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A circular stability test for general polynomials *

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Abstract: We extend a new stability test proposed recently for discrete system polynomials [1] to polynomials with complex coefficients. The method is based on a three-term recursion of a conjugate symmetric sequence of polynomials. The complex version has the same relative improved efficiency as the real version in comparison to the classical Schur–Cohn formulation for counting the number of zeros of a polynomial with respect to the unit circle. Furthermore, if desirable, the complex text can be carried out using only real polynomials and arithmetic.

Keywords: Stability criterion for discrete-time systems, Zeros location in the unit circle, Complex symmetric polynomials, Three-term polynomial recursions.

1. Introduction

The problem under consideration is counting the number of zeros of a polynomial $D_n(z)$ with complex coefficients

$$D_n(z) = d_0 + d_1 z + \cdots + d_n z^n = d_n \prod_{i=1}^{n} (z - z_i), \quad D_n(1) \neq 0 \text{ is real}, \tag{1}$$

inside, on, and outside the unit circle (IUC, UC, and OUC zeros)

$$C = \{ z \mid z = e^{j\psi},\ \psi \in [0, 2\pi] \}. \tag{2}$$

The reason for the assumptions made on $D_n(1)$ in 91) will become clear later. It is noted that these assumptions are not restrictive in practice (cf. also [1], Remark 4.1): zeros at $z = 1$ are easily detected and removed; making $D_n(1)$ real may require a rescaling of the polynomial, e.g. its multiplication by $\overline{D}_n(1)$, where overbar means complex conjugate.

This problem has been originally solved by Schur (necessary and sufficient conditions for IUC zeros) [2] and Cohn (extension to UC and OUC zeros) [3]. It has been treated extensively also by Marden [4] and Jury [5] (the Jury–Marden stability table) as well as many others [6]. A different solution to this problem has been introduced by the author in [7,8,1]. It is based on a three-term recursion of symmetric polynomials rather than the Schur–Cohn two-term recursion of asymmetric (no specific form) polynomials. The new formulation was found to be more efficient in solving the zero location for real polynomials by approximately a factor of 2. The purpose of this paper is to establish the complex version of this formulation.

After a short preliminary study of properties of (conjugate) symmetric and antisymmetric polynomials, we follow the outline of the paper [1] and show how the formulas, theorems, proofs, etc. there extend in a natural manner to polynomials with complex coefficients. A remarkable outcome is that, in spite of the complex numbers arithmetics, the crucial step of counting the IUC and OUC zeros still involves the same and simple real arithmetic of the real case. So, the relative actual saving in number of real arithmetics

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(compared to the Schur–Cohn test) is somewhat even better in this general case. Also, we show that the complex algorithm can, optionally, be carried out using only real polynomials and arithmetic. We often compare the results presented here with [1]; when appropriate we refer to [1] for more details and discussion or even state results without proof when these can be found in [1] in a (formally) identical form.

2. Preliminaries

We denote by $D_n^{\#}(z)$ the reciprocal polynomial of $D_n(z)$,

$$D_n^{\#}(z)=\bar{d}_n+\bar{d}_{n-1}z+\cdots+\bar{d}_0z^n=z^n\bar{D}_n(z^{-1}). \tag{3}$$

A polynomial $P_n(z)$ with complex coefficients will be called (conjugate) symmetric if

$$P_n^{\#}(z)=P_n(z) \tag{4a}$$

and (conjugate) antisymmetric if

$$P_n^{\#}(z)=-P_n(z). \tag{4b}$$

It is easy to see from the fact that the zeros of $D_n^{\#}(z)$ are $\bar{z}_i^{-1}$, $i=1,\ldots,n$, that:

Lemma 1. *A polynomial is either symmetric or antisymmetric if and only if it has only UC zeros z_c or reciprocal pair (RP) zeros $(z_r, \bar{z}_r^{-1})$ (with the same multiplicity).*

Subsequently, we shall always imply in any count of zeros their multiplicity. Another obvious feature of symmetric or antisymmetric polynomials is stated next:

Lemma 2. *Let a polynomial $P_n(z)$ of complex coefficients be written as the sum*

$$P_n(z)=S_n(z)+\mathrm{j}A_n(z), \tag{5}$$

where $S_n(z)$ and $A_n(z)$ are two real polynomials. Then, $P_n(z)$ is conjugate symmetric (antisymmetric) if and only if $S_n(z)$ is real symmetric (antisymmetric) and $A_n(z)$ is real antisymmetric (symmetric, respectively). A real antisymmetric polynomial, say $A_n(z)$, can further be factored into

$$A_n(z)=(z-1)S_{n-1}(z) \tag{6}$$

where $S_n(z)$ is symmetric.

We define for an arbitrary polynomial $P_n(z)$ the following $\hat{P}_n(z)$ (a polynomial in $z^{1/2}$ and $z^{-1/2}$) as its 'balanced polynomial':

$$\hat{P}_n(z)=z^{-n/2}P_n(z). \tag{7}$$

An important feature of symmetric and antisymmetric polynomials is revealed by their corresponding balanced polynomials.

Theorem 3. *A polynomial $P_n(z)$ with complex coefficients is symmetric if and only if*

$$\operatorname{Im}\hat{P}_n(z)=0 \quad \text{for all } z\in C; \tag{8a}$$

and is antisymmetric if and only if

$$\operatorname{Re}\hat{P}_n(z)=0 \quad \text{for all } z\in C. \tag{8b}$$

Proof. See the appendix.

3. The regular formulation

The polynomial $D_n(z)$ of (1) can be written as the sum of a symmetric and an antisymmetric polynomial:

$$D_n(z) = \tfrac{1}{2}\left[D_n(z) + D_n^{\#}(z)\right] + \tfrac{1}{2}\left[D_n(z) - D_n^{\#}(z)\right]. \tag{9}$$

We would like to define, as in [1],

$$T_n(z) = D_n(z) + D_n^{\#}(z) \quad \text{and} \quad T_{n-1}(z) = \left[D_n(z) - D_n^{\#}(z)\right]/(z-1). \tag{10}$$

The second definition imposes on the antisymmetric part of $D_n(z)$ to have a zero at $z = 1$. In view of Lemma 2, this requires its imaginary-part real polynomial to vanish at $z = 1$. Then, since the imaginary-part polynomial of the symmetric part of $D_n(z)$ in (9) (a real antisymmetric polynomial) is always zero at $z = 1$, the polynomial $T_{n-1}(z)$ is well defined if and only if Im $D_n(1) = 0$. We already assumed in (1) and shall assume throughout that $D_n(z)$ is real and nonzero at $z = 1$.

We now define a sequence of polynomials $\{T_k(z)\}_{k=0}^{n}$ by (10) and the recursion

$$T_k(z) = (\delta_k + \bar{\delta}_k z)T_{k-1}(z) - zT_{k-2}(z), \quad k = n, n-1, \ldots, 2, \tag{11a}$$

with

$$\delta_k = T_k(0)/T_{k-1}(0). \tag{11b}$$

This recursion is refered to as regular if the following normal conditions hold:

$$T_{n-i}(0) \neq 0, \quad i = 1, \ldots, n. \tag{12}$$

We shall restrict ourselves temporarily to normal conditions. Singular cases, when (12) does not hold, and their treatment will be discussed in Section 5.

Remark 1. It may be observed that the only difference between the three-term recursion in (11) and in [1] is that all the real symmetric polynomials in [1] are replaced by (complex) conjugate symmetric polynomials. Thus the term $(\delta_k + \delta_k z)$ there is replaced by the first degree polynomial $(\delta_k + \bar{\delta}_k z)$ that remains symmetric also for complex δ_k's.

It is not difficult to show now:

Lemma 4. *The polynomials $\{T_k(z)\}_{k=0}^{n}$ defined for a polynomial $D_n(z)$ of complex coefficients with real nonzero value at $z = 1$ by (10)–(12) are all symmetric, have exactly their indicated degree (the only acceptable exception is for $T_n(z)$ to have a simple zero at $z = 0$ and thus be of degree $n - 1$), and*

$$\sigma_k = T_k(1) \tag{13}$$

are real and nonzero for all $k = 1, \ldots, n$.

Proof. Symmetry of two polynomials in the three-term recursion (11) implies symmetry of the third. Starting with the two symmetric polynomials (10), the recursion (11) implies that all subsequent polynomials are also symmetric and (assuming normal conditions) of exactly their claimed degree. Finally, a symmetric polynomial is always real valued at $z = 1$ (Lemma 2).

We define the following second sequence $\{D_k(z)\}_{k=0}^{n}$ by $D_0(z) = \frac{1}{2}T_0(z) = \frac{1}{2}\sigma_0$ and

$$D_k(z) = \tfrac{1}{2}T_k(z) + \tfrac{1}{2}(z-1)T_{k-1}(z). \tag{14}$$

It is obvious from here and (13) that

$$D_k(1) = \tfrac{1}{2}T_k(1) \tag{15}$$

are real and nonzero and from (10) that the n-th degree polynomial here becomes the polynomial under investigation (1).

Remark 2. In Section 5 we shall show that zeros of $D_n(z)$ on the unit circle occur also as zeros of subsequent $D_k(z)$, $k = n, n-1, \ldots, s > 1$, and they imply a singularity $T_{s-1} \equiv 0$. Therefore, regular conditions guarantee that no polynomial in the sequence $\{D_k(z)\}_{k=0}^{n}$ has UC zeros. This establishes the first statement in the following theorem.

Theorem 5. *Provided that the recursion* (11) *is regular not one of the polynomials in the sequence* $\{D_k(z)\}_{k=0}^{n}$ *has UC zeros. Denote by* (α_k, γ_k) *the number of* (IUC, OUC) *zeros of* $D_k(z)$, $k < n$ $(\alpha_k + \gamma_k = k)$, *then the number of zeros of* $D_{k+1}(z)$ *with respect to C are*:
(a) $(\alpha_k + 1, \gamma_k)$ *if* $\operatorname{sgn} D_{k+1}(1) = \operatorname{sgn} D_k(1)$,
(b) $(\alpha_k, \gamma_k + 1)$ *if* $\operatorname{sgn} D_{k+1}(1) = -\operatorname{sgn} D_k(1)$.

The next theorem, our main result, can be easily deduced from Theorem 5 and (15).

Theorem 6. *Given* $D_n(z)$, *a polynomial with complex coefficients and nonzero real* $D_n(1)$ *value, and provided the sequence* $\{T_k(z)\}_{k=0}^{n}$ *defined for it is regular, the number of IUC and OUC zeros of* $D_n(z)$ *are respectively* $n - \nu_n$ *and* ν_n, *where*

$$\nu_n = \operatorname{Var}\{\sigma_n, \sigma_{n-1}, \ldots, \sigma_0\} \tag{16}$$

with σ_k *defined in* (13) *and* Var *denoting the number of sign changes in the indicated sequence of real numbers.*

Remark 3. Theorems 5 and 6 are comparable with Theorems 2.1 and 2.2 in [1]. The distribution of zeros for complex polynomials is obtained from a sequence that is defined in the same way and, remarkably, remains real even in this general case. (In fact, as described in Section 4, if desirable, the stability test can be performed completely in real arithmetics and polynomials.) Theorem 3 admits the next extension of the proof in [1] for this generalization.

Proof of Theorem 5. The proof is baed on the argument principle. Let $T_k(z)$, $\hat{T}_k(z)$ and $R_k(\psi)$ be, respectively, the k-th symmetric polynomial, its balanced polynomial, and the real-valued function (a polynomial in $\cos\frac{1}{2}\psi$ and $\sin\frac{1}{2}\psi$) that the latter takes on C by Theorem 3.

