A discrete stability equation theorem and method of stable model reduction

Y. BISTRITZ

School of Engineering, Tel-Aviv University, Tel-Aviv, Israel

Received 20 December 1981
Revised 15 March 1982

A method to derive stable reduced-order models for a stable discrete-time invariant system is presented. The method is based on a unit circle stability criterion. This criterion is the z-plane equivalent of a stability criterion for continuous time systems and is proven by using the bilinear transformation. The resulting model reduction method forms the discrete analogue of the stability equation and Michailov's methods for the simplification of continous time systems, that were recently suggested in various, closely related, versions.

Keywords: Discrete modelling, Unit circle stability criterion, Model reduction.

1. Introduction

A well known problem in simplification of both continuous and discrete-time linear systems by Padé type reduced order models is that stable systems may yield unstable models. One way to overcome this problem is by deriving models that approximate the system in the Padé sense only partially and have a predetermined stable characteristic equation. Efforts have thus been made to derive methods that determine appropriate lower order stable characteristic polynomials. One obvious possibility is to compute the eigenvalues of the high order system and retain the dominant modes. Other methods that have been suggested for continuous systems such as the Routh method [1] and its generalization, the Padé–Hurwitz method [2], do not require the knowledge of the exact eigenvalues but approximate them indirectly. Another approach that has recently been presented for continuous systems derives the low order Hurwitz polynomials using a stability criterion called the stability equation or the Michailov criterion [3–7]. The purpose of this paper is to present a similar method for the reduction of discrete systems. This method derives Padé type models in the z-plane that are stable for a stable system. It is based on a z-plane version of the stability criterion that underlies the continuous methods. Some of the methods in [3–7] differ in the techniques used to derive the 'numerators' (Padé equations, continued fractions or stability equations). They all derive the same denominators (the treatment in [7] being a little more general) by a method that is based on the stability equation criterion and is restated here for reference purposes.

Theorem 1 (see e.g. [8], p. 416). *A real polynomial*

$$\Delta(s)=\sum_{i=0}^{\nu}\delta_i s^i=\delta_\nu\prod_{i=1}^{\nu}(s-s_i),\quad \delta_i>0,\ \forall i, \tag{1}$$

is Hurwitz (Re $s_i<0$, $\forall i$) *if and only if the zeros and poles of*

$$\rho(s)=\frac{\Delta(s)-\Delta(-s)}{\Delta(s)+\Delta(-s)} \tag{2}$$

are simple, located on the imaginary $s=\mathrm{j}\omega$ *axis and they mutually separate each other.*

A polynomial $\Delta(s)$ is therefore Hurwitz iff $\rho(s)$ can be written as

$$\rho(s)=\frac{ks\prod_{i=1}^{m}(s^2+\omega_{2i}^2)}{\prod_{i=1}^{n}(s^2+\omega_{2i-1}^2)},\quad k>0,\quad 0<|\omega_1|<|\omega_2|<\cdots<|\omega_\nu|, \tag{3}$$

where $\nu=m+n+1$, $m=n-1$ or $m=n$. The 'gain' is $k=\delta_{\nu-1}/\delta_\nu$ and $k>0$ is necessary and sufficient, subject to the rest of the conditions in the theorem, for all the coefficients of $\Delta(s)$ to have the same sign [8] (the condition $\delta_{\nu-1}/\delta_\nu>0$ may replace the $\delta_i>0$, $\forall i$ in (1)).

The reduction methods in [3–7] use, for a $\hat{\nu}$-th order model, $\hat{\nu}<\nu$, the polynomial $\Delta(s)$ that is determined by adding the numerator of $\hat{\rho}(s)$ to the denominator, where

$$\hat{\rho}(s)=\frac{\hat{k}s\prod_{i=1}^{\hat{m}}(s^2+\omega_{2i}^2)}{\prod_{i=1}^{\hat{n}}(s^2+\omega_{2i-1}^2)},\quad \hat{k}>0, \tag{4}$$

$\hat{\nu}=\hat{m}+\hat{n}+1$, $\hat{m}=\hat{n}-1$ or $\hat{m}=\hat{n}$ and $\hat{k}$ is chosen to satisfy

$$\left.\frac{\hat{\rho}(s)}{s}\right|_{s=0}=\left.\frac{\rho(s)}{s}\right|_{s=0}.$$

Clearly the resulting polynomial $\Delta(s)$ is Hurwitz by the sufficiency part of Theorem 1.

We shall derive an analogous criterion for the stability of discrete system characteristic polynomials and apply it in the derivation of reduced order models in the discrete z-plane. We shall adopt the term stable polynomials for a polynomial $D(s)$ whose zeros are all inside the unit circle $|z|=1$, where by the term Hurwitz polynomial we shall mean that a polynomial in the form (1) satisfies Re $s_i<0$ $\forall i$. We mainly concern here the approximation of stable $D(z)$ polynomials by lower order stable polynomials. We shall therefore apply the result to form Padé type model reduction for single input and output (scalar) systems. By this we avoid intentionaly the discussion of some (solvable) dimensional problems [9] that may occur in certain extensions to multivariable systems of scalar methods (e.g. [4]). Partial Padé approximation techniques to fit a scalar numerator to a predetermined denominator by matching the first time moments of the system are well established and will not be repeated here.

2. The discrete stability equation criterion

Let $D(z)$ be a real ν-th order polynomial

$$D(z)=\sum_{i=0}^{\nu}d_iz^i=d_\nu\prod_{i=1}^{\nu}(z-z_i). \tag{5}$$

Using the bilinear transformation,

$$z=\frac{1-s}{1+s},\qquad s=\frac{z-1}{z+1}, \tag{6a,b}$$

a discrete version of Theorem 1 can be derived as follows. Let $\Delta(s)$ be the polynomial of (1) whose zeros are the bilinear mapping of the zeros of D(z), $s_i=(z_i-1)/(z_i+1)$. Evidently, $D(z)$ is stable if and only if $\Delta(s)$ is Hurwitz. Define the rational function

$$\rho_\nu(z)=\frac{D(z)-\widetilde{D(z)}}{D(z)+\widetilde{D(z)}} \tag{7}$$

where $\widetilde{D(z)}$ is the reciprocated polynomial

$$\widetilde{D(z)} = \sum_{i=0}^{\nu} d_{\nu-i} z^i = z^\nu D(z^{-1}) = d_\nu \prod_{i=1}^{\nu} (1 - z z_i). \tag{8}$$

It can be verified by substitution of $z_i = (1 - s_i)/(1 + s_i)$ and $z = (1 - s)/(1 + s)$ in $D(z)$ and $\widetilde{D(z)}$ that $\rho_\nu(z)$ of (7) is mapped into $\rho_\nu(s)$ of (2) for $\Delta(s)$ whose zeros are the bilinear mapping of the zeros of $D(z)$. The bilinear transformation also maps the unit circle

$$C = \{ z \mid z = e^{j\Omega}, \Omega \in [-\hat{\pi}, \hat{\pi}] \} \tag{9}$$

onto the $s = j\omega$ axis by the following one-to-one mapping dictated by (6b)

$$j\omega = j \operatorname{tg}\left(\frac{\Omega}{2}\right). \tag{10}$$

We may therefore state the following discrete stability criterion:

Theorem 2. *A real polynomial $D(z)$,*

$$D(z) = \sum_{i=0}^{\nu} d_i z^i = d_\nu \prod_{i=1}^{\nu} (z - z_i), \quad |d_\nu| > |d_0|,$$

has all its zeros inside the unit circle, $|z_i| < 1$, if and only if the zeros and poles of $\rho_\nu(z)$, (7), are simple, located on the unit circle, $|z| = 1$, and mutually separate each other.

Remark. The condition $|d_\nu| > |d_0|$ is necessary for a polynomial to be stable. It can be traced back to the $\delta_{\nu-1}/\delta_\nu > 0$ condition of (3) following Theorem 1. An interesting immediate corollary is that for a polynomial $D(z)$ satisfying $|d_\nu| < |d_0|$, Theorem 2 states necessary and sufficient conditions for $D(z)$ to be anti-stable ($|z_i| > 1$, $\forall i$).

3. Application to model reduction

The stability equation method for the simplification of discrete systems follows from Theorem 2 in a similar manner by which the continuous methods [3–7] follow from Theorem 1. To be more specific we state the following discrete model reduction problem. Given a stable high order discrete system

$$G(z) = \sum_{i=0}^{\mu} n_i z^i / D(z) \tag{11}$$

where $D(z)$ is a stable polynomial in the form (5), $\mu < \nu$. A model of reduced order $\hat{\nu} < \nu$ is required with transfer function

$$\hat{G}(z) = \sum_{i=0}^{\hat{\mu}} \hat{n}_i z^i / \hat{D}(z), \qquad \hat{D}(z) = \sum_{i=0}^{\hat{\nu}} \hat{d}_i z^i \tag{12}$$

such that $\hat{D}(z)$ is a stable polynomial of order $\hat{\nu}$, $\hat{\mu} < \hat{\nu}$ and $\hat{G}(z)$ matches the first $\hat{\mu}$ time moments of $G(z)$.

A solution to this problem that applies the result of Theorem 2 to derive $\hat{D}(z)$ and the well known partial Padé techniques is obtained as follows. Given the stable polynomial $D(z)$ it is readily found from the structure of $\rho_\nu(z)$ and the result of Theorem 2 that $\rho_\nu(z)$ can be written in the following forms:

$$\rho_{2m}(z) = \frac{(z-1)(z+1)\beta_{2m-2}(z)}{\alpha_{2m}(z)}, \qquad \rho_{2m+1}(z) = \frac{(z-1)\beta_{2m}(z)}{\alpha_{2m}(z)} \tag{13a,b}$$

with

$$\beta_{2l}(z)=\prod_{i=1}^{l}\left(z^2-2z\cos\Omega_{2i}+1\right),\quad l=m-1,m,\qquad \alpha_{2m}(z)=\prod_{i=1}^{m}\left(z^2-2c\cos\Omega_{2i-1}+1\right)\quad (13\text{c,d})$$

where $z_i=e^{j\Omega_i}$, $i=1,2,\ldots,\nu-1$, represents the zeros and poles of $\rho_\nu(z)$ on the unit circle, which for a real polynomial appear in complex conjugate pairs $(z-e^{j\Omega_i})(z-e^{-j\Omega_i})=(z^2-2z\cos\Omega_i+1)$ and by Theorem 2 satisfy

$$-1<\cos\Omega_{\nu-1}<\cdots<\cos\Omega_2<\cos\Omega_1<1. \qquad (13\text{e})$$

The $\hat{\nu}$-th order polynomial $\hat{D}(z)$ is determined by the relation

$$\frac{\hat{D}(z)-z^{\hat{\nu}}\hat{D}(z^{-1})}{\hat{D}(z)+z^{\hat{\nu}}\hat{D}(z^{-1})}=\hat{k}\rho_{\hat{\nu}}(z) \qquad (14)$$

where $\hat{k}>0$ is a constant whose determination is discussed later and $\rho_{\hat{\nu}}(z)$ is constructed from the $\hat{\nu}$ dominant quadratic terms $(z^2-2z\cos\Omega_i+1)$, $i=1,\ldots,\hat{\nu}$, of (13) and has the proper structure (13a) or (13b) according to the parity of $\hat{\nu}$. It therefore follows from the sufficiency part of Theorem 2 that the polynomial $\hat{D}(z)$ is stable.

The construction of a lower order stable polynomial requires only the knowledge of the values of $\cos\Omega_i$ (13). One obvious way to find these values is by the computation of the zeros and poles of $\rho_\nu(z)$. Instead, we show next how the special structure of $\rho_\nu(z)$ can be used to replace the numerator and denominator of $\rho_\nu(z)$ by polynomials of half order whose zeros are exactly the required $\cos\Omega_i$'s.

The polynomials $\beta_{2l}(z)$ and $\alpha_{2m}(z)$ of (13c, d) are both of even order and with zeros in reciprocated pairs, z_i, z_i^{-1}. As a result, their coefficients p_i and q_i in

$$\alpha_{2k}(z)=\sum_{i=0}^{2k}p_iz^i,\qquad \beta_{2k}(z)=\sum_{i=0}^{2k}q_iz^i \qquad (15)$$

have to satisfy $p_i=p_{2k-i}$ and $q_i=q_{2k-i}$. The polynomial $\alpha_{2k}(z)$ (and similarly $\beta_{2k}(z)$) can thus be written by

$$\alpha_{2k}(z)=z^i\left[p_0(z^k+z^{-k})+p_1(z^{k-1}+z^{-(k-1)})+\cdots+p_k\right] \qquad (16)$$

or

$$z^{-k}\alpha_{2k}(z)=\sum_{i=0}^{k-1}2p_iT_i(z)+p_kT_0(z) \qquad (17)$$

where we have defined

$$T_n(z)=\tfrac{1}{2}(z^n+z^{-n}). \qquad (18)$$

The transformation

$$x=\tfrac{1}{2}(z+z^{-1}) \qquad (19)$$

maps $T_n(z)$ into the well known Chebyshev polynomials $T_n(X)$ [10], which can also be obtained using the recursion

$$T_{n+2}(x)=2xT_{n+1}(x)-T_n(x),\qquad T_0(x)=1,\quad T_1(x)=x. \qquad (20)$$

The right hand side of (17) can be transformed into the variable x by substituting $T_i(z)$ by $T_i(x)$. Equations (15)–(20) represent therefore a procedure to perform the change of variables (19) in $\rho_\nu(z)$,

$$z^{-m}\alpha_{2m}(z)\to A_m(x)=\sum_{i=0}^{m}a_ix^i=\sum_{i=0}^{m-1}2p_iT_i(x)+p_m, \qquad (21\text{a})$$

$$z^{-l}\beta_{2l}(z)\to B_l(z)=\sum_{i=0}^{l}b_ix^i=\sum_{i=0}^{l-1}2q_iT_i(x)+q_l,\quad l=m-1,m. \qquad (21\text{b})$$

The transformation (19) maps the unit circle (9) onto the real interval $[-1, 1]$. It places each reciprocal pair of zeros and poles of $\rho_\nu(z)$ into their common projection on the real axis. The polynomials $B_l(x)$ and $A_m(x)$ found in (21), rewritten as

$$B_l(x) = b_l \prod_{i=1}^{l} (x - x_{2i}), \quad l = m-1, m, \qquad A_m(x) = a_m \prod_{i=1}^{m} (x - x_{2i-1}) \tag{22}$$

have zeros given by

$$x_i = \tfrac{1}{2}\left(z_i + z_i^{-1}\right) = \cos \Omega_i, \quad i = 1, \ldots, \nu - 1. \tag{23}$$

The computation of the zeros of these two polynomials, of half the order of the respective $\alpha_{2m}(z)$ and $\beta_{2l}(z)$ polynomials, gives the required information for the construction of the lower order stable polynomials.

It follows from the above that the stable approximation for $D(z)$ can be found by the following steps.

Step (i) Given $D(z)$, form the numerator and denominator of $\rho_\nu(z)$ (7), (8)

$$Q(z) = D(z) - \widetilde{D(z)}, \qquad P(z) = D(z) + \widetilde{D(z)} \tag{24}$$

and obtain $\beta_{2m-2}(z)$ or $\beta_{2m}(z)$ and $\alpha_{2m}(z)$ by performing one polynomial division by $(z-1)$ followed by one polynomial division by $(z+1)$, as indicated by the relations of (13a, b),

$$Q(z) = (z-1)(z+1)\beta_{2m-2}(z), \quad P(z) = \alpha_{2m}(z), \qquad \nu = 2m, \tag{25a}$$

or

$$Q(z) = (z-1)\beta_{2m}(z), \quad P(z) = (z+1)\alpha_{2m}(z), \qquad \nu = 2m+1. \tag{25b}$$

The division of a polynomial having a zero at $z = 1$ or at $z = -1$ by $(z-1)$ or by $(z+1)$, respectively, is a simple task,

$$C(z) = \sum_{i=0}^{n} c_i z^i \Rightarrow \begin{cases} C(z)/(z-1) = \sum_{i=0}^{n-1} e_i z^i, & e_0 = -c_0,\ e_i = e_{i-1} - c_i, \\ C(z)/(z+1) = \sum_{i=0}^{n-1} f_i z^i, & f_0 = c_0,\ f_i = c_i - f_{i-1}. \end{cases} \tag{26}$$

Step (ii) Replace $\beta_{2l}(z)$ by $B_l(x)$, $l = m-1$, $l = m$, and $\alpha_{2m}(z)$ by $A_m(x)$, using the procedure outlined by (15)–(20) and compute the zeros of $B_l(x)$ and $A_m(x)$. These zeros are real, lie inside the interval $[-1, 1]$, properties that may be advantegeous in the computation and they separate each other,

$$-1 < x_{\nu-1} < \cdots < x_{\hat{\nu}-1} < \cdots < x_2 < x_1 < 1 \tag{27}$$

where $x_1, x_3, \ldots$ are the 'poles' (zeros of $A_m(x)$) and $x_2, x_4, \ldots$ are the 'zeros' (zeros of $B_l(x)$).

Step (iii) Construct the $\hat{\nu}$-th order stable polynomial $\hat{D}(z)$ by adding numerator to denominator in $\rho_{\hat{\nu}}(z)$, constructed as follows:

$$\rho_{2\hat{m}}(z) = \frac{\hat{k}(z-1)(z+1)\prod_{i=1}^{\hat{m}-1}\left(z^2 - 2zx_{2i} + 1\right)}{\prod_{i=1}^{\hat{m}}\left(z^2 - 2zx_{2i-1} + 1\right)}, \quad \hat{k} > 0, \tag{28a}$$

$$\rho_{2\hat{m}+1}(z) = \frac{\hat{k}(z-1)\prod_{i=1}^{\hat{m}}\left(z^2 - 2zx_{2i} + 1\right)}{(z+1)\prod_{i=1}^{\hat{m}}\left(z^2 - 2zx_{2i-1} + 1\right)}, \quad \hat{k} > 0, \tag{28b}$$

with $\hat{\nu} = \hat{m} + \hat{n} + 1$. The choice of $\hat{k} > 0$ such that

$$\left.\frac{\rho_{\hat{\nu}}(z)}{z-1}\right|_{z=1} = \left.\frac{\rho_{\nu}(z)}{z-1}\right|_{z=1} \tag{29}$$

yields polynomials that are the z-plane equivalents of those that are found by the continuous methods [3–7].

The derivation of the lower order model $\hat{G}(z)$ of (12) is completed by determining a numerator so as to have the first $\hat{\mu}$ terms of the expansion of $\hat{G}(z)$ and $G(z)$ about $z = 1$ agreeing. This stands to matching the first time moments of the system and is straightforwardly carried out by well known Padé techniques (e.g. [2]).

We have by Theorem 2 that any choice of $\hat{k} > 0$ in (28) yields a stable polynomial. This freedom can be used to find $\hat{k}$ that minimizes some measure of error between the stable model and the system. The root-locus investigation of the location of the approximated stable eigenvalues in comparison to the original eigenvalues may also be revealing. One should however be aware of the additional complexity that is implied by such optimizations in the above otherwise simple method to derive stable reduced-order models.

The quality of the reduced-order models stems from their retention of stability and from the Padé sense of approximation [12,13]. Models of increasing order $\hat{\nu}$ form better approximations in the mathematical semi-norm sense of [13] and reproduce the system for $\hat{\nu} = \nu$. However, like in other Padé methods, the exact measure of closeness of the approximation to the high order stable system is not known in advance. If it is important to have models that form minimal approximations in a definite least square or Chebyshev sense and if the more computational effort required for this purpose is afforded, the method of this paper can be combined with the algorithms for rational approximations in the complex z-plane that are described in [14]. It is shown there that the convergence and the rate of convergence of the best L_2 or L_∞ approximations depend to a great extent on the quality of the initial approximation. The models determined by the algorithm of the present paper represent an easy method to derive appropriate initial conditions for the iterative algorithms in [14]. The combination of the two methods may therefore suggest a powerful procedure for the derivation of the best L_2 or L_∞ norm stable reduced-order models for a given stable discrete system. It is noted however that models with simple choice of k to satisfy (29), with $D(z)$ determined by the retention of the dominant quadratic terms of $\rho_\nu(z)$ in (13) and with the matching of the first time moments of the system (all these selections characterize approximation of low frequency features, $\Omega \to 0$, $z \to 1$), have been found to yield very satisfactory approximation. The next example illustrates the method and its approximating qualities.

Example. Consider the scalar discrete system of [11] whose transfer function $G(z) = N(z)/D(z)$ is given by

$$N(z) = 1.682z^7 + 1.116z^6 - 0.21z^5 + 0.152z^4 - 0.516z^3 - 0.262z^2 + 0.044z - 0.006, \tag{30a}$$

$$D(z) = 8z^8 - 5.046z^7 - 3.348z^6 + 0.63z^5 - 0.456z^4 + 1.548z^3 + 0.786z^2 - 0.132z + 0.018. \tag{30b}$$

The zeros of $D(z)$ are found to be

$$0.87965568 \pm j\,0.24463066, \qquad -0.05405313 \pm j\,0.65568454,$$
$$0.07733240 \pm j\,0.10778302, \qquad -0.58755979 \pm j\,0.09547938.$$

We shall demonstrate the method of this paper by deriving the second and the third reduced-order models. We have by (7), (24), (15) for this case

$$\alpha_8(z) = D(z) + D(z) = 4.009 - 2.589z - 1.281z^2 + 1.089z^3 + 0.456z^4 + 1.089z^5 - 1.281z^6 - 2.589z^7 + 4.009z^8,$$

$$\beta_6(z) = \frac{D(z) - D(Z)}{(z-1)(z+1)} = 3.991 - 2.457z + 1.924z^2 - 2.916z^3 + 1.924z^4 - 2.457z^5 + 3.991z^6.$$

Using (15)–(20) we find

$$z^{-4}\alpha_8(z) \to A_4(x) = 8.018T_4(x) - 5.178T_3(x) - 2.562T_2(x) + 1.089T_1(x) - 0.456T_0(x),$$

$$A_4(x) = 8.018x^4 - 2.589x^3 - 8.6585x^2 + 2.214x + 1.2655.$$

Similarly $z^{-3}\beta_6(z)$ transforms to

$$B_3(x) = 7.982x^3 - 2.457x^2 - 5.0225x + 0.4995.$$

The 'poles' (zeros of $A_4(x)$) x''_{2i-1} and the 'zeros' (zeros of $B_3(x)$) x'_{2i} of the stability equation are given by

$$x''_1 = 0.96977417, \quad x''_3 = 0.58941292, \quad x''_5 = -0.29260273, \quad x''_7 = -0.943633588,$$

$$x'_2 = 0.9168992, \quad x'_4 = 0.096701851, \quad x'_6 = -0.705782266.$$

They form the real parts of the complex pairs of poles and zeros of $\rho_8(z)$ on the unit circle. These are sketched in Figure 1.

A second-order polynomial $\hat{D}_2(z)$ is derived from $\hat{\rho}_2(z)$, the second-order rational function (28) that retains x''_1,

$$\hat{\rho}_2(z) = \frac{4(z-1)(z+1)}{33.084285(z^2 - 2\cdot 0.96977417z + 1)},$$

where we satisfy (29) by letting $\hat{k} = k_2/k_1$ and choose k_1 and k_2 to retain the values at $z = 1$ of respectively the denominator and the numerator divided by $(z-1)$ (this choice also satisfies $\hat{D}(1) = D(1)$);

$$\hat{D}_2(z) = 37.084285z^2 - 64.168571z + 29.084285.$$

A third-order polynomial $\hat{D}_3(z)$ is derived via (28) by retention of x''_1 and x'_2,

$$\hat{\rho}_3(z) = \frac{48.134313(z-1)(z^2 - 2.09168992z + 1)}{16.542142(z+1)(z^2 - 2.096977417z + 1)},$$

$$\hat{D}_3(z) = 64.67645z^3 - 151.94508z^2 + 120.86079z - 31.592171.$$

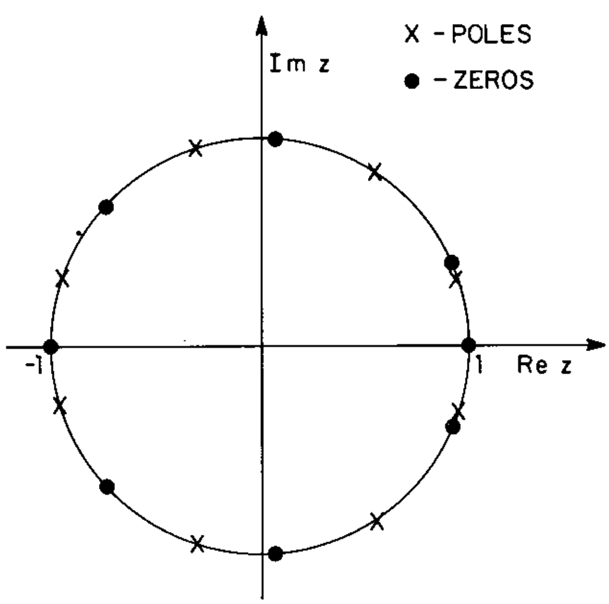

Fig. 1. Pole–zero pattern for the $\rho(z)$ of the polynomial $D(z)$, equation (30b).

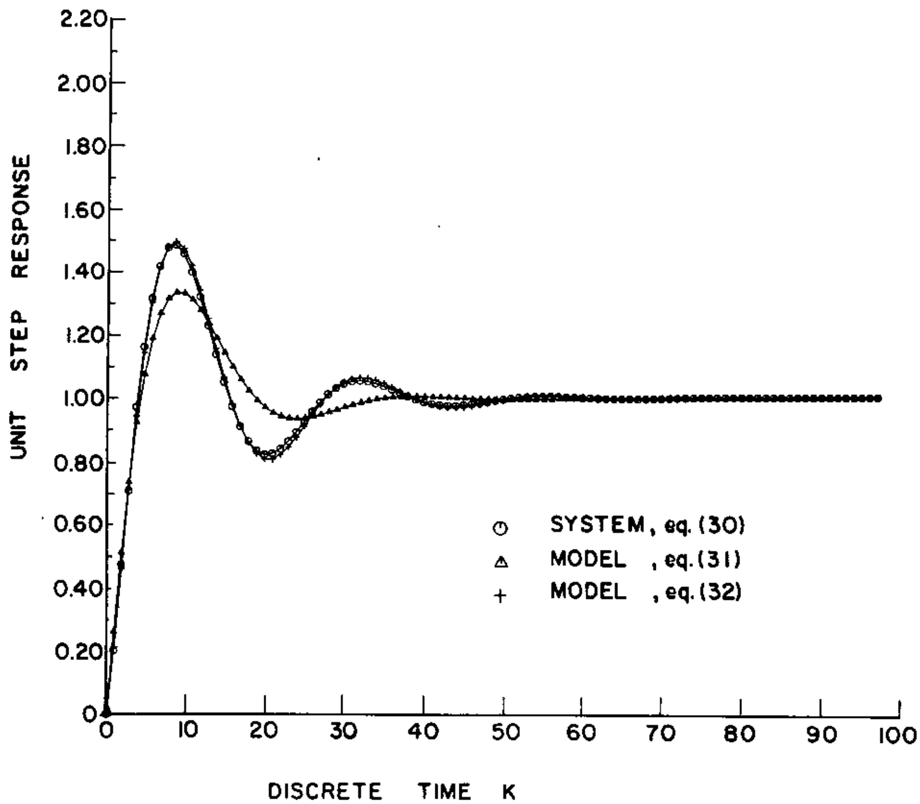

Fig. 2. Unit step responses for the system and the reduced 2-nd and 3-rd order models.

The second- and the third-order models that have $\hat{D}_2(z)$ and $\hat{D}_3(z)$ for denominators and match respectively 2 and 3 time moments, are given by

$$\hat{G}_2(z)=\frac{0.269652z-0.215721}{z^2-1.730344z+0.784275}, \tag{31}$$

$$\hat{G}_3(z)=\frac{0.22172214z^2-0.2733654z+0.082567}{z^3-2.349311z^2+1.868698z-0.488467}. \tag{32}$$

The poles of $\hat{G}_2(z)$ are $0.865172\pm j0.189083$ and the poles of $\hat{G}_3(z)$ are $0.88525591\pm j0.24545971$, 0.5789868; they tend to approximate the dominant poles of $G(z)$. The unit step response for the system and the two reduced-order models are given in Figure 2.

4. Concluding remarks

A method for the reduction of discrete-time linear systems that yields stable models for stable systems has been presented. The method is based on a unit circle criterion of stability and forms the z-plane equivalent of the stability-equation methods of model reductions suggested in [3–7]. The method has been presented for single input and output systems but can be extended to multivariable systems regarding the reduced-order stable polynomials as the characteristic equation of some proper matrix-valued partial Padé approximation method. If a better transient response approximation is required the method can be extended to derive biased models in which the lower-order stable polynomial is determined from a $\hat{\rho}_{\bar{p}}(z)$ that retains also some quadratic terms $(z^2 - 2z \cos \Omega_i + 1)$ that correspond to high-frequency values of $|\Omega_i|$. Another interesting possible extension is the use of the free 'gain' $\hat{k}$ in $\hat{\rho}_{\bar{p}}(z)$ to optimize the resulting lower-order stable model or to satisfy some further specifications. The method has however been found to yield good approximation even for the non-optimal indicated choice of $\hat{k}$. Without the use of the optimization option the method is computationally straightforward.

References

[1] M.F. Hutton and B. Friedland, Routh approximation for reducing order of linear time-invariant systems, *IEEE Trans. Automat. Control* 20 (1975) 324–337.
[2] Y. Bistritz and U. Shaked, Stable linear systems simplification via Padé approximations to Hurwitz polynomials, *ASME J. Dynamic Syst. Measurement Control* 103 (1981) 279–284.
[3] T.C. Chen, C.Y. Chang and K.W. Han, Reduction of transfer functions by the stability equation method, *J. Franklin Inst.* 308 (1979) 389–404.
[4] T.C. Chen, C.Y. Chang and K.W. Han, Model reduction using the stability equation method and the Padé approximation method, *J. Franklin Inst.* 309 (1980) 473–490.
[5] T.C. Chen, C.Y. Chang and K.W. Han, Model reduction using the stability equation method and the continued fraction method, *Int. J. Control* 32 (1980) 81–94.
[6] T.C. Chen, C.Y. Chang and K.W. Han, Stable reduced order Padé approximants using stability equation method, *Elect. Lett.* 16 (1980) 345–346.
[7] B.W. Wan, Linear model reduction using Michailov criterion and Padé approximation techniques, *Int. J. Control* 33 (1981) 1073–1089.
[8] E.A. Guillemin, *The Mathematics of Circuit Analysis* (M.I.T. Press, Cambridge, MA, 1949).
[9] Y. Bistritz and U. Shaked, Minimal Padé model reduction of multivariable systems, *Proc. IEEE C.D.C. Conf.*, Dec. 16–18 (1981).
[10] Y. Bistritz and G. Langholz, Model reduction by Chebyshev polynomial techniques, *IEEE Trans. Automat. Control* 24 (1979) 741–747.
[11] C. Hwang, Y.P. Shih and R.Y. Hwang, A combined time and frequency domain method for model reduction of discrete systems, *J. Franklin Inst.* 311 (1981) 391–402.
[12] G.A. Baker, *Essentials of Padé Approximations* (Academic Press, New York, 1975).
[13] M.J. Goldman and C.T. Leondes, Truncation of the mixed Cauer form, the best approximation in a definite mathematical sense, *Int. J. Control* 15 (1977) 972–982.
[14] Y. Bistritz and G. Langholz, Model reduction by best Chebyshev approximations in the complex plane, *Int. J. Control* 30 (1979) 277–289.