A direct Routh stability method for discrete system modelling

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Received 21 April 1982

The Routh approximation method which has been suggested for the reduction of stable discrete time linear systems to guarantee stable models, uses the bilinear transformation. A stability theorem in the z-plane is presented which is shown to be an equivalent of the Routh criterion. An efficent method that avoids the bilinear transformation is presented by which the Routh disrete models are derived directly in the z-plane.

Keywords: Discrete modelling, Discrete stability criterion, Model reduction.

1. Introduction and problem statement

The approximation of stable systems by reduced order Padé type models may yield unstable models both in the case of continuous and of discrete-time systems. Solutions for this stability problem in the continuous case can be obtained by the Routh [1], [2], the Hurwitz [3], [4] and the Padé–Hurwitz [5], [6] methods. These methods guarantee stable models for stable systems that approximate the slow eigenvalues [1]–[4] or both some slow and some fast eigenvalues [5], [6] of the high order system eigenvalues. The Routh and the Hurwitz methods which have been shown to be equivalent and to form a special case of the Padé–Hurwitz methods [5], [6] were applied in [4] and [7], using the bilinear transformation, to solve also the stability problem that is encountered in discrete system approximations. The purpose of this correspondence is to show how these same models can be derived, more efficiently, entirely in the z-plane.

The problem that is treated in [4] and [7] and that is resolved in this correspondence is the following. Given a high order discrete-time stable system of order ν that is described by the z-transfer function

$$G(z)=\frac{N(z)}{D_\nu(z)}=\frac{n_0+n_1z+\ldots+n_\mu z^\mu}{d_0+d_1z+\ldots+d_\nu z^\nu},\quad \mu<\nu, \tag{1}$$

a model of lower order $\hat{\nu}<\nu$ is sought that has a transfer function

$$\hat{G}(z)=\frac{\hat{N}(z)}{D_{\hat{\nu}}(z)}=\frac{\hat{n}_0+\hat{n}_1z+\ldots\hat{n}_{\hat{\mu}} z^{\hat{\mu}}}{\hat{d}_0+\hat{d}_1z+\ldots+\hat{d}_{\hat{\nu}} z^{\hat{\nu}}},\quad \hat{\mu}<\hat{\nu}. \tag{2}$$

The polynomial $D_\nu(z)$ is stable, that is, all its zeros reside inside the unit circle $|z|=1$. The reduced degree polynomial $D_{\hat{\nu}}(z)$ is the stable polynomial that is determined such that its bilinear transformation is the $\hat{\nu}$-th Routh approximant of the bilinear transformation of $D_\nu(z)$. Let $\Delta_{\hat{\nu}}(s)$ and $\Delta_\nu(s)$ denote the polynomials whose zeros are the bilinear transformation

$$z=\frac{1+s}{1-s},\quad s=\frac{z-1}{z+1} \tag{3a, b}$$

of the zeros of $D_{\hat{\nu}}(z)$ and $D_\nu(z)$, respectively. The determination of $D_\nu(z)$ in [4] and [7] involves the following sequence of operations:

$$D_\nu(z)\xrightarrow[\text{(3a)}]{\text{bilinear}}\Delta_\nu(s)\xrightarrow{\text{Routh}}\Delta_{\hat{\nu}}(s)\xrightarrow[\text{(3b)}]{\text{bilinear}}D_{\hat{\nu}}(z). \tag{4}$$

We are mainly concerned here with the replacement of the indirect procedure of (4) by a direct derivation of $D_\mu(z)$ from $D_\nu(z)$. We discuss, therefore, in (1)–(2) the single-input single-output case only. By treating scalar systems we intensionally avoid the discussion of some problems in certain multivariable extensions suggested for such type of model reduction methods [8]. The numerator $\hat{N}(z)$ of (2) is chosen such that $G(z)$ matches the first μ time moments of $G(z)$. The determination of such scalar numerators for predetermined denominators is well established and will not be repeated [2], [9].

2. A direct discrete Routh method

Assume a real polynomial $D_\nu(z)$,

$$D_\nu(z) = \sum_{i=0}^{\nu} d_i z^i = d_\nu \prod_{i=1}^{\nu} (z - z_i), \tag{5}$$

and define for $D_\nu(z)$ the two polynomials

$$V_0(z) = \tfrac{1}{2}\left[D_\nu(z) - \tilde{D}_\nu(z)\right], \qquad U_0(z) = \tfrac{1}{2}\left[D_\nu(z) + \tilde{D}_\nu(z)\right] \tag{6a, b}$$

where $\tilde{D}_\nu(z)$ is the reciprocated polynomial

$$\tilde{D}_\nu(z) = \sum_{i=0}^{\nu} d_{\nu-i} z^i = d_\nu \prod_{i=1}^{\nu} (1 - z z_i). \tag{7}$$

Let the polynomial

$$\Delta_\nu(s) = \sum_{i=1}^{\nu} \delta_i s^i = \delta_\nu \prod_{i=1}^{\nu} (s - s_i) \tag{8}$$

be the polynomial whose zeros s_i are the bilinear transformation (3b) of the zeros z_i of $D_\nu(z)$. We define

$$\rho_\nu(s) = \frac{\Delta_\nu(s) - \Delta_\nu(-s)}{\Delta_\nu(s) + \Delta_\nu(-s)}, \tag{9}$$

and we have that

$$z - z_i = \frac{2(s - s_i)}{(1-s)(1-s_i)}, \qquad 1 - z z_i = \frac{2(-s - s_i)}{(1-s)(1-s_i)}$$

and therefore that

$$\rho_\nu(z) = \frac{V_0(z)}{U_0(z)} = \frac{\{\Pi_{i=1}^{\nu}(z - z_i)/(1 - z z_i)\} - 1}{\{\Pi_{i=1}^{\nu}(z - z_i)/(1 - z z_i)\} + 1} = \frac{\{\Pi_{i=1}^{\nu}(s - s_i)/(-s - s_i)\} - 1}{\{\Pi_{i=1}^{\nu}(s - s_i)/(-s - s_i)\} + 1} = \rho_\nu(s). \tag{10}$$

The Routh stability table for the polynomial $\Delta_\nu(s)$ is well known to be equivalent to the following continued fraction expansion of $\rho_\nu(s)$ about $s = 0$:

$$\rho_\nu(s) = \frac{1}{\gamma_1/s} + \frac{1}{\gamma_2/s} + \cdots + \frac{1}{\gamma_\nu/s}, \tag{11}$$

where a necessary and sufficient condition for $\Delta_\nu(s)$ with positive coefficients, $\delta_i > 0$, $\forall i$, to be Hurwitz (i.e. $\operatorname{Re} s_i < 0$, $\forall i$) is that the above expansion is valid with $\gamma_i > 0$, $\forall i$ (e.g. [10]). Applying the bilinear transformation (3) on (11), using (10), we immediately obtain the following discrete stability theorem.

Theorem. *A real polynomial $D_\nu(z)$ has all its zeros inside the unit circle $z = 1$ if and only if in the following continued fraction expansion of the ratio between $V_0(z)$ and $U_0(z)$ of* (6a, b),

$$\frac{V_0(z)}{U_0(z)} = \frac{1}{\gamma_i \frac{z+1}{z-1}} + \frac{1}{\gamma_2 \frac{z+1}{z-1}} + \cdots + \frac{1}{\gamma_\nu \frac{z+1}{z-1}}, \tag{12}$$

$\gamma_i > 0$ *for all* $i = 1,\ldots,\nu$.

If $D_\nu(z)$ is a stable polynomial, the truncation of the expansion (12) after $\hat{\nu}$ terms determines a polynomial $D_{\hat{\nu}}(z)$ which is also stable. The polynomial $\Delta_{\hat{\nu}}(s)$ of (8) is, in this case, a Hurwitz polynomial and the truncation of (11) after $\hat{\nu}$ terms determines a Hurwitz polynomial which is the bilinear correspondent of $D_{\hat{\nu}}(z)$. On the other hand we have that the expansion (11) is a possible technique for the derivation of the Routh approximation for stable model reduction of continous systems [6]. (The original representation in [1] uses the truncation of a continued fraction expansion about $s = \infty$ of the $\rho_\nu(s)$ function which is defined for the reciprocated polynomial $\tilde{\Delta}_\nu(s)$ instead of $\Delta_\nu(s)$.) It follows therefore that the polynomial $D_{\hat{\nu}}(z)$ which is determined by the truncation of (12) is identical (up to a scaling factor) to the polynomials derived in [4] and [7] for the stable discrete model reduction problem. The expansion (12) indicates a method for the derivation of $D_{\hat{\nu}}(z)$ directly in the z-plane. The algorithm for this is as follows:

Algorithm. As $D_{\hat{\nu}}(s)$ depends only on $(\gamma_1,\ldots,\gamma_{\hat{\nu}})$ it can be derived by the following recursion that can be shown from (12) by induction or by using standard continued fraction techniques [11]:

$$D_i(z) = \gamma_i(z+1)\,D_{i-1}(z) + (z-1)\,D_{i-2}(z), \quad \text{starting with} \quad D_0(z) = z-1, \quad D_{-1}(z) = 1. \tag{13}$$

The formulation (13) already represents an advantage over the techniques in [4] and [7] that calculate the polynomial $\Delta_{\hat{\nu}}(s)$ from $(\gamma_1,\ldots,\gamma_{\hat{\nu}})$ and then transforms it to the z-plane. It is noted that (13) yields in the process of the derivation of $D_{\hat{\nu}}(z)$ also all the lower degree Routh stable approximants, $D_i(z)$, $i < \hat{\nu}$, which in the indirect formulation of [4] and [7] requires a separate transformation for each of the polynomials $\Delta_i(s)$.

We show next that a recursive easy derivation of γ_i, $i = 1, 2$, also follows from (12). For this purpose we use the following nesting structure that the expansion (12) satisfies:

$$\frac{V_{i-1}(z)}{U_{i-1}(z)} = \frac{1}{\gamma_i \dfrac{z+1}{z-1} + \dfrac{V_i(z)}{U_i(z)}}, \tag{14}$$

where each $V_i(z)/U_i(z)$ represents a function of a form similar to $V_0(z)/U_0(z)$ that gives rise to a continued fraction of the type (12) which has length $(\nu - i)$ and the coefficients $(\gamma_{i+1}, \gamma_{i+2},\ldots,\gamma_\nu)$. Therefore similar to $V_0(z)$, all $V_i(z)$ satisfy $V_i(1) = 0$, and we may rewrite the above equation as

$$\frac{V_{i-1}(z)/(z-1)}{U_{i-1}(z)} = \frac{U_i(z)}{(z+1)\gamma_i U_i(z) + (z-1)\,V_i(z)}. \tag{15}$$

This equation requires separate equality of the numerators and the denominators. We have from the numerator that

$$U_i(z) = V_{i-1}(z)/(z-1) \tag{16}$$

and from the denominator, first by setting $z = 1$, that

$$\gamma_i = \frac{1}{2}\frac{U_{i-1}(1)}{U_i(1)} \tag{17}$$

and then by eliminating $V_i(z)$ that

$$V_i(z) = \left[U_{i-1}(z) - \gamma_i(z+1)U_i(z)\right]/(z-1). \tag{18}$$

Equations (16)–(18) together with (13) represent a sequential algorithm to derive the Routh-bilinear stable polynomials $D_i(z)$, $i = 1, 2, \ldots$ directly in the z-plane. It produces $D_{\hat{\nu}}(z)$ at the step $i = \hat{\nu}$ and reproduces $D_\nu(z)$ at the final step $i = \nu$. Following are two comments that concern the implementation of the algorithm presented by (16)–(18) and (13).

(i) Equations (16) and (18) require the division of polynomials that have zeros at $z = 1$ by $(z-1)$. This is performed as follows:

$$A(z)=\sum_{i=0}^{n} a_i z^i \quad \Rightarrow \quad A(z)/(z-1)=\sum_{i=0}^{n-1} b_i z^i, \quad b_0=-a_0, \quad b_i=b_{i-1}-a_i. \tag{19}$$

(ii) The polynomials $U_i(z)$ and $V_i(z)$ have special mirror $(a_i = a_{n-i})$ and anti-mirror $(a_i = -a_{n-i})$ structures respectively, that can be used to carry out the calculation of only the first half of their coefficients.

Example. Let the high order system be given the transfer function [7]

$$G(z)=\frac{1.682\,z^7+1.116\,z^6-0.21\,z^5+0.152\,z^4-0.516\,z^3-0.262\,z^2+0.044\,z-0.006}{8\,z^8-5.046\,z^7-3.348\,z^6+0.63\,z^5-0.456\,z^4+1.548\,z^3+0.786\,z^2-0.132\,z+0.018}. \tag{20}$$

We have therefore

$$V_0(z)=-3.991+2.457\,z+2.067\,z^2+0.459\,z^3-0.459\,z^5-2.067\,z^6-2.457\,z^7+3.991\,z^8,$$

$$U_0(z)=4.009-2.589\,z-1.281\,z^2+1.089\,z^3-0.456\,z^4+1.089\,z^5-1.281\,z^6-2.589\,z^7+4.009\,z^8.$$

To derive a second order model we perform the algorithm for $i = 1, 2$:

$$U_1(z)=3.991+1.534\,z-0.533\,z^2-0.992\,z^3-0.992\,z^4-0.533\,z^5+1.534\,z^6+3.991\,z^7,$$

$$\gamma_1=\tfrac{1}{2}U_0(1)/U_1(1)=0.125, \qquad D_1(z)=0.125(z+1)+(z-1)=1.125\,z-0.875,$$

$$V_1(z)=-3.510-0.2305\,z+1.176\,z^2-0.104\,z^3+0.104\,z^4-1.176\,z^5+0.2305\,z^6+3.510\,z^7,$$

$$U_2(z)=3.510+3.741\,z+2.565\,z^2+2.669\,z^3+2.565\,z^4+3.741\,z^5+3.510\,z^6,$$

$$\gamma_2=\tfrac{1}{2}U_1(1)/U_2(1)=0.1793681,$$

$$D_2(z)=0.1793681(z+1)(1.125\,z-0.875)+(z-1)(z+1)=1.201789\,z^2-1.955158z+0.84053,$$

where $D_2(z)$ has the same zeros as the denominator obtained in reference [7] using the indirect approach. Completing a numerator such that $G(z)$ fits the first two time moments of $G(z)$ we get the following second order stable model:

$$\hat{G}(z)=\frac{-0.298503+0.373124\,z}{0.701497-1.626873\,z+z^2}. \tag{21}$$

Conclusions

The direct Routh discrete method yields models equivalent with those derived in [4] and [7] using the bilinear approximation. The method inherits therefore the favourable feature of stability preservation without calculation of any eigenvalues. Some further features of the direct approach which represent computational advantages over the indirect approach are as follows:

(1) The reduced order models are derived completely in the z-plane. The application of the bilinear transformation, that is required twice in the indirect approach, is eliminated.

(2) The derivation of a $\hat{\nu}$-th order model yields also the stable denominators for all lower models $\hat{\nu}-i$, $i = 1, 2 \ldots$.

(3) The algorithm converges in ν steps to the original system. An improved approximation of order $(\hat{\nu}+1)$ can be derived from the previous $\hat{\nu}$-th order results in a single further step.

References

[1] M.F. Hutton and B. Friedland, Routh approximations for reducing order of linear, time invariant systems, *IEEE Trans. Automat. Control* 20 (1975) 324–337.

[2] Y. Shamash, Model reduction using the Routh stability criterion and the Padé approximation technique, *Internat. J. Control* 21 (1975) 475–484.

[3] R.K. Appiah, Linear model reduction using Hurwitz polynomial approximations, *Internat. J. Control* 28 (1978) 477–488.

[4] R.K. Appiah, Padé methods of Hurwitz polynomial approximation with application to linear system reduction, *Internat. J. Control* 29 (1979) 39–48.

[5] Y. Bistritz and U. Shaked, The Padé–Hurwitz method of model reduction, in: *Proc. 17th Ann. Allerton Conf. on Communication Control and Computing,* Monticello, Illinois (1979) pp. 82–91.

[6] Y. Bistritz and U. Shaked, Stable linear systems simplification via Padé approximations to Hurwitz polynomials; *ASME J. Dynamic Systems Measurement and Control* 103 (1981) 279–284.

[7] C. Hwang, Y.P. Shih, and R.Y. Huang, A combined time and frequency domain method for model reduction of discrete systems, *J. Franklin Inst.* 311 (1981) 391–402.

[8] Y. Bistritz and U. Shaked, Minimal Padé model reduction of multivariable systems, in: *Proc. of the IEEE 20th C.D.C. Conf.* (1981) pp. 672–675.

[9] Y. Shamash, Continued fraction methods for the reduction of discrete time dynamic systems, *Internat. J. Control* 20 (1974) 267–275.

[10] E.A. Guillemin, *The Mathematics of Circuit Analysis,* (MIT Press, Cambridge, MA, 1949).

[11] H.S. Wall, *Analytical Theory of Continued Fractions* (Chelsea, New York, 1967).