To prove part (a) we consider the quotient

$$f(z) = \frac{zD_k(z)}{D_{k+1}(z)}. \tag{17}$$

By the assumption in (a), $f(1)$ is real and positive. We proceed to show that for no value of $z \in C$ may $f(z)$ take real values which are negative. We have

$$f(z) = \frac{z[T_k(z) + (z-1)T_{k-1}(z)]}{T_{k+1}(z) + (z-1)T_k(z)} = \frac{z^{1/2}[\hat{T}_k(z) + (z^{1/2} - z^{-1/2})\hat{T}_{k-1}(z)]}{\hat{T}_{k+1}(z) + (z^{1/2} - z^{-1/2})\hat{T}_k(z)}.$$

The last expression becomes for $z = e^{j\psi}$,

$$f(e^{j\psi}) = \frac{e^{j\psi/2}[R_k(\psi) + j\,2\sin\frac{1}{2}\psi R_{k-1}(\psi)]}{R_{k+1}(\psi) + j\,2\sin\frac{1}{2}\psi R_k(\psi)}$$
$$= \frac{[\cos\frac{1}{2}\psi R_k(\psi) - 2\sin^2\frac{1}{2}\psi R_{k-1}(\psi)] + j\sin\frac{1}{2}\psi[R_k(\psi) + 2\cos\frac{1}{2}\psi R_{k-1}(\psi)]}{R_{k+1}(\psi) + j\,2\sin\frac{1}{2}\psi R_k(\psi)}.$$

Now, $f(e^{j\psi_0})$ may take real values either if the two imaginary parts are zero for ψ,

$$\sin \tfrac{1}{2}\psi_0\left[R_k(\psi_0) + 2\cos \tfrac{1}{2}\psi_0 R_{k-1}(\psi_0)\right] = 0, \qquad 2\sin \tfrac{1}{2}\psi_0 R_k(\psi_0) = 0, \tag{18}$$

or if the two real parts are zero,

$$\cos \tfrac{1}{2}\psi_0 R_k(\psi_0) - 2\sin^2\tfrac{1}{2}\psi_0 R_{k-1}(\psi_0) = 0, \qquad R_{k+1}(\psi_0) = 0. \tag{19}$$

Solutions to (18) other than $\sin\frac{1}{2}\psi_0 = 0$ ($f(z)$ was already found positive at $z = 1$), require $R_k(\psi_0) = R_{k-1}(\psi_0) = 0$ and imply that $D_k(z)$ has a UC zeros at $z = e^{j\psi_0}$, against the regularity assumption. Next, setting (19) into $f(e^{j\psi_0})$ gives

$$f(e^{j\psi_0}) = \frac{1}{2} + \frac{\cos\frac{1}{2}\psi_0 R_{k-1}(\psi_0)}{R_k(\psi_0)} = \frac{1}{2} + \frac{\cos^2\frac{1}{2}\psi_0}{2\sin^2\frac{1}{2}\psi_0} > 0.$$

Thus $f(z)$ takes positive real values at solutions to (19).

So, we have shown that as z traverses the unit circle C, $f(z)$ does not encircle the origin. Applying the argument principle, $D_{k+1}(z)$ has as many IUC zeros as $zD_k(z)$, namely $\alpha_{k+1} = \alpha_k + 1$.

The proof of part (b) uses, instead of (17), the quotient

$$g(z) = \frac{D_k(z)}{D_{k+1}(z)}$$

and proceeds similarly. First, by assumption, $g(1)$ is real and negative. Then, for $z \in C$,

$$g(e^{j\psi}) = \frac{\cos \frac{1}{2}\psi R_k(\psi) + 2\sin^2\frac{1}{2}\psi R_{k-1}(\psi) + j\sin \frac{1}{2}\psi\left[-R_k(\psi) + 2\cos \frac{1}{2}\psi R_{k-1}(\psi)\right]}{R_{k+1}(\psi) + j\,2\sin \frac{1}{2}\psi R_k(\psi)}.$$

The two imaginary parts cannot be simultaneously zeros for that would imply a UC zero of $D_k(z)$. The two real parts may become zero at values of ψ_0 for which

$$g(e^{j\psi_0}) = -\frac{1}{2} - \frac{\cos^2\frac{1}{2}\psi_0}{2\sin^2\frac{1}{2}\psi_0} < 0.$$

Since $g(z)$ takes on C either complex or real negative values, it cannot encircle the origin as z traverses C. Therefore $D_{k+1}(z)$ has as many IUC zeros as $D_k(z)$, $\alpha_{k+1} = \alpha_k$ or $\gamma_{k+1} = \gamma_k + 1$.

4. Computational complexity

The stability test for complex polynomials can be carried out by only real arithmetics. Let $\{S_k(z)\}_{k=0}^n$ and $\{A_k(z)\}_{k=0}^n$ be the sequences of real symmetric and real antisymmetric polynomials resulting from the decomposition (5) of each conjugate symmetric $T_k(z)$. The stability test can be equivalently accomplished by simultaneously propagating the following interlacing pair of recursions:

$$S_k(z) = \delta_k^R(z+1)S_{k-1}(z) + \delta_k^I(z-1)A_{k-1}(z) - zS_{k-2}(z), \tag{20a}$$

$$A_k(z) = \delta_k^R(z+1)A_{k-1}(z) - \delta_k^I(z-1)S_{k-1}(z) - zA_{k-2}(z), \tag{20b}$$

where δ_k^R and δ_k^I are the real and imaginary pats of δ_k.

The test starts with $S_k(z)$ and $A_k(z)$ found for $k = n, n-1$ from (10), then performs (20) for $k = n, n-1, \ldots, 2$. The zeros distribution is given by Theorem 6 with

$$\sigma_k = S_k(1). \tag{21}$$

Two variations on the derivation of the sequence for the sign-variation rule are possible. Setting $z = 1$ into

(20a) or (11a) shows that, starting with σ_n, σ_{n-1}, the rest of the sequence can be found by

$$\sigma_{k-2} = 2\delta_k^{\mathrm{R}}\sigma_{k-1} - \sigma_k, \quad k = n, \ldots, 2. \tag{22}$$

Alternatively, since the sign rule (16) is insensitive to a common scaling factor, σ_k can be replaced with $\hat{\sigma}_k = \sigma_k/\sigma_0$ calculated successively by

$$\hat{\sigma}_k = 2\delta_k^{\mathrm{R}}\hat{\sigma}_{k-1} - \hat{\sigma}_{k-2}, \quad k = 1, \ldots, n, \tag{23}$$

starting with $\hat{\sigma}_0 = 1$ and $\hat{\sigma}_{-1} := 0$.

The calculation of each coefficient of a polynomial $T_k(z)$ requires 4 (real) multiplications and 8 (real) additions but it is sufficient to find only one half of the coefficients, employing the involved symmetry properties. The computation complexity is consequently $M_n = \mathrm{O}(n^2)$ multiplications and $A_n = \mathrm{O}(2n^2)$ additions. (We use here $f = \mathrm{O}(\alpha n^2)$ to mean $(f/n^2) \to \alpha$ for large n.)

The Schur–Cohn test for complex polynomials can be performed (at best, after dropping normalization factors which are not necessary in the context of a stability test) in 4 multiplications and 4 additions per coefficient, but *all* the coefficients of all the polynomials have to be computed, resulting with a total of $M_0 = \mathrm{O}(2n^2)$ and $A_0 = \mathrm{O}(2n^2)$. Similarly, the Jury–Marden stability table for complex polynomials, that is based on this formulation and appears in general in versions that require $\mathrm{O}(4n^2)$ multiplications and $\mathrm{O}(3n^2)$ additions [6,9], can also be adjusted to attain the values M_0 and A_0. So, both in its complex and in its real forms, by comparison to the classical formulation, the new formulation involves half the number of entries, requires approximately half the number of multiplications, but the same number of additions.

5. Singular cases

The recursion (11) encounters a singularity when a $T_{s-1}(z)$ with $T_{s-1}(0) = 0$ occurs.

$$\begin{aligned} T_n(z) &= (\delta_n + \bar{\delta}_n z)T_{n-1}(z) - zT_{n-2}(z), \\ &\vdots \\ T_k(z) &= (\delta_k + \bar{\delta}_k z)T_{k-1}(z) - zT_{k-2}(z), \\ &\vdots \\ T_{s+1}(z) &= (\delta_{s+1} + \bar{\delta}_{s+1} z)T_s(z) - zT_{s-1}(z). \end{aligned} \tag{24}$$

We already know from Theorem 6 that singular cases indicate that not all the zeros are inside the unit circle (but not vice versa). Following [1] we classify the singular cases into two classes; the case when $T_{s-1}(z) \equiv 0$ is called a singularity of the first type, and the case when $T_{s-1}(0) = 0$ but $T_{s-1}(z)$ is not identically zero is called a singularity of the second type.

Singularities of the first type

If $T_{s-1}(z) = 0$ then (24) shows that $T_s(z)$ is a factor of all preceding polynomials $T_{s+1}(z), \ldots, T_n(z)$ and consequently it is also a factor of all $D_k(z)$, $k = s, s+1, \ldots, n$, defined in (14). Therefore $D_n(z)$ has (the zeros of $T_s(z)$ as) s UC and RP zeros. Conversely, if $D_n(z)$ has a total of s UC or RP zeros then they are zeros of $T_n(z)$ and $T_{n-1}(z)$ and (24) shows that they are common factor of all subsequent $T_k(z)$ till $T_{s-1}(z)$ that must then be identically zero (its degree $s-1$ cannot accommodate s zeros). This establishes an 'if and only if' relation between the first-type singularity and $D_n(z)$ having UC or RP zeros. It also justifies the first statement in Theorem 5. Note that regarding $(z_r, \bar{z}_r^{-1})$ rather than (z_r, z_r^{-1}) as a reciprocal pair of zeros is the only difference from [1] of the current characterization of the first-type singularity.

The treatment of first-type singularities: Choose the remaining s symmetric polynomials to be the polynomials derived by (10) and (11) for

$$D_{s-1}(z)=K\left[\frac{\mathrm{d}T_s(z)}{\mathrm{d}z}\right]^{\#}, \tag{25}$$

where K is any scaling factor that arranges $D_{s-1}(1)$ to be real and of sign opposite to $T_s(1)$.

Theorem 7. *Following a first-type singularity by the above procedure, the number of IUC zeros is given by* $\alpha_n = n - \nu_n$ *where* ν_n *is given by* (16). *The number of UC zeros is* $\beta_n = 2\nu_s - s$ *where*

$$\nu_s = \mathrm{Var}\{\sigma_s, \sigma_{s-1}, \ldots, \sigma_0\}. \tag{26}$$

The number of reciprocal pairs is $s - \nu_s$.

The proof follows from a theorem due to Cohn [3; 2, Theorem (45,2)], and from the previously established properties of the regular recursion (cf. [1, Theorem 4.3]).

Singularities of the second type

A second-type singularity is not specific to any special pattern of zeros position (except that it implies OUC zeros). As it turns out the procedure offered in [1] and its proof there are valid without any change of requirements also for conjugate reciprocal and conjugate symmetric polynomials. We shall repeat the procedure here for the completeness of the current presentation.

The treatment of second-type singularities: Resume the recursion after replacing $T_s(z)$ and $T_{s-1}(z)$ by

$$T_s(z)+(z-1)T_{s-1}(z)[z^q - z^{-q}] \tag{27a}$$

and

$$T_{s-1}(z)[K+z^q+z^{-q}], \quad K>2, \tag{27b}$$

respectively, where q is the number of zeros of $T_q(z)$ at $z=0$ and K is an arbitrary (>2) real constant.

Theorem 8 (cf. [1, Theorem 4.4]). *Following second-type singularities by the above procedure, the number of IUC zeros is given by* $n-\nu_n$, *where* ν *is given by* (16). *If a first-type singularity is not apparