.e.:

$$UB = \sum_{j=1}^{600} p_j U_j.$$

Provided that  $p_j, j = 1, \dots, 600$  represent the “real” unknown distribution of the machine net working time, the above expression is an upper bound for the profit value of the stochastic optimal control. The reason for this is that when calculating the numerical upper bound,  $UB$ , we assume that whatever the future scenario is, its net working time is known in advance. Therefore, we can apply the optimal control in each scenario. As a result, we achieve a higher profit in comparison to a practical setting when the future scenario and its net working time are unknown.

It is interesting to reveal that for the particular considered problem, the pseudo-stochastic approach provides a solution, which is very close to an unknown solution of stochastic optimal control: in the vicinity of less than 0.5%.

### 4.3. Large-scale system

This section demonstrates the applicability of the developed method to a system where the machine has to process a large number of products. We took  $I = 200$  and  $T = 1000$ , with the parameters  $a_i$  and  $t_i$  being chosen randomly,  $a_i$  within the range  $[0.05, 1.0]$  and  $t_i$  within the range  $[1.0, 50.0]$ . In Appendix C we discuss the complexity of the method

**Table 7.** Comparison of the profit function according to the pseudo-stochastic and the certainty equivalence methods

Method	Profit function
Pseudo-stochastic, (case 3)	2.01
Certainty equivalence	0.84

and find that with respect to the number of products, the complexity is  $O(I^2)$ . Table 5 presents the values of the profit function for this case.

### 4.4. Short planning horizon

In comparison with the certainty equivalence method, we forecast more accurately the net working time of the machine,  $T'$ , since our  $M(t)$  takes into account the entire probability distributions, rather than only their first moments (compare Figs. 2 and 3). The difference is especially evident on short planning horizons, where the values of

$$T' = \int_0^T M(t)dt$$

for the two methods significantly differ. In this section we consider the planning horizon  $T = 100$ , and the probability distributions of case 3 above. The net working time, according to certainty equivalence, is  $T' = 90$  and according to our procedure, is  $T' = 84.7$ . The importance of accurate calculation of the net working time is exemplified by the situation when the capacity of the machine maintenance crew is planned in advance. The planned net repair time of the machine is  $T - T'$ . The profit function in such a situation includes, in addition to Equation (20), a penalty term of, say, 0.01 per time unit overage of repair time and a penalty term of, say, 0.1 per time unit underage of the repair time in each scenario. Tables 6 and 7 present the values of  $X_i, i = 1, \dots, 10$  and of the profit function for the two methods.

The evident difference in the profit function as seen in Table 7 results from the different computations of the net working time of the machine using the two methods. Accordingly, since the net working time is used here for the maintenance plan, it directly affects the profit function. When the net working time is not included in the profit function and the planning horizon is relatively large, it appears that the certainty equivalence method provides accurate enough results. In such cases, one will prefer to use the simpler certainty equivalence method.

**Table 6.** Comparison of the produced quantities according to the pseudo-stochastic and the certainty equivalence methods

Method	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	$X_6$	$X_7$	$X_8$	$X_9$	$X_{10}$
Pseudo-stochastic, (case 3)	1.87	0.91	0.20	2.19	0	1.66	0.60	0.60	0	0.15
Certainty equivalence	1.95	0.97	0.23	2.28	0	1.74	0.64	0.64	0	0.18

## 5. Discussion and conclusions

In this paper, a pseudo-stochastic approach for modeling a dynamic system was proposed. The suggested approach has several advantages: (i) it models the stochastic phenomena by a simple, deterministic formulation that can be analyzed by the maximum principle; (ii) the model directly considers the stochastic information in the system dynamics by indexing each future event and estimating its influence on the expected realization; (iii) the model can directly handle several sources of uncertainty as long as they can be represented by random variables with known distributions; (iv) any type of distribution can be easily integrated into the suggested modeling approach; and (v) the suggested approach does not assume steady-state conditions of the system dynamics.

In a particular case study, we have applied an off-line optimal scheduling of multiple parts on a single machine, which is prone to random failures and repairs. The numerical analysis has shown that for the particular problem we consider the obtained value of the profit function is very close to that of the stochastic optimal control. We also observed that for a large planning horizon the net working times, as computed by the pseudo-stochastic method and by the certainty equivalence principle, become closer to one another. As a result, an on-line application of the two methods may yield similar results. In more complex systems, it is not always possible to determine a lucid upper bound for the optimal stochastic control. As a result, the accuracy of the pseudo-stochastic method in those cases is hard to verify.

