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A NEW UNIT CIRCLE STABILITY CRITERION

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Abstract

New necessary and sufficient conditions for a real polynomial to have all its zeros inside the unit circle are derived. The conditions are obtained by the study of certain new forms of z-domain continued fraction expansions. They induce an effective procedure for testing the stability of discrete systems that reminds in many ways the Routh scheme for Hurwitz polynomials. A table form is also presented for the stability criterion. The table has half the size and involves half the amount of computation of the Jury-Marden table.

1. Introduction

An important problem in discrete system analysis and design is to find necessary and sufficient conditions for the zeros of the characteristic polynomial of the system to lie inside the unit circle. The problem of distribution of the zeros of a polynomial with respect to the unit circle was originally solved by Cohn [1] and later simplified and put into a tabular form by Marden [2] and Jury [3]. The Jury-Marden table is customarily considered as a discrete analogue of the Routh table which is used to determine the stability of continuous time systems. However, a closer inspection of the two tabular forms and their associated conditions hardly support this analogy. The Jury-Marden table requires the computation of twice the number of entries then a comparable Routh table for a polynomial of the same degree and it has four times its size. The necessary and sufficient conditions imposed on the respective tables also seem less elegant in the Jury-Marden presentation. Furthermore, the Routh conditions are closely related to certain continued fraction expansions of Hurwitz polynomials (see [4] or theorem 2 below). In fact the Routh array is simply a row by row inscriptions of the polynomials involved in performing the successive steps of these associated expansions. The Jury Marden table on the other hand is not related in a similar way to any z-domain continued fraction expansion. It can be said, at most, to be equivalent to a certain division procedure [3] that cannot be expressed in closed and explicit continued fraction form.

In an attempt to reduce the amount of computation that is presently required for testing the stability of a discrete system by the coefficients of its characteristic polynomial and in order to make it comparable with the less effort required for the corresponding continuous-systems situation, one may reasonably search for necessary and sufficient conditions for unit circle stability that are related to some adequate z-domain continued fraction expansions.

Apparently, the only stability criterion discussed the literature, which is based on a z-domain continued fraction expansion, is the bilinear conversion of the s-plane continued fraction forms that imply the Routh criterion. Such expansions, which are in terms of $(z+1)/(z-1)$ or $(z-1)/(z+1)$, do not constitute an appropriate parallelism to the meaning of the s and s^{-1} terms in the continuous case. It is known that the forward and backward differences play in discrete system theory roles that are comparable with the derivative and integrative senses of s and s^{-1}. Therefore, as plausible candidates for the sought analogy, one may attempt continued fraction expansions in terms of z-1 or $1-z^{-1}$.

The present paper introduces continued fraction forms for the tangent function of a polynomial D(z) [6] that involve terms of (z-1) and $(1-z^{-1})$. It is shown that for a polynomial with inside the unit circle (IUC) zeros these form exist and have positive coefficients. Complementary conditions are derived subsequently under which these forms imply also IUC zeros. The computational aspects of the new conditions are studied and the paper culminates on a new tabular array for testing the stability of discrete time systems from the coefficients of its characteristic polynomial. The parallelism between the interrelation of the new stability table and the z-plane continued fraction expansion forms and the similar relations for the Routh array and z-plane continued fraction expansions for Hurwitz polynomial is remarkable. This has also a most desirable consequence - the new unit circle stability table requires an amount of computation that is comparable with that in the Routh test for a polynomial of the same degree.

2. Unit Circle Stability Conditions

Denote by $D_n(z)$ the real polynomial given by

$$D_n(z)=\sum_{i=o}^{n} d_i z^i = d_n \prod_{i=1}^{n} (z-z_i) \quad , \quad d_n>0 \qquad (1)$$

The polynomial $D_n(z)$ is called stable if all its zeros are IUC, $|z_i|<1$, $i=1,\dots,n$. Our presentation will frequently consider both z-and s-plane stability conditions. To avoid ambiguity we shall use the term stable polynomial only in the discrete context whereas for the anologous s-plane case, the polynomial $H_n(s)$

$$H_n(s)=\sum_{i=o}^{n} h_i s^i = h_n \prod_{i=1}^{n} (s-s_i) \quad , \quad h_n>0 \qquad (2)$$

will be called Hurwitz if $\mathrm{Re}\, s_i<0$ $i=1\dots n$.

The tangent function of $D_n(z)$ is defined to be

$$\rho_n(z) = \frac{D_n(z)-\tilde{D}_n(z)}{D_n(z)+\tilde{D}_n(z)} \tag{3}$$

where $\tilde{D}_n(z)$ denotes the reciprocated polynomial of $D_n(z)$

$$\tilde{D}_n(z)= \sum_{i=o}^{n} d_{n-i} z^i = z^n D_n(z^{-1}) \tag{4}$$

It is noticed that the zeros of $\tilde{D}_n(z)$ are the inverse of the zeros of $D_n(z)$ (z^{-1} replace $z_i \neq 0$ and each $z_i=0$ reduces by one of the actual degree of $D_n(z)$). We shall make use of the following result.

