Transactions Briefs

Z-Domain Continued Fraction Expansions for Stable Discrete Systems Polynomials

YUVAL BISTRITZ

Abstract —**A z-plane continued fraction expansion (CFE) that is related to the first Cauer s-plane CFE via Bruton's LDI transformation is considered. Necessary and sufficient conditions are imposed on the CFE for a polynomial to be stable (have all its zeros inside the z-plane unit circle). The implementation of this CFE in a tabular form establishes the Routh-like stability table in [1] first derived in a conference paper [2]. The application of this stability table is now extended to also count zeros outside the unit circle, making it compatible in this respect with the related second table form in [3]. However, the closer analogy of the present formulation to the s-plane Cauer CFE's and Routh table suggest additional merits of this formulation to the design of digital networks (e.g., switched-capacitor filters). A brief account of three related alternative CFE's is included.**

I. Introduction

The Routh method for testing the characteristic polynomial of a continuous-time system for stability may be looked upon as an application of Euclid's algorithm to the even and odd parts of the polynomial. This process is also recognized as the expansion of the rational function formed by the odd over even parts (the "tangent function") into (Cauer) simple continued fractions. The resulting continued fraction expansion (CFE), interpreted properly, yields also a number of important procedures of two-element-kind ladder networks [4].

The purpose of this paper is to develop a theory for discrete systems and digital networks that provides the basis for a z-domain CFE, a discrete stability table and two-element-kind ladder structures such that each in separate possess a meaningful analogy to its s-plane correspondent, and such that the inter-relation between the three objects retain the notion that exists among the corresponding components in the s-plane. This development has, beside its theoretical interest, some practical merits. It yields a Routh-like computationally less expensive, stability table [1]. Another rewarding outcome stems from the possibility to simulate the LC ladder structure and obtain corresponding low sensitivity digital ladders [5]. We regard Cauer CFE as the link between the Routh table and the LC ladder forms. Therefore, we presently concentrate on devising appropriate z-domain CFE's that are closely related to the Cauer CFE's.

The content of the paper is as follows; A brief review, in Section II, of some s-plane and z-plane stability conditions as required for later reference is followed by the introduction in Section III of CFE for the discrete tangent function of polynomial that proceeds in forward and backward difference terms, $(z-1)$ and $(1-z^{-1})$. It is obtained by applying the LDI transformation [6] to the first Cauer CFE. The transformation yields at once necessary, but not sufficient conditions for stability. A subsequent study in Section IV of the CFE yields complementary conditions to imply also stability. The necessary and sufficient conditions for stability, as imposed on the CFE, when arranged in a tabular array yield the stability table [1] (not repeated here). The stability conditions here are close, but different from, the table in [3]. In Section V we show how the number of zeros of a polynomial outside the unit circle is also obtainable by the current formulation. It is pointed out that the stability table of [1] is compatible with the table in [3] for the zero location problem but that it has a slightly less desirable appearance. It is indicated, however, that this is made up for by its more direct relevance to the design of digital networks that follow s-plane prototypes [5] (e.g., switched-capacitor filters [6], [14]). The last section contains a brief presentation of three more z-domain CFE's. One which corresponds to the table in [3] and two CFE's of the second Cauer form types.

Manuscript received September 6, 1983; revised April 29, 1985.
The author was with the Department of Electrical Engineering, Tel-Aviv University, Israel. He is now with the Department of Electrical Engineering, Stanford University, Stanford, CA 94305-2099.

II. Z-Plane versus S-Plane Stability

A stability test for discrete systems deals with the location of the zeros of a real polynomial

$$D_n(z)=d_0+d_1z+\cdots+d_nz^n,\qquad d_n>0 \tag{1}$$

with respect to the unit circle C ($|z|=1$). A polynomial is called stable if it has all its zeros inside the unit circle (IUC). More generally, the polynomial may also have zeros on the unit circle (UC) or outside the unit circle (OUC). A stability test involves necessary and sufficient conditions for a polynomial to be stable. The zero location problem generalizes the problem by questioning the number of IUC, UC, and OUC zeros of a polynomial. The corresponding problems for continuous-time systems deals with the distribution of the zeros of a real polynomial, say $H_n(s)$ between the left half and the right half of the complex s-plane. The polynomial $H_n(s)$ is called Hurwitz if all its zeros lie in the open left half plane. Necessary and sufficient condition for a polynomial to be Hurwitz or more generally its zero distribution can be obtained by the Routh table (cf. [9]). Equivalent conditions that are required for later reference are summarized by:

Lemma 1: the real polynomial $H_n(s)$ is Hurwitz if and only if the following equivalent conditions hold:

(i) The s-plane tangent function $\rho_n(s)$, defined for $H_n(s)$ by

$$\rho_n(s)=\frac{H_n(s)-H_n(-s)}{H_n(s)+H_n(-s)} \tag{2}$$

can be written in the form

$$\rho_n(s)=\frac{Ks\prod_{i=1}^{l}\left(s^2+\omega_{2i}^2\right)}{\prod_{i=1}^{m}\left(s^2+\omega_{2i-1}^2\right)},\qquad K>0,\ 0<\omega_1^2<\omega_2^2<\cdots<\omega_n^2 \tag{3}$$

where $n=m+l+1$; $l=m-1$ or $l=m$.

(ii) The expansion of $\rho_n(s)$ in the following first Cauer CFE (about $s=\infty$) exists

$$\rho_{2m}(s) \quad \text{or} \quad \rho_{2m+1}^{-1}(s) = \frac{1}{\gamma_n s} + \frac{1}{\gamma_{n-1} s} + \cdots + \frac{1}{\gamma_1 s} \tag{4}$$

and the coefficients γ_i are positive for all $i=1,\cdots,n$.

The sequence of polynomial in the above Cauer CFE form the rows in the Routh table [8] and regarding $\rho_n(s)$ as the driving function, the above are conditions for its realization by an LC ladder [4].

A discrete tangent function $\rho_n(z)$ can be defined by

$$\rho_n(z) = \frac{D_n(z) - D_n^*(z)}{D_n(z) + D_n^*(z)} \tag{5}$$

where D_n^* is the reciprocal polynomial $z^n D_n(z^{-1})$. A stable polynomial is featured by a $\rho_n(z)$ with interlacing poles and zeros on C (comparable with condition (i) in Lemma 1).

Lemma 2: A polynomial $D_n(z)$ with $d_n > |d_0|$ is stable if and only if $\rho_n(z)$ can be written for $n=2m+1$ and $n=2m$, respectively, in the form

$$\rho_{2m+1}(z) = \frac{K(z-1)\prod_{i=1}^{m}(z^2 - 2z\cos\Omega_{2i}+1)}{(z+1)\prod_{i=1}^{m}(z^2-2z\cos\Omega_{2i-1}+1)}, \quad K>0 \tag{6a}$$

$$\rho_{2m}(z) = \frac{K(z-1)(z+1)\prod_{i=1}^{m-1}(z^2 - 2z\cos\Omega_{2i}+1)}{\prod_{i=1}^{m}(z^2-2z\cos\Omega_{2i-1}+1)}, \quad K>0 \tag{6b}$$

where

$$-1 < \cos\Omega_{n-1} < \cos\Omega_{n-2} < \cdots < \cos\Omega_2 < \cos\Omega_1 < 1. \tag{6c}$$

This lemma was first shown in [9]. In its form above it was obtained in [10] by applying the bilinear transformation $s=(z-1)/(z+1)$ to (3) in Lemma 1.

Application of the bilinear transformation to condition (ii) of Lemma 1 yields a bilinear CFE for $\rho_n(z)$ [11], [12]. The family of all bilinear CFE, as derived in [12], admits some quite effective schemes to test stability and determine zeros distribution. However, CFE's that proceed in terms of $(z-1)/(z+1)$ or $(z+1)/(z-1)$ do not make an appropriate analogy to the meaning of s and s^{-1} in analog circuits and continuous systems [13], [7].

III. A New CFE for Stable Polynomials

Consider the following Bruton's LDI (lossless digital integrator) [6], [7]:

$$s = \tfrac{1}{2}(z^{1/2} - z^{-1/2}). \tag{7}$$

It represents a conformal transformation that, letting $z=re^{j\Omega}$, can be shown to map co centric circles of radii r of the z-plane into ellipses with focuses at $\pm j$ and axes of lengths

$$\tfrac{1}{2}(r^{1/2}+r^{-1/2}) \quad \text{and} \quad \tfrac{1}{2}(r^{1/2}-r^{-1/2}).$$

Radial lines are mapped into confocal hyperboles orthogonal to the ellipses. A graphical illustration is shown in Fig. 1. The unit circle $C=\{z|z=e^{j\Omega},\ \Omega\in(-\pi,\pi)\}$ is mapped by (7) into the interval of the imaginary axis $J=\{s|s=j\omega,\ \omega\in(-1,1)\}$ via the

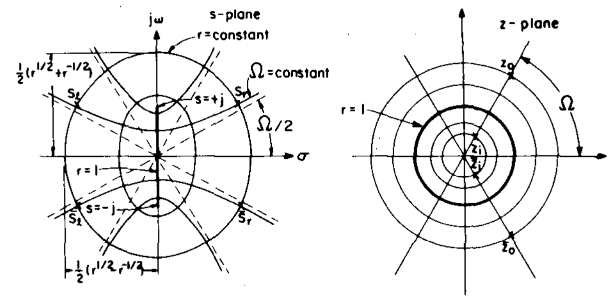

Fig. 1. The LDI transformation (lossless digital integrator).

function

$$\omega = \sin(\Omega/2). \tag{8}$$

This function is one to one and increasing and maps $\Omega\in(-\pi,\pi)$ onto $\omega\in(-1,1)$. Unfortunately, except to this sub-mapping of C to J, the LDI transformation is not one to one nowhere else. The interior and the exterior of C are each mapped into both the left half and right half of the s-plane. Referring for illustration to Fig. 1; the point z_i is mapped into both s_l and $\bar{s}_r$ $(=-s_l)$; the point $\bar{z}_i$ into $\bar{s}_l$ and s_r; z_0 $(=z_i^{-1})$ into s_l and s_r; and $\bar{z}_0$ into s_l and $\bar{s}_r$. Thus the LDI transformation, unlike the bilinear one, is not a stability preserving mapping. This discouraging fact has often been the reason for preferring synthesis procedures based on the bilinear transformation [6], [14]. We shall subsequently apply the LDI transformation to the Cauer CFE (4). In view of its above properties, we shall rest in our manipulations only on the mapping between the J interval and C, not expecting a too obvious necessary and sufficient conditions simply by setting (7) into (4). Indeed, while a set of necessary conditions for stability (from which the comparable conditions in [7] can be deduced) will be obtained with relative ease, the derivation of complementary sufficient conditions will turn out to be quite cumbersome.

Assume a stable $D_n(z)$ and examine the mapping of a typical product in (3) and (6) by (8) $z^{-1}(z^2-2z\cos\Omega_k+1) \leftrightarrow 4[s^2+\sin^2(\Omega_k/2)]$. Therefore, we have

$$z^{m-1}\frac{\prod_{i=1}^{l}(z^2-2z\cos\Omega_{2i}+1)}{(z+1)\prod_{i=1}^{m}(z^2-2z\cos\Omega_{2i-1}+1)} \leftrightarrow \frac{\prod_{i=1}^{l}(s^2+\omega_{2i}^2)}{\prod_{i=1}^{m}(s^2+\omega_{2i-1}^2)} \tag{9}$$

with $l=m-1$ or $l=m$, where by (6c) and (8)

$$0<\omega_1^2<\cdots<\omega_{n-1}^2<1, \qquad \omega_k = \sin(\Omega_k/2). \tag{10}$$

Consequently, we have for $n=2m+1$ (we skip an obvious parallel derivation for $n=2m$), from (6a),

$$z^{-1/2}(z+1)\rho_{2m+1}(z) \leftrightarrow \frac{Ks\prod_{i=1}^{m}(s^2+\omega_{2i}^2)}{\prod_{i=1}^{m}(s^2+\omega_{2i-1}^2)}. \tag{11}$$

By the equivalence of the two conditions in Lemma 1, the right-hand side of (11) is the tangent function of a Hurwitz polynomial and, therefore, admits the CFE (4). Thus setting (7) into (4), we have from (11) for $n=2m+1$

$$z^{-1/2}(z+1)\rho_{2m+1}(z)$$
$$= \gamma_{2m+1}\frac{1}{2}(z^{1/2}-z^{-1/2}) + \frac{1}{\gamma_{2m}\frac{1}{2}(z^{1/2}-z^{-1/2})} + \cdots + \frac{1}{\gamma_1\frac{1}{2}(z^{1/2}-z^{-1/2})}.$$

Multiplying the two sides of the last expression by $z^{-1/2}$, repeating the last steps also for $n=2m$ and defining $\delta_i := \frac{1}{2}\gamma_i > 0$, $i=1,\cdots,n$, the next theorem has been proven.

Theorem 1: If the polynomial $D_n(z)$ is stable then its tangent function has the following CFE's for $n=2m+1$ and $2m$, respectively,

$$(z+1)^{-1}\rho_{2m+1}^{-1}(z) = \frac{1}{\delta_{2m+1}(z-1)} + \frac{1}{\delta_{2m}(1-z^{-1})} + \cdots + \frac{1}{\delta_1(z-1)} \tag{12a}$$

$$(z+1)^{-1}\rho_{2m}(z) = \frac{1}{\delta_{2m}(z-1)} + \frac{1}{\delta_{2m-1}(1-z^{-1})} + \cdots + \frac{1}{\delta_1(1-z^{-1})} \tag{12b}$$

and the coefficients δ_i are all positive, $\delta_i > 0$, $i=1,\cdots n$.

Remark 1: The positivity of all δ_i is not sufficient for stability. A counter example is provided by the numerical example in [1] (we shall soon relate the CFE of the last theorem to the stability table in [1]). This difference from the situation in applying the bilinear transformation to $\rho(s)$ stems from the fact that the LDI transformation does not preserve stability.

Remark 2: A CFE closely related to (12) and necessary conditions for stability constitutes the principal theorem in [7]. The relation between the rational function, say $R_n(z)$, expanded there into CFE of the form (12) with coefficients r_i and $\rho_n(z)$ here is

$$R_{2m+1}(z) = \tfrac{1}{2}(z-1) + \tfrac{1}{2}(z+1)\rho_{2m+1}(z)$$
$$R_{2m}(z) = \tfrac{1}{2}(z-1) + \tfrac{1}{2}(z+1)\rho_{2m}^{-1}(z).$$

The coefficients r_i and δ_i are related, therefore, by $r_n = \frac{1}{2}(\delta_n + 1)$, $r_{n-1} = 2\delta_{n-1}$, $r_{n-2} = \frac{1}{2}\delta_{n-2},\cdots$,etc. Thus $\delta_i > 0$, $i=1,\cdots,n$ imply $r_i > 0$, $i=1,\cdots,n$ but not vice versa. The former remark and the forthcoming results also provide explanation why the argument in [7] may not be reversed and $r_i > 0$, $i=1,\cdots,n$ are necessary but not also sufficient for stability.

IV. Sufficient Conditions for Stability

We next proceed to derive additional conditions together with which the existence of (12) and $\delta_i > 0$ also imply stability. A real polynomial can always be written as the sum of a symmetric and an antisymmetric polynomial

$$D_n(z) = \tfrac{1}{2}[D_n(z) + D_n^*(z)] + \tfrac{1}{2}[D_n(z) - D_n^*(z)] = \tfrac{1}{2}S_n(z) + \tfrac{1}{2}A_n(z) \tag{13}$$

where $S_n(z)$ and $A_n(z)$ are called symmetric and antisymmetric polynomials, if $S_n^*(z) = S_n(z)$ and $A_n^*(z) = -A_n(z)$, respectively. A polynomial of odd degree can always be written in the form

$$D_{2m+1}(z) = \tfrac{1}{2}A_{2m+1}(z) + \tfrac{1}{2}(z+1)S_{2m}(z). \tag{14a}$$

Similarly, a polynomial of even degree can always be written in the form

$$D_{2m}(z) = \tfrac{1}{2}S_{2m}(z) + \tfrac{1}{2}(z+1)A_{2m-1}(z). \tag{14b}$$

Consider next a sequence of $n+1$ polynomials $\{D_k(z)\}_{k=0}^n$ defined for a given set of n positive numbers $\delta_1,\cdots,\delta_n$ by the following assignment; $D_k(z)$ is the polynomial of degree k whose tangent function $\rho_k(z)$ has a CFE of the form (12) with $\{\delta_1,\cdots,\delta_k\}$. In other words, $D_{2i+1}(z)$ is defined via (14a) in association with $\rho_{2i+1}(z)$ and $D_{2i}(z)$ via (14b) in association with $\rho_{2i}(z)$, respectively, by

$$(z+1)\rho_{2i+1}(z) = \frac{A_{2i+1}(z)}{S_{2i}(z)} \tag{15a}$$

$$(z+1)^{-1}\rho_{2i}(z) = \frac{A_{2i-1}(z)}{S_{2i}(z)}. \tag{15b}$$

The nested structure of these partial CFE's implies the relations

$$(z+1)\rho_{2i+1}(z) = \delta_{2i+1}(z-1) + z\rho_{2i}(z)/(z+1) \tag{16a}$$

$$(z+1)\rho_{2i}^{-1}(z) = \delta_{2i}(z-1) + z\rho_{2i-1}^{-1}(z)/(z+1). \tag{16b}$$

Using (15), we find that $S_{2i}(z)$ is common for $D_{2i}(z)$ and $D_{2i+1(z)}$ and $A_{2i-1}(z)$ is common for $D_{2i-1}(z)$ and $D_{2i}(z)$. Therefore, the sequence $\{D_k(z)\}_{k=0}^n$ is completely defined by another sequence of exactly $n+1$ polynomials, which are symmetric (for even degrees) or antisymmetric (for odd degrees). The polynomials in this sequence, that we shall denote by $\{T_k(z)\}_{k=0}^n$, are defined for $\{D_k(z)\}$ by

$$T_k(z) = D_k(z) + (-1)^k D_k^*(z), \qquad k=1,\cdots,n \tag{17}$$

and $\{T_k(z)\}$ reciprocally defines $\{D_k(z)\}$ by

$$D_k(z) = \tfrac{1}{2}T_k(z) + \tfrac{1}{2}(z+1)T_{k-1}(z), \qquad k=1,\cdots,n. \tag{18}$$

The new sequence $\{T_k(z)\}_{k=0}^n$ can be generated, by (14)–(16), from $\{\delta_1,\cdots,\delta_n\}$ by the recursion

$$T_{k+1}(z) = \delta_{k+1}(z-1)T_k(z) + zT_{k-1}(z), \qquad k=0,\cdots,n. \tag{19}$$

Starting with $T_{-1}(z)=0$ and $T_0(z)=1$. Note that, for given $\{\delta_1,\cdots,\delta_n\}$, (19) together with (18) show an alternative constructive definition for the sequence $\{D_k(z)\}$.

The next lemma is a key result for complementing the positivity of $\{\delta_i\}$ into also sufficient conditions for stability of $D_n(z)$

Lemma 3: Assume $\{\delta_1,\cdots,\delta_n\}$ are n positive numbers and let $\{D_k(z)\}_{k=0}^n$ be the sequence of polynomials constructed through (19), (18).

(i) If $D_{2i-1}(z)$ is stable and $D_{2i+1}(-1) < 0$ then $D_{2i+1}(z)$ is stable, $i=1,2,\cdots$; $2i+1<n$.

(ii) $D_{2i-2}(z)$ is stable and $D_{2i}(-1) > 0$ then $D_{2i}(z)$ is stable, $i=1,2,\cdots$; $2i<n$.

Proof: The proof uses the necessary and sufficient conditions in Lemma 2, condition (i) of Lemma 1, and the way the LDI transformation maps J onto C. Let $\omega_i(k)$ denote the ω_i values for a $\rho_k(s)$ expressed as in (3). Assume $\delta_1,\cdots,\delta_k > 0$ and consider the function $\rho_k(s)$ defined for $\gamma_i = 2\delta_i > 0$, $i=1,\cdots,k$ by (4). This $\rho_k(s)$ corresponds to a Hurwitz polynomial and, therefore, has a structure (3) with

$$0 < \omega_1^2(k) < \omega_2^2(k) < \cdots < \omega_n^2(k). \tag{20}$$

Define for $\omega_i(k)$ the real numbers

$$x_i(k) = 2\omega_i^2(k) - 1 \tag{21}$$

then, repeating the steps preceding Theorem 1, the LDI transformation determines a $\rho_k(z)$ given by the form (6) with $x_i(k) = \cos\Omega_k$ and we have by the monotonicity of the mapping (8),

$$-1 < x_1(k) < \cdots < x_{k-1}(k). \tag{22}$$

To prove that $D_k(z)$ is stable, by the sufficiency part of Lemma 2, the remaining crucial point is to show for (22) also that

$x_{k-1}(k)<1$. This part of the proof follows from the given stability of $D_{k-2}(z)$ and $(-1)^k D_k(-1)>0$, by incorporating similar to the above manipulations also of the sequences $\{x_i(k-2)\}$ and $\{x_i(k-1)\}$, that correspond to $\rho_{k-2}(z)$ and $\rho_{k-1}(z)$, respectively, (see [2] for more details).

The necessary and sufficient conditions for stability to complement Theorem 1 is now stated.

Theorem 2: The polynomial $D_n(z)$ is stable if and only if the CFE (12) exists and has positive coefficients $\delta_1,\cdots,\delta_n>0$ and the sequence associated with it $\{D_k(z)\}_{k=0}^n$, or equivalently $\{T_k(z)\}_{k=0}^n$, satisfies:

(i) for $n=2m+1$

$$D_{2i+1}(-1)<0 \text{ or } T_{2i+1}(-1)<0, \qquad i=1,\cdots,m \tag{23a}$$

(ii) for $n=2m$

$$D_{2i}(-1)>0 \text{ or } T_{2i}(-1)>0, \qquad i=1,\cdots,m. \tag{23b}$$

Proof: The conditions on $\{D_k(z)\}_{k=0}^n$ follow immediately by repetitive application of Lemma 3. The equivalent conditions on $\{T_k(z)\}_{k=0}^n$ follow from (18).

The sequence $\{T_k(z)\}_{k=0}^n$ can be obtained for $D_n(z)$ without actual reference to the sequence $\{D_k(z)\}_{k=0}^n$ or to the CFE (12). Given $D_n(z)$,

(i) $T_n(z)$ and $T_{n-1}(z)$ are formed by

$$T_n(z)=D_n(z)+(-1)^n D_n^*(z)$$
$$T_{n-1}(z)=\left[D_n(z)-(-1)^n D_n^*(z)\right]/(z+1). \tag{24}$$

(ii) The rest of the sequence $T_{n-2}(z)$, $T_{n-3}(z),\cdots$ are constructed by successive use of the following recursion, that can be deduced from (19), for $i=n,n-1,\cdots,1$

$$\delta_i=-\frac{T_i(0)}{T_{i-1}(0)}$$
$$T_{i-2}(z)=z^{-1}\left[T_i(z)-\delta_i(z-1)T_{i-1}(z)\right]. \tag{25}$$

The algorithm (24), (25) is best carried out in the table form of [1]. Theorem 2 provides a proof for the stability conditions there. The rows in [1] are the coefficients of $\{T_k(z)\}_{k=0}^n$ in descending power of z order (first row corresponding to $T_n(z)$ till last row corresponding to $T_0(z)$). The row sums σ_i defined for the table in [1] are related to $\{T_k(z)\}_{k=0}^n$ by $\sigma_i=\hat{\sigma}_{n-i}$, $i=0,\cdots,n$ where

$$\hat{\sigma}_i := T_i^*(-1)=(-1)^i T_i(-1). \tag{26}$$

Reference [1] contains an efficient special algorithm for the table form and discusses computational aspects of the stability test. An important point is that the symmetry properties of the polynomials $T_k(z)$ admit the actual computation of only half of the entries of the table and implies a significant saving in computation (similar to [3]). The above algorithm and the table hold a remarkable similarity to the Routh table (better than in [3]) that correspond to the first Cauer CFE (4). The first and second rows in a Routh table for a polynomial $H_n(s)$ are formed by the coefficients of $[H_n(s)+(-1)^n H_n(-s)]$ and $[H_n(s)-(-1)^n H_n(-s)]/s$, respectively, written in descending order of $x=s^2$ with self evident analogy to (24). The construction of the rest of the rows also reveals a Routh-like pattern [1].

V. Extension to Unstable Polynomials

The algorithm here and in [1] are closely related and compatible in computation with the stability table in [3]. We next briefly show how the stability table in [1] is equally applicable also to count OUC (outside the unit circle) zeros. Thereafter, the need for the current alternative formulation will be justified by its possession of additional important features for many filter design techniques related to analog filters or using switched capacitors [5]. First we show that Theorem 2 can be restated as follows.

Theorem 3: The polynomial $D_n(z)$ is stable if and only if $\hat{\sigma}_i=(-1)^i T_i(-1)$ have the same sign for all $i=0,\cdots,n$, namely,

$$\operatorname{var}\{\hat{\sigma}_n,\cdots,\hat{\sigma}_0\}=0 \tag{27}$$

where Var denotes the number of sign variations.

Proof: We only need to show that any two of the following three conditions implies the third:

(i) $$\delta_1,\cdots,\delta_n>0$$

(ii) $$(-1)^{2i+1}T_{2i+1}(-1)>0 \tag{23a}$$

(iii) $$(-1)^{2i}T_{2i}(-1)>0. \tag{23b}$$

This follows immediately from

$$\delta_i=(\hat{\sigma}_{i-2}+\hat{\sigma}_i)/2\hat{\sigma}_{i-1} \tag{28}$$

a relation that is verified by setting $z=-1$ into (19).

We say that the polynomial $D_n(z)$ obeys normal conditions if the recursion (24), (25) can be completed. (It is singular if a $T_i(0)=0$ interrupts the construction). Normal conditions are also equivalent to saying that the CFE (12) exists or that the stability table, as presented in [1], can be constructed.

Theorem 4: Assume that $D_n(z)$ obeys normal conditions and let

$$\operatorname{var}\{\hat{\sigma}_n,\cdots,\hat{\sigma}_0\}=\nu \tag{29}$$

then $D_n(z)$ has $n-\nu$ IUC, no UC and ν OUC zeros.

A proof for this extension of Theorem 3 will not be given. It requires tools beyond the CFE context of this paper. It can be shown, as in [3], by Rouché's theorem or by a Cauchy index theorem applied to the imaginary J interval [16].

The singular cases can be classified, interpreted and handled, again, in ways similar to [3]. These details, that can be deduced from [3], are omitted. They are of limited importance for network realization (the simple CFE breaks down in singular cases; singular conditions may occur only for unstable polynomials) and the table in [3] has a more convenient form for merely the zero location problem (all-symmetric rows; not so many minus signs to memorize; formation of initial two rows not depending on parity of n).

Our purpose here was to emphasize that the current formulation with the table [1], in spite of a slightly less desirable appearance, is compatible with [3] in computation and broadness to deal with the general zero location problem. The differences between the two formulations reflects an effort to make the current one as similar as possible to the classical continuous-time systems stability conditions, Cauer CFE's, and LC ladders. As a result the current formulation is believed to be of more interest for the design of digital networks in relation with s-plane prototypes [5].

VI. Some Related CFE's

We conclude the paper by briefly presenting some additional CFE's, other than the principal CFE (12). They may too find applications in the design of digital networks.

It is well known that any CFE can be presented in more than one form. The CFE in (12) and the three presented in the next theorem are four different CFE's in the meaningful sense that they can not be obtained one from the other by CFE equivalence transformations [14]. Only necessary conditions for stability (as in [7] or here in Theorem 1) are stated.

Theorem 5: If the polynomial $D_n(z)$ of (1) is stable then its tangent function ρ_n has the following CFE's:

(i) *2nd form* $(z+1)\rho_{2m+1}(z)$ or $z(z+1)^{-1}\rho_{2m}(z)$ can be expanded into

$$\frac{1}{\delta_n'/(z-1)} + \frac{1}{\delta_{n-1}'/(1-z^{-1})} + \cdots + \left(\frac{1}{\delta_1'/(z-1)}\right)\Bigg|_{\text{if } n=2m+1} + \left(\frac{1}{\delta_1'/(1-z^{-1})}\right)\Bigg|_{\text{if } n=2m} \qquad (30)$$

and δ_i', $i=1,\cdots,n$ are all positive.

(ii) *3rd form* $(z-1)^{-1}\rho_n(z)$ can be expanded into

$$\delta_n(z+1) - \frac{1}{\delta_{n-1}(z+1)} - \cdots - \left(\frac{1}{\delta_1(z+1)}\right)\Bigg|_{\text{if } n=2m+1} - \left(\frac{1}{\delta_1(1+z^{-1})}\right)\Bigg|_{\text{if } n=2m} \qquad (31)$$

and δ_i $i=1,\cdots,n$ are all positive.

(iii) *4th form* $(z-1)\rho_{2m+1}^{-1}(z)$ or $z(z-1)^{-1}\rho_{2m}(z)$ can be expanded into

$$\frac{1}{\delta_n'/(z+1)} - \frac{1}{\delta_{n-1}'/(1+z^{-1})} - \cdots - \left(\frac{1}{\delta_1'/(z+1)}\right)\Bigg|_{\text{if } n=2m+1} - \left(\frac{1}{\delta_1'/(1+z^{-1})}\right)\Bigg|_{\text{if } n=2m} \qquad (32)$$

and δ_i $i=1,\cdots,n$ are all positive.

Proof: The second form can be verified by replacing in the proof of Theorem 1 the s-plane CFE (4) by the second Cauer CFE form (an expansion of $\rho_n(s)$ about infinity) [4]. The third form is easily recognized as the CFE in association with the table in [3]. The fourth CFE is a "second Cauer form" for the third form.

Remark 3: The third CFE can also be obtained from the first form (12) (and the fourth from the second) by a $-\pi/2$ rotation of the z-plane coordinates. Really, the stability table in [3] is related to a sequence of polynomials defined on the real interval $[-1,1]$ in much the same way that the presentation here was derived from its relation to the s-plane imaginary interval J. This is yet another way to perceive the special appropriateness of the first CFE and the table in [1] for s-plane-related digital filter designs.

References

[1] Y. Bistritz, "A stability new test for linear discrete systems in a table form," *IEEE Trans. Circuits Syst.*, vol. CAS-30, pp. 917–919, 1983.
[2] ____, "A new unit circle stability criterion," in *Proc. Symp. on Mathematical Theory of Networks and Systems*, Beer Sheva, Israel, pp. 69–87, June 1983.
[3] ____, "Zero location with respect to the unit circle of discrete-time linear system polynomials," *Proc. IEEE*, vol. 72, pp. 1131–1142, Sept. 1984.
[4] L. Weinberg, *Network Analysis and Synthesis.* New York: McGraw-Hill. 1962.
[5] Y. Bistritz, "Digital Cauer type ladders for stable filters, submitted to the IEEE Symp. on Circuits and Systems.
[6] L. T. Bruton, "Low sensitivity digital ladder filters," *IEEE Trans. Circuits Syst.*, vol. CAS-22, pp. 168–176, 1975.
[7] A. M. Davis, "A new Z domain continued fraction expansion," *IEEE Trans. Circuits Syst.*, vol. CAS-29, pp. 658–662, 1982.
[8] S. Barnett, *Polynomials and Linear Control Systems*, Marcel Dekker, 1983.
[9] H. W. Schussler, "A stability theorem for discrete systems," *IEEE Trans. Acoustics, Speech, Signal Processing*, vol. ASSP-24, pp. 87–98, 1976.
[10] Y. Bistritz, "A discrete stability equation theorem and method of stable model reduction," *Systems Control Lett.*, vol. 1, pp. 373–381, 1982.
[11] J. Szczupak, S. K. Mitra, and E. I. Jury, "Some new results on discrete systems stability," *IEEE Trans. Acoustics, Speech, Signal Processing*, vol. ASSP-25, pp. 101–107, 1977.
[12] Y. Bistritz, "Direct bilinear Routh stability criteria for discrete systems," *Systems Contr. Lett.*, vol. 4, pp. 265–271, 1984.
[13] A. Fettweis, "Digital filters structures related to classical filter networks, *Arch. Elec. Ubertrangung.*, vol. 25, pp. 79–89, 1971.
[14] R. Gregorian, K. W. Martin, and G. S. Temes, "Switched-capacitor circuit design," *Proc. IEEE*, vol. 71, pp. 941–966, 1983.
[15] H. S. Wall, *Analytical Theory of Continued Fractions.* New York: Chelsea, 1967.
[16] Y. Bistritz, "A Cauchy index approach for zero location of polynomials with respect to the unit circle," to be in the Proc. IEEE Conf. on Decision and Control, Fort Lauderdale, FL, Dec. 1985.

Analysis and Realization of Cascaded Transmission-Line Networks by the Transfer Scattering Matrix

TAKAO KOMURO AND MICHIO AKIYAMA

Abstract —**An approach to cascaded, uniform, lossless transmission-line networks by the transfer scattering matrix in the z-domain is presented. This approach can be applied to network problems treated in both the time and the frequency domains and is well suited for a computer programming. It is shown that the reflection function and the transfer function are obtained by simple matrix multiplications and conversely, the network is synthesized by the use of the transfer scattering matrix determined from the realizable function.**

I. Introduction

The analysis and realization of networks consisting of cascaded sections of uniform lossless transmission lines of equal electrical length have received much attention because such networks result from equivalent circuit representation in microwave network theory, optics, and acoustics. Various methods of the analysis and realization of these networks have been proposed in the frequency domain [1]–[5] and in the time domain [6]–[11].

The purpose of this paper is the presentation of an approach to these cascaded transmission-line networks by the transfer scattering matrix (**T**-matrix) in the z-domain. Since the **T**-matrix is based on the traveling wave theory, the approach can be applied to network problems treated in both the time and the frequency domains. The approach is suitable for processing systematically by a computer because the procedures for the analysis and realization can be performed by the matrix calculation.

The properties of the **T**-matrix of the transmission-line network is first given. Subsequently, it is shown that the reflection function (the reflected impulse response) and the transfer function (the transmitted impulse response) are obtained by simple matrix multiplications and conversely, the network is synthesized by the use of the **T**-matrix determined from the given realizable function. In the case of the transfer function of a symmetric network, two networks of which the reflection functions have opposite sign are synthesized with ease.

Manuscript received September 18, 1984.
T. Komuro is with the Department of Electronic Engineering, Ibaraki University, Hitachi, Ibaraki, 316 Japan.
M. Akiyama is with the Department of Information Science, Ibaraki University, Hitachi, Ibaraki, 316 Japan.

0098-4094/85/1100-1166$01.00 ©1985 IEEE