Further research is required in order to generalize the above observation and to determine the conditions under which the profit function values of the above approaches are close. A possible strategy to improve the pseudo-stochastic approach, in the case of significant differences between these approaches, is to reapply it each time new significant information is gathered and affects our knowledge of future events.

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## Appendixes

### Appendix A

**Lemma A1.** *The rate of the expected machine state at time  $t$  is given by:*

$$\dot{M}(t) = \sum_{n=1}^{\infty} [\pi_n^{R(s)}(t) - \pi_n^{F(s)}(t)].$$

**Proof.** Let us introduce two binary random variables:  $A_n(t) = \{0, 1\}$ , which defines whether the  $n$ th repair occurs (or not) by time  $t$ ; and  $B_n(t) = \{0, 1\}$ , which defines whether the  $n$ th failure occurs (or not) by time  $t$ .

By definition, the probability mass functions of these variables are:

$$P_{A_n(t)}(x) = \begin{cases} \int_0^t \pi_n^{R(s)}(\tau) d\tau & \text{if } x = 1 \\ 1 - \int_0^t \pi_n^{R(s)}(\tau) d\tau & \text{if } x = 0, \end{cases}$$

$$P_{B_n(t)}(x) = \begin{cases} \int_0^t \pi_n^{F(s)}(\tau) d\tau & \text{if } x = 1 \\ 1 - \int_0^t \pi_n^{F(s)}(\tau) d\tau & \text{if } x = 0. \end{cases}$$

Let us introduce a new random variable,  $C_n(t)$ , as  $C_n(t) = A_n(t) - B_n(t)$ , which is used in obtaining the machine status at time  $t$ . Let  $M^0$  be the initial machine state (zero or one). Then,  $C_n(t) = -1$  when  $M^0 = 1$  and by time  $t$  the  $n$ th failure has occurred whereas the  $n$ th repair has not occurred and  $C_n(t) = 1$  when  $M^0 = 0$  and by time  $t$  the  $n$ th repair has occurred, whereas the  $n$ th failure has not occurred. Otherwise,  $C_n(t) = 0$ . The machine status at time  $t$ ,  $C(t)$  can be regarded as the accumulation of differences  $C_n(t)$ :

$$C(t) = M^0 + \sum_{n=1}^{\infty} C_n(t), \quad \forall t.$$

The expected machine state at each time  $t$ ,  $M(t)$  is, thus, the expectation of the right-hand side of the above expression, i.e.:

$$\begin{aligned} M(t) &= M^0 + E \left[ \sum_{n=1}^{\infty} [A_n(t) - B_n(t)] \right] \\ &= M^0 + \sum_{n=1}^{\infty} E[A_n(t) - B_n(t)] \\ &= M^0 + \sum_{n=1}^{\infty} \left[ \int_0^t \pi_n^{R(s)}(\tau) d\tau - \int_0^t \pi_n^{F(s)}(\tau) d\tau \right]. \end{aligned}$$

By taking the time derivative of the last expression we obtain the dynamics from which the lemma immediately follows. ■

The proof can be extended to the general case presented in Equation (4) by appropriately representing the affecting random variables.

### Appendix B

To compute function  $M(t)$  from Equation (9), one has to know theoretically the probability density functions of all future events, since each event (failure or repair), no matter how far beyond the planning horizon it is expected to occur, has a positive probability to occur within the planning horizon. The number of all future events is infinite. Practically, however, only a limited number of future events,  $N$ , (usually in the order of the number of events expected within the planning horizon) is necessary to obtain the optimal

solution. This number is called the “effective information horizon” (EIH) and is discussed in details in Herbon *et al.* (2003). For the example in Section 4.1, the number of pairs (failure and repair) expected within the planning horizon is 10. The EIH was found to be  $N = 25$ , since for a larger  $N$  the function  $M(t)$  within the planning horizon does not change significantly.

Having determined  $N$ , the following steps are needed to calculate  $M(t)$  (for the case  $M(0) = 1$  (the case  $M(0) = 0$  is similar):

*Step 1.* Set  $n = 0$  and set the probability density function of the first failure  $\pi_1^F(t) = \varphi_F(t)$ , (we omit argument  $s$  for simplicity of representation).

*Step 2.* Set  $n = n + 1$ . Calculate the probability density function of the absolute time of the  $n$ th recovery,  $\pi_n^R(t)$ , as the convolution of  $\pi_n^F(t)$  with  $\varphi_R(t)$ , i.e.:

$$\pi_n^R(t) = \int_0^t \pi_n^F(y) \varphi_R(t - y) dy, \quad t \in [0, T], \quad (\text{A1})$$

and calculate the probability density function of the absolute time of the  $(n+1)$ th failure,  $\pi_{n+1}^F(t)$ , as the convolution of  $\pi_n^R(t)$  with  $\varphi_F(t)$ , i.e.:

$$\pi_{n+1}^F(t) = \int_0^t \pi_n^R(y) \varphi_F(t - y) dy, \quad t \in [0, T]. \quad (\text{A2})$$

*Step 3.* If  $n \leq N$ , then go to Step 2, otherwise go to Step 4.

*Step 4.* Calculate the sum

$$\sum_{n=1}^N [\pi_n^R(t) - \pi_n^F(t)], \quad t \in [0, T], \quad (\text{A3})$$

and integrate the result on the interval  $[0, T]$  from left to right starting with the initial value  $M(0) = 1$ .

### Appendix C

The problem

$$\text{Max } u(\mathbf{x}) \sum_{i=1}^I a_i h(X_i), \quad (\text{A4})$$

subject to

$$\sum_{i=1}^I X_i t_i \leq T' \quad \text{and} \quad X_i \geq 0, \quad (\text{A5})$$

where  $h(X_i)$  is a monotone increasing concave function and parameters  $t_i > 0$  and  $a_i > 0$ ,  $i = 1, \dots, I$ . The optimal solution of the problem satisfies the following properties.

**Property 1.** The constraint  $\sum_{i=1}^I X_i t_i \leq T'$  is active, i.e.:

$$\sum_{i=1}^I X_i t_i = T'. \quad (\text{A6})$$

**Proof.** The property immediately follows from the fact that  $h(X_i)$  is an increasing function. ■

**Property 2.** If  $a_i/t_i \geq a_j/t_j$ , and  $X_i = 0$ , then  $X_j = 0$ .

**Proof.** The Lagrangian of the problem is:

$$L = \sum_{i=1}^I a_i h(X_i) - \lambda \left( \sum_{i=1}^I X_i t_i - T' \right), \quad (\text{A7})$$

where  $\lambda \geq 0$  is the Lagrange multiplier. By differentiating the Lagrangian w.r.t.  $X_i$ , we obtain that:

$$X_i = \left[ \frac{\partial h}{\partial X_i} \right]^{-1} \left( \frac{\lambda t_i}{a_i} \right). \quad (\text{A8})$$

By taking into account the non-negativity of  $X_i$ , we finally have:

$$X_i = \max \left\{ 0, \left[ \frac{\partial h}{\partial X_i} \right]^{-1} \left( \frac{\lambda t_i}{a_i} \right) \right\}. \quad (\text{A9})$$

The property follows from Equation (A9) and the concavity of  $h$ . ■

From Equations (A6) and (A9) it follows that the Lagrange multiplier satisfies the equation:

$$\sum_{i=1}^I t_i \max \left\{ 0, \left[ \frac{\partial h}{\partial X_i} \right]^{-1} \left( \frac{\lambda t_i}{a_i} \right) \right\} = T'. \quad (\text{A10})$$

Based on the properties of the optimal solution proved above, the following algorithm solves the problem.

*Step 1.* Sort the products in the decreasing order of  $a_i/t_i$ . Re-number the products in this order. Set index  $m = 0$ .

*Step 2.* Set  $m = m + 1$ .

*Step 3.* Calculate  $\lambda$  from:

$$\sum_{i=1}^m t_i \left[ \frac{\partial h}{\partial X_i} \right]^{-1} \left( \frac{\lambda t_i}{a_i} \right) = T',$$

and calculate

$$X_{m+1} = \left[ \frac{\partial h}{\partial X_{m+1}} \right]^{-1} \left( \frac{\lambda t_{m+1}}{a_{m+1}} \right).$$

*Step 4.* If  $X_{m+1} > 0$ , then go to Step 2. Otherwise, set  $X_i = 0$  for  $i = m + 1, \dots, I$

and

$$X_i = \left[ \frac{\partial h}{\partial X_i} \right]^{-1} \left( \frac{\lambda t_i}{a_i} \right) \text{ for } i = 1, \dots, m. \text{ Stop.}$$

The complexity of the algorithm is  $O(I^2)$ , since Steps 2 and 3 comprise two loops with respect to the number of products.

*Example:* If  $h(X_i) = \ln(X_i + 1)$  as in Section 4, then  $\lambda$  is calculated in Step 3 as:

$$\lambda = \frac{\sum_{i=1}^m a_i}{T' + \sum_{i=1}^m t_i},$$

and the non-zero production quantities are calculated in Step 4 as:

$$X_i = \frac{a_i}{\lambda t_i} - 1.$$

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