<u>Theorem 1 [6]</u>. A real polynomial $D_n(z)$, $d_n > |d_o|$, is stable if and only if the zeros and poles of its tangent function $\rho_n(z)$ are simple, located on the unit circle $|z|=1$ and mutually separate each other.

This theorem implies the following [6]. The polynomial $D_n(z)$, $d_n > |d_o|$ is stable if and only if $\rho_n(z)$ can be written in the form

$$\rho_{2m+1}(z) = \frac{K(z-1) \prod_{i=1}^{m} (z^2-2z \cos\Omega_{2i}+1)}{(z+1) \prod_{i=1}^{m}(z^2-2z \cos\Omega_{2i-1}+1)} \qquad K > 0 \tag{5a}$$

for $n=2m+1$ and

$$\rho_{2m}(z) = \frac{K(z-1)(z+1) \prod_{i=1}^{m-1} (z^2-2z \cos\Omega_{2i}+1)}{\prod_{i=1}^{m}(z^2-2z \cos\Omega_{2i-1}+1)} \qquad K > 0 \tag{5b}$$

for $n=2m$

where

$$1<\cos\Omega_{n-1}<\cos\Omega_{n-2}<\ldots<\cos\Omega_2<\cos\Omega_1<1 \tag{5c}$$

We shall also use some s-plane conditions on Hurwitz polynomials that are summarized in the next theorem. These are long established results that are brought here for reference convenience.

<u>Theorem 2</u>: The following statements are all equivalent

(i) The real polynomial $H_n(s)$ is Hurwitz

(ii) The tangent function $\rho_n(s)$ defined for $H_n(s)$

$$\rho_n(s)= \frac{H_n(s)-H_n(-s)}{H_n(s)+H_n(-s)} \tag{6a}$$

can be written in the form

$$\rho_n(s)= \frac{Ks \prod_{i=1}^{\ell} (s^2+\omega_{2i}^2)}{\prod_{i=1}^{m} (s^2+\omega_{2i-1}^2)} \qquad K>0 \tag{6b}$$

where $n=m+\ell+1$, $\ell=m-1$ or $\ell=m$ and

$$0<\omega_1^2<\omega_2^2<\ldots<\omega_n^2 \tag{6c}$$

(iii) The following continued fraction expansions exist and $\gamma_i>0$ for all $i=1,\ldots,n$

$$\rho_{2m+1}(s) = \gamma_{2m+1}s + \frac{1}{\gamma_{2m}s} + \ldots + \frac{1}{\gamma_1 s}, \quad n = 2m+1 \tag{7a}$$

$$\rho_{2m}(s) = \frac{1}{\gamma_{2m}s} + \frac{1}{\gamma_{2m-1}s} + \ldots + \frac{1}{\gamma_1 s}, \quad n = 2m \tag{7b}$$

The well known Routh table is in fact a tabular presentation of the polynomials involved in the step by step performance of the expansions depicted in (7).

The next theorem presents the new z-domain continued fraction expansion for stable polynomials.

<u>Theorem 3</u> If the polynomial $D_n(z)$ is stable then its tangent function has the following continued expansion (for n=2m+1 and n=2m respectively).

$$(z+1)\rho_{2m+1}(z)=\delta_{2m+1}(z-1)+ \frac{1}{\delta_{2m}(1-z^{-1})} + \ldots + \frac{1}{\delta_1(z-1)} \tag{8a}$$

$$(z+1)^{-1}\rho_{2m}(z)= \frac{1}{\delta_{2m}(z-1)} + \frac{1}{\delta_{2m-1}(1-z^{-1})} + \ldots + \frac{1}{\delta_1(1-z^{-1})} \tag{8b}$$

with positive coefficients δ_i for all i, $\delta_i>0$, $i=1,\ldots n$

<u>Proof</u>: Consider the transformation

$$s = \frac{1}{2} (z^{1/2}-z^{-1/2}) \tag{9}$$

that maps the unit circle C

$$C=\{z \mid z=e^{i\Omega} \quad \Omega\varepsilon[-\Pi,\Pi]\} \tag{10}$$

one to one and onto the s-plane imaginary axis interval J defined by

$$J=\{s \mid s=j\omega \qquad \omega\varepsilon[-1,1]\} \tag{11}$$

If $D_n(z)$ is stable then by theorem 1 its tangent function $\rho_n(z)$ has the structure (5a,c) or (5b,c) for n=2m+1 or n=2m, respectively. A typical product in these expressions are mapped by (9) as follows

$$z^{-1}(z^2-2z\cos\Omega_k+1) \leftrightarrow 4[s^2+\sin^2(\Omega_k/2)] \tag{12}$$

Therefore

$$\frac{z^{m-\ell}\prod^{\ell}(z^2-2z\cos\Omega_{2i}+1)}{\prod^{m}(z^2-2z\cos\Omega_{2i-1}+1)} \leftrightarrow \frac{\prod^{\ell}(s^2+\omega_{2i}^2)}{\prod^{m}(s^2+\omega_{2i-1}^2)} \tag{13}$$

where

$$\omega_k=\sin(\Omega_k/2) \tag{14}$$

and condition (5c) is converted into

$$0<\omega_1^2<\ldots<\omega_{n-1}^2<1 \tag{15}$$

For n=2m+1 we consequently have from (5a) that

$$z^{-1/2}(z+1)\rho_{2m+1}(z) \leftrightarrow \frac{Ks\prod^{\ell}(s^2+\omega_{2i}^2)}{\prod^{m}(s^2+\omega_{2i-1}^2)} \tag{16}$$

By theorem 2 (the equivalence of (ii) and (iii) there), the right hand side of (16) can be expanded into the form (7a). Setting (9) into (6a) we have from (16) the following equality

$$z^{-1/2}(z+1)\rho_{2m+1}(z)= \\ = \gamma_{2m+1}\frac{1}{2}(z^{1/2}-z^{-1/2})+\frac{1}{\gamma_{2m}\frac{1}{2}(z^{1/2}-z^{-1/2})}+\ldots+\frac{1}{\gamma_1\frac{1}{2}(z^{1/2}-z^{-1/2})} \tag{17a}$$

Repeating the last steps for n=2m we obtain

$$(z+1)^{-1}z^{1/2}\rho_{2m}(z)= \frac{1}{\gamma_{2m}\frac{1}{2}(z^{1/2}-z^{-1/2})}+\ldots+\frac{1}{\gamma_1\frac{1}{2}(z^{1/2}-z^{-1/2})} \tag{17b}$$

setting $\delta_i=\frac{1}{2}\gamma_i>0$ in (17a,b), the required forms (8a,b) follow from (17a,b) by multiplying the two sides of (17a) and of (17b), respectively by $z^{1/2}$ and $z^{-1/2}$.

It is emphasized that the positivity of $\delta_i>0$ in (8) is not sufficient for a polynomial to be stable. This is different from the situation in the continuous case of theorem 2 and in contrast to the bilinear-Routh continued fraction expansion forms of [5]. The dissimilarity may not come as a surprise if one is aware of the fact that, unlike the bilinear transformation, the interior and the exterior of the z-plane unit circle are each mapped by (9) into both the left and the right halves of the s-plane. We shall establish in the sequel the additional conditions together with which the expansions (8) and $\delta_i>0$ also imply stability.

A real polynomial $D_n(z)$ can always be written as the sum of a symmetric and an asymmetric polynomial,

$$D_n(z)= \frac{1}{2}[D_n(z)+\tilde{D}_n(z)]+ \frac{1}{2}[D_n(z)-\tilde{D}_n(z)]= \frac{1}{2}S_n(z)+ \frac{1}{2}A_n(z) \quad (18a)$$

where a polynomial $S_n(z)$ is called symmetric if it is equal to its reciprocal polynomial, namely

$$\tilde{S}_n(z)=S_n(z) \;, \quad S_n(z)=\sum_{i=o}^{n} s_i z^i \quad \text{with} \quad s_i=s_{n-i} \;\; i=0,1,\ldots \quad (18b)$$

and a polynomial $A_n(z)$ is called asymmetric if it is the minus sign of its reciprocated polynomial, namely

$$A_n(z)=-A_n(z) \qquad A_n(z)=\sum_{i=o}^{n} a_i z^i \quad \text{with} \quad a_i=-a_{n-i}, \;\; i=0,1\ldots \quad (18c)$$

A polynomial of odd degree can always be written in the form

$$D_{2m+1}(z)= \frac{1}{2} A_{2m+1}(z)+ \frac{1}{2}(z+1)S_{2m}(z) \quad (19a)$$

because a symmetric polynomial of odd degree must have a zero at $z=-1$. Similarly, a polynomial of even degree can always be written in the form

$$D_{2m}(z)= \frac{1}{2} S_{2m}(z)+ \frac{1}{2}(z+1)A_{2m-1}(z) \quad (19b)$$

because an asymmetric polynomial of even degree must have a zero at $z=-1$.

Consider next a sequence of n+1 polynomials $\{D_k(z)\}_{k=o}^{n}$ defined for a given set of n positive real numbers $\delta_1, \delta_2,\ldots,\delta_n>0$ by the following assignment; $D_k(z)$ is the polynomial of degree k whose tangent function $\rho_k(z)$ has a continued fraction expansion of the form (8a,b) with coefficients of expansion $\{\delta_1\ldots,\delta_k\}$. In other words, odd degree polynomials $D_{2i+1}(z)$ in the sequence are defined in association with $\delta_{2i+1}(z)$ by

$$(z+1)\rho_{2i+1}(z) = \frac{A_{2i+1}(z)}{S_{2i+1}(z)} = \delta_{2i+1}(z) + \frac{1}{\delta_{2i}(1-z^{-1})} + \ldots + \frac{1}{\delta_1(z-1)} \quad (20a)$$

and even degree polynomials $D_{2i}(z)$ in this sequence are defined via

$\rho_{2i}(z)$ by

$$(z+1)^{-1}\rho_{2i}(z)= \frac{A_{2i-1}(z)}{S_{2i}(z)} = \frac{1}{\delta_{2i}(z-1)} + \dots + \frac{1}{\delta_1(1-z^{-1})} \qquad (20b)$$

The structure of these continued fraction forms implies the relations

$$(z+1)\rho_{2i+1}(z)=\delta_{2i+1}(z-1)+z\rho_{2i}(z)/(z+1) \qquad (21a)$$

$$(z+1)\rho_{2i}(z)=\delta_{2i}(z-1)+z\rho_{2i-1}^{-1}(z)/(z+1) \qquad (21b)$$

Expressing therefore the polynomials $D_k(z)$ by the decomposition of (19) into sums of symmetric and asymmetric polynomials, we find that $S_{2i}(z)$ is common for $D_{2i}(z)$ and $D_{2i+1}(z)$ and $A_{2i-1}(z)$ is common for $D_{2i-1}(z)$ and $D_{2i}(z)$.

If we define for $\{D_k(z)\}_{k=o}^{n}$ a second sequence of n polynomial $\{T_k(z)\}_{k=o}^{n}$

$$T_{2i+1}(z)=D_{2i+1}(z)-\tilde{D}_{2i+1}(z) \qquad i=0,\dots,m-1(m) \qquad (22a)$$

$$T_{2i}(z)=D_{2i}(z)+\tilde{D}_{2i}(z) \qquad i=0,\dots,m \qquad (22b)$$

then $\{D_k(z)\}$, are also reciprocally defined by $\{T_k(z)\}$

$$2D_k(z)=T_k(z)+(z+1)T_{k-1}(z) \qquad k=1,\dots,n \qquad (23)$$

The new sequence $\{T_k(z)\}_{k=o}^{n}$ can, by (19)-(22), be generated from $\{\delta_1,\dots,\delta_n\}$ successively by the rule

$$T_{k+1}(z)=\delta_{k+1}(z-1)T_k(z)+zT_{k-1}(z) \qquad k=1,\dots,n \qquad (24)$$

starting with $T_o(z)=1$ and $T_1(z)=\delta_1(z-1)$.

Given $\{\delta_1,\dots,\delta_n\}$, equations (24) and (23) show a constructive definition for the associated sequence of polynomials $\{D_k(z)\}_{k=o}^{n}$.

<u>Theorem 4</u>: Assume $\{\delta_1,\dots,\delta_n\}$ are n positive real numbers and let $\{D_i(z)\}_{i=o}^{n}$ be the sequence of polynomial constructed through (23) and (24).

(i) If $D_{2k-1}(z)$ is stable and $D_{2k+1}(-1)<0$ then $D_{2k+1}(z)$ is also stable, $k=1,2,\dots;\ 2k+1<n$

(ii) If $D_{2k-2}(z)$ is stable and $D_{2k}(-1)>0$ then $D_{2k}(z)$ is also stable, $k=1,2,...;\quad 2k<n$

Proof: We can restate theorem 1 as follows. A polynomial $D_n(z)$ is stable if and only if its tangent function can be written in the form

$$\rho_{2m+1}(z)=\frac{K(z-1)\prod^{m}(z^2+zX_{2i}+1)}{(z+1)\prod^{m}(z^2+zX_{2i-1}+1)}\quad,\qquad K>0 \tag{25a}$$

when $n=2m+1$ or when $n=2m$

$$\rho_{2m}(z)=\frac{K(z-1)(z+1)\prod^{m-1}(z^2+zX_{2i}+1)}{\prod^{m}(z^2+zX_{2i-1}+1)}\quad,\qquad K>0 \tag{25b}$$

where X_i are real numbers

$$-2<X_1<X_2<...<X_{n-1}<2 \tag{25c}$$

We shall use in the sequel the notation $X_i(k)$ to refer to the real numbers X_i involved in a structure $\rho_k(z)$ of the form (25) in association with the polynomial $D_k(z)$ of the sequence.

Consider the first part of the theorem $D_{2k+1}(z)$ is defined for $\delta_1,...\delta_{2k+1}>0$. Let $\gamma_i=2\delta_i>0$, $i=1,...,2k+1$ and consider $\rho_{2k+1}(s)$, the function defined by (7), with these numbers taken as coefficients in the expansion. By the equivalence of (7) and (6) in theorem 2, $\rho_{2k+1}(s)$ has a structure depicted by (6) for some real numbers $\omega_i^2(2k+1)$ that satisfy

$$0<\omega_i^2(2k+1)<\omega_2^2(2k+1)<...<\omega_{2k+1}(2k+1) \tag{26}$$

where $2k+1$ in $\omega_i^2(2k+1)$ is used as an index to associate this sequence with the $(2k+1)$th order case. Define for $\omega_i^2(2k+1)$ the real numbers

$$X_i(2k+1)=4\omega_i^2(2k+1)-2 \tag{27}$$

By reversing part of the proof in theorem 3, it can be shown that the transformation (9) maps $\rho_{2k+1}(s)$ into

$$(z+1)\rho_{2k+1}(z) = \frac{K(z-1)\prod^{k}[z^2-zX_{2i}(2k+1)+1]}{\prod^{k}[z^2-zX_{2i-1}(2k+1)+1]} \quad , \quad K > 0 \tag{28}$$

where from (26) and (27)

$$-2<X_1(2k+1)<X_2(2k+1)<\ldots<X_{2k}(2k+1) \tag{29}$$

If (28) and (29) are compared with (25a) and (25c), it is observed that to prove the stability of $D_{2k+1}(z)$ it remains to show that in (29)

$$X_{2k}(2k+1)<2 \tag{30}$$

holds as well. Repeating the above reasoning with regard to $D_{2k}(z)$ we have that, given $\delta_1,\ldots,\delta_{2k}>0$, $D_{2k}(z)$ is associated with $_{2k}(z)$ which has a structure (25b) for $X_i(2k)$ that satisfy

$$-2<X_1(2k)<X_2(2k)<\ldots<X_{2k-1}(2k) \tag{31}$$

The polynomial $D_{2k-1}(z)$ is by assumption stable. Therefore it has a tangent function $\rho_{2k-1}(z)$ of the form (25a) and $X_i(2k-1)$ that satisfy

$$-2<X_1(2k-1)<X_2(2k-1)<\ldots<X_{2k-2}(2k-1)<2 \tag{32}$$

From the relations (21), the zeros of $\rho_{2k-1}(z)$ are also zeros of $\rho_{2k}(z)$ and the poles of $\rho_{2k}(z)$ are also poles of $\rho_{2k+1}(z)$. Therefore substituting

$$X_{2i}(2k)=X_{2i}(2k-1) \tag{33a}$$

and

$$X_{2i-1}(2k)=X_{2i-1}(2k+1) \tag{33b}$$

into (31) we obtain

$$-2<X_1(2k+1)<\ldots<X_{2k-3}(2k+1)<X_{2k-2}(2k-1)<X_{2k-1}(2k+1) \tag{34}$$

It is given that $D_{2k+1}(-1)<0$, therefore, $T_{2k+1}(-1)<0$ by (23). Similarly $T_{2k-1}(-1)<0$ because $D_{2k-1}(-1)<0$ is a necessary condition for any stable polynomial of the form (1). We can then conclude from (24) that $T_{2k}(-1)>0$.

We now are in a position to prove (30). Equations (34) and (32) imply

$X_{2k-3}(2k+1)<2.$ (35)

Consequently in (26) we either have (30) or one of the three only remaining possibilities

(a) $X_{2k-1}(2k+1)<2<X_{2k}(2k+1)$

(b) $X_{2k-2}(2k+1)<2<X_{2k-1}(2k+1)$

(c) $X_{2k-3}(2k+1)<2<X_{2k-2}(2k+1)$

Possibilities (a) and (b) contradict $T_{2k+1}(-1)<0$ because by comparing (22a) and (25a), $T_{2k+1}(-1)$ is given by

$$T_{2k+1}(-1)=-2K[2-X_{2k}(2k+1)]\prod^{k-1}[2-X_{2i}(2k+1)]=$$
$$=[2-X_{2k}(2k+1)].(-P) \quad , \quad P>0 \qquad (36)$$

Possibility (c) contradicts $T_{2k}(-1)>0$ because from (22b) and (25b) it can be written as

$$T_{2k}(-1)=[2-X_{2k-1}(2k)]\prod^{k-1}[2-X_{2i-1}(2k)]=$$
$$=[2-X_{2k-1}(2k)].P \quad , \quad P>0 \qquad (37)$$

where the second equality follows from (31), (33a) and (35). Therefore (30) must hold and it implies with (28) and (29) that $D_{2k+1}(z)$ is stable. The second part of the theorem can similarly be proven. △ △ △

The next theorem is our main result. It complements theorem 4 into necessary and sufficient conditions for stability.

<u>Theorem 5</u>: Given the polynomial $D_n(z)$ and the sequence $\{D_k(z)\}_{k=0}^{n}$ constructed by (23) and (24), $D_n(z)$ is stable if and only if

(i) for n=2m+1, the expansion (25a) exists and has positive coefficients $\delta_1,\ldots,\delta_{2m+1}>0$ and

$$D_{2k+1}(-1)<0 \quad \text{for} \quad k=0,\ldots,m \qquad (38)$$

(ii) for n=2m, the expansion (25b) exists and has positive coefficients $\delta_1,\ldots\delta_{2m}>0$ and

$$D_{2k}(-1)>0 \quad \text{for} \quad k=0,\ldots,m \qquad (39)$$

<u>Proof</u>: Assume first that $D_{2m+1}(z)$ is stable. By theorem 3 we have $\delta_1,\ldots,\delta_{2m+1}>0$. Let $D_{2k+1}(z)$, k<m be any polynomial in the se-

quence. Its tangent function must have then the form (28), and (29) holds. It is evident from the proof of theorem 4 that the maximal values of the real numbers $X_i(k)$ for tangent functions $\rho_k(z)$ of successive degrees k form a strictly increasing sequence. Thus in (29) we also have $X_{2k}(2k+1)<X_{2m}(2m+1)<2$. Consequently, $D_{2k+1}(z)$ is stable for any $k=0,\ldots,m$ and (38) follows as a necessary condition that is satisfied by stable polynomials of the form (1).

Assume next that $D_{2m+1}(z)$ has the expansion (8a) with $\delta_1,\ldots,\delta_{2m+1}>0$ and (38) holds. Consider the sequence of odd degree polynomials $D_{2i+1}(z)$, $i=0,\ldots,m$. It starts with $D_1(z)$ that is given by setting $k=1$ in (23) and using (24)

$$2D_1(z)=(\delta_1+1)z+z-1$$

$D_1(z)$ has a zero at $(\delta_1-1)/(\delta_1+1)$ that is inside the unit circle for $\delta_1>0$. Therefore by theorem 4 all the odd degree subsequent polynomials are stable and particularly $D_{2m+1}(z)$ is stable.

The proof for the even order part is similar; If $D_{2m}(z)$ is stable then $\delta_1,\ldots,\delta_{2m}>0$ by theorem 3 and it also follows that $D_{2i}(z)$, $i=1,\ldots,m$ are all stable and (39) necessarily holds. For the converse, if (8b), $\delta_1\ldots\delta_{2m}>0$ and (39) all hold, then all $D_{2i}(z)$, $i=1,\ldots,m$ are stable by theorem 4 because δ_1, $\delta_2>0$ is sufficient for $D_2(z)$, defined by (23) and (24), to be stable.

3. Computational Procedures

We suggest in this section some computational schemes that implement the new necessary and sufficient conditions of the previous section into useful stability tests for a polynomial $D_n(z)$ given in the form (1).

<u>Algorithm 1</u>: The sequence of polynomials $\{T_i(z)\}_{i=0}^{n}$, defined for

$D_n(z)$ by (22) and (24) can be constructed successively in descending degree order. First $T_n(z)$ and $T_{n-1}(z)$ are formed

(i) when $n=2m+1$

$$T_n(z)=D_{2m+1}(z)-\tilde{D}_{2m+1}(z) \tag{40a}$$

$$T_{n-1}(z)=[D_{2m+1}(z)+\tilde{D}_{2m+1}(z)]/(z+1) \tag{41a}$$

(ii) when $n=2m$

$$T_n(z)=D_{2m}(z)+\tilde{D}_{2m}(z) \tag{40b}$$

$$T_{n-1}(z)=[D_{2m}(z)-\tilde{D}_{2m}(z)]/(z+1) \tag{41b}$$

where (40a,b) were defined by (22), and (41a,b) can be verified from (22) (19) and (23). Next, the other polynomials of the sequence are constructed according to the following scheme that can be deduced from (24),

$$\delta_i=-T_i(0)/T_{i-1}(0) \tag{42}$$

$$T_{i-2}(z)=z^{-1}[T_i(z)-\delta_i(z-1)T_{i-1}(z)] \tag{43}$$

<u>The conditions of stability</u>; The polynomial $D_n(z)$ is stable if and only if

(i) when $n=2m+1$,

$$\delta_i>0,\ i=1,\dots,2m+1 \tag{44a}$$

$$T_{2i+1}(-1)<0,\ i=0,\dots,m \tag{45a}$$

(ii) when $n=2m$

$$\delta_i>0\ i=1,\dots,2m \tag{44b}$$

$$T_{2i}(-1)>0\ i=0,\dots,m \tag{45b}$$

Conditions (44) and (45) follow from theorem 5 where for the validity of (45a,b) note that $T_{2i+1}(-1)=D_{2i+1}(-1)-\tilde{D}_{2i+1}(-1)=2D_{2i+1}(-1)$ and $T_{2i}(-1)=D_{2i}(-1)+\tilde{D}_{2i}(-1)=2D_{2i}(-1)$, using (22) and (4).

<u>Remark 4.1</u>: Advantage can be taken of the fact that $T_{2i}(z)$ and $T_{2i+1}(z)$ are symmetric, (18b), and asymmetric, (18c), polynomials. It is sufficient to compute only one half of their coefficients for the above algorithm.

<u>Remark 4.2</u>: The construction of $T_{n-1}(z)$ in (41a,b) requires the elimination of a (z+1) factor from a polynomial. Such an operation

involves only simple additive arithmetic

$$F_n(z)=\sum_{i=o}^{n} f_i z^i \rightarrow F_n(z)/(z+1)=\sum_{i=o}^{n-1} e_i z^i \tag{46a}$$

where

$$e_o=f_o \quad , \quad e_i=f_i-e_{i-1} \tag{46b}$$

Algorithm 2 - A tabular form: Use for the polynomials $\{T_i(z)\}_{i=0}^{n}$ the following explicit notation

$$T_k(z)=\sum_{i=o}^{k} \beta_{n-k,k-i} z^{k-i} \qquad i=0,\ldots,n \tag{47}$$

Consider the array of n+1 rows formed by the coefficients of these polynomials

$$\begin{array}{llllllll}
\beta_{00} & \beta_{01} & \beta_{02} & & & & (\beta_{0,n-1}) & (\beta_{0,n}) \\
& \beta_{10} & \beta_{11} & & & & (\beta_{1,n-1}) & \\
& & \cdot\,\beta_{i-1,0} & \cdots & \beta_{i-1,k+1} & & (\beta_{i-1,n+1-i}) & \\
& & & \beta_{i,0} \;\cdots\; \beta_{i,k} & \beta_{i,k+1} & & (\beta_{i,n-i}) & \\
& & & & \Downarrow & & & \\
& & & \cdots\cdots & \beta_{i+1,k} & & & \\
& & & & & \beta_{n,0} & &
\end{array} \tag{48}$$

The first two rows are obtained from the coefficients of $T_n(z)$ and $T_{n-1}(z)$ given by (40a,b) and (41a,b).

$$T_n(z)=\beta_{00}z^n+\beta_{01}z^{n-1}+\ldots+\beta_{0,n} \tag{49a}$$

$$T_{n-1}(z)=\beta_{10}z^{n-1}+\beta_{11}z^{n-2}+\ldots+\beta_{1,n-1} \tag{49b}$$

The next rows are constructed by a scheme that is deduced from (42) and (43) as follows. First we conveniently redefine $\tilde{\delta}_i=\delta_{n-i+1}$, that is

$$(\tilde{\delta}_1,\ldots,\tilde{\delta}_n)=(\delta_n,\ldots,\delta_1)$$

From equations (42) and (47) we have

$$\delta_{n-i+1}=-T_{n-i+1}(0)/T_{n-i}(0)=-\beta_{i-1,n-i+1}/\beta_{i,n-i}=\beta_{i-1,0}/\beta_{i,0}$$

where the last equality follows from the opposite symmetries of two polynomial of adjacent degrees. We have therefore

$$\tilde{\delta}_i=\beta_{i-1,0}/\beta_{i,0} \tag{50}$$

Next, by substituting (47) into (43) and comparing coefficients of similar powers of z, we obtain after some manipulation of the indices,

$$\beta_{i+1,k}=\beta_{i-1,k+1}+\tilde{\delta}_i(\beta_{i,k} - \beta_{i,k+1}) \qquad i=1,\ldots,n-1 \tag{51}$$

Summarizing, the first two rows of table (48) is formed by writing the coefficients of $T_n(z)$ and $T_{n-1}(z)$ in descending powers of z^k and the table is completed by the rules (50) and (51). The scheme (51) can alternatively be replaced by a determinant rule of the type familiar from the Routh table,

$$\beta_{i+1,k}= -\frac{1}{\beta_{i,0}} \begin{vmatrix} \beta_{i-1,0} & \beta_{i-1,k+1} \\ \beta_{i,0} & \beta_{i,k+1}-\beta_{i,k} \end{vmatrix} \tag{52}$$

This form is also schematically indicated by the arrows in (48).

<u>The conditions of stability in the table form</u>: The polynomial $D_n(z)$ is stable if and only if (regardless of the parity of n), the first entries of all the rows are positive,

$$\beta_{i,o}>0 \qquad i=0,\ldots,n \tag{53}$$

and the next summations for rows 1,3,...

$$\sigma_0=\beta_{00}-\beta_{01}+\beta_{02}-\beta_{03}+\beta_{04}-\beta_{05}+\cdots\pm\beta_{0,n}$$
$$\sigma_2=\beta_{20}-\beta_{21}+\beta_{22}-\beta_{23}+\cdots\pm\beta_{0,n-2} \tag{54a}$$
$$\vdots$$
$$\sigma_{2m}=\beta_{2m,0}(-\beta_{2m,1})$$

are all positive

$$\sigma_{2i}>0 \qquad i=0,\ldots,m \tag{54b}$$

To verify condition (53), note that the necessary condition for stability of a polynomial $D_n(z)$ given by (1), $d_n>|d_o|$ is equivalent to $\beta_{oo}>0$. Therefore the condition (44) that $\tilde{\delta}_i=\delta_{n-i+1}>0$ for all $i=1,\ldots,n$, is equivalent via (50) and $\beta_{oo}>0$ to (53). In the second condition (54), σ_{2i} are equal to $\tilde{T}_{n-2i}(-1)$ where $\tilde{T}_k(z)$ is the reciprocated polynomial of $T_k(z)$. Therefore (54b) presents the conditions

$$\sigma_{2i}=\tilde{T}_{n-2i}(-1)=(-1)^n T_{n-2i}(-1)>0 \qquad i=0,\ldots,m$$

which becomes (45a) when n=2m+1 and (45b) when n=2m.

Remark 4.3: The entries of any row can be multiplied by a common positive real number without affecting the stability test. The property may be convenient for hand computation.

Remark 4.4: The scheme presented by (50) and (51) involves less arithmetics then (56) (even after the unnecessary division in (52) by $\beta_{i,0}>0$ is deleted).

The computational saving admitted by the symmetry and asymmetry of the polynomials $T_{2i}(z)$ and $T_{2i+1}(z)$, $i=0,\ldots,m$, has already been mentioned in remark 4.1. Its impact on the table form is as follows. The odd number rows 1,3,5... of the table have the symmetry of the first row (are all asymmetric for $n=2m+1$ and symmetric for $n=2m$). The even number rows 2,4,6,... have the opposite symmetry of the second row (are all symmetric for $n=2m+1$ and asymmetric for $n=2m$). The right half entries of a symmetric or asymmetric row are, respectively, 'mirror' and 'anti-(minus sign) mirror' reflections of the left half entries. It is sufficient therefore to calculate only the left half of the table. (We put in (48) and in the examples below in brackets the entries of the right half of table to remind their redundancy). The number of entries to be calculated is then equal to the number of entries involved in a Routh table for a polynomial of the same degree. Using the scheme (50) and (51) for its construction, the table requires n division operations for $\tilde{\delta}_i$ and one additional multiplication per each entry in (the left halves of) rows, $3,4,\ldots,n+1$. The number of elementary multiplicative operations is equal to the demands of the Routh table (the additive arithmetic is however higher - one extra substraction per each term in the left half of the table). Once familiarity with the new table is gained, the right half of the table can be totally dropped and only the left half, with size and pattern of the Routh table, be handled.

By comparison with the Jury-Marden table, it is obvious from the above account that the new table involves half the number of entries (one fourth of the size, if its left half, is only considered) and consequently half of the multiplicative operations of the former table.

To complete the comparison with the Routh table and criterion, the following additional similarities and differences are summarized.

(1) The condition of positivity of all the entries of the first column appears also in the Routh array but $\sigma_{2i}>0$ $i=0,\dots,m$ is a second and extra required condition.

(2) The determinant rule of (52) is slightly different. In (52) and also in (51) the difference term $\beta_{i,k+1}-\beta_{i,k}$ replaces the would be only a $\beta_{i,k+1}$ term in the Routh scheme.

(3) The first two rows are constructed from the symmetric and asymmetric parts of $D_n(z)$. The symmetric and asymmetric parts of $D_n(z)$ play the respective roles of the even and odd parts of the Hurwitz polynomials that form the first and second rows of the Routh table. More explicitely it can be memorized that $D_n(z)+\tilde{D}_n(z)$ and $[D_n(z)-\tilde{D}_n(z)]/(z+1)$ stay for $H_n(s)+H_n(-s)$ and $[H_n(s)-H_n(-s)]1/s$ expressed as functions of $x=s^2$ in the Routh table.

<u>Example 1</u>: Consider the polynomial

$$D_5(z)=1.5z^5+13.5z^4+28.5z^3-3.5z^2-4.5z-0.5$$

We have by (40a) and (41a)

$$T_5(z)=D_5(z)-\tilde{D}_5(z)=2z^5+18z^4+32z^3-32z^2-18z-2$$

$$T_4(z)=[D_5(z)-\tilde{D}_5(z)]/(z+1)=(z^5+9z^4+25z^3+25z^2+9z+1)/(z+1)$$
$$=z^4+8z^3+17z^2+8z+1$$

The table of (48) is constructed from $T_5(z)$ and $T_4(z)$ using (52) or (50) and (51)

$\sigma_0 = 32$	2	18	32	(-32)	(-18)	(-2)
	1	8	17	(8)	(1)	
$\sigma_2 = -20$	4	14	(-14)	(-4)		
	5.5	2.4	(-5.5)			
$\sigma_4 = 12/11$	6/11	(-6/11)				
	35					

In the construction of the table the right halves of all rows were completed by alternating 'mirror' and 'anti-mirror' reflection pattern that the table always satisfies. The first condition (53) that the first entries of all rows are positive is satisfied for this example but the second condition (54), that $\sigma_{2i}>0\ \forall i$ is violated by $\sigma_2<0$. Therefore not all the zeros of $D_5(z)$ are inside the unit circle. In fact the zeros of $D_5(z)$ are given by 0.454, -0.256, -0.146, -5.460, -3.591. The condition $\beta_{i,0}>0$ i=1,...,n and $\tilde{\delta}_i>0$, i=1,...,n are clearly equivalent. This example therefore illustrates also the fact that $\tilde{\delta}_i>0$ in the expansions (8) of theorem 3 are not sufficient for stability.

<u>Example 2</u>: Consider the polynomial

$$D_4(z)=z^4-1.368z^3+0.4126z^2+0.08z+0.00025$$

Using (40b) and (41b)

$$T_4(z)=D_4(z)+\tilde{D}_4(z)=1.0025z^4-1.288z^3+0.8252z^2-1.288z+1.0025$$

$$T_3(z)=[D_4(z)-\tilde{D}_4(z)]/(z+1)=0.9975z^3-2.4455z^2+7.4455z-0.9975$$

The table is

$\sigma_0 = 5.4062$	1.0025	-1.288	0.8252	(-1.288)	(1.0025)
	0.9975	-2.4455	(2.4455)	(-0.9975)	
$\sigma_2 = 8.4348$	2.1723	-4.0903	(2.1723)		
	0.4303	(-0.4303)			
$\sigma_4 = 0.2542$	0.2542				

Conditions (53) and (54) are both satisfied and therefore the polynomial is stable.

4. Conclusions

The paper has presented new necessary and sufficient conditions for a polynomial to have all its zeros inside the unit circle. A new stability test for characteristic polynomial of discrete systems is presented in a continued fraction and in table form. The established relations are very similar to well known analogous relations between the Routh table and the continued fraction expansions of Hurwitz polynomials. The new stability test requires half the amount of computations of the former Jury-Marden table and its similarity to the Routh table make it easy to remember. The new table can be extended also to obtain the number of zeros inside and outside the unit circle [7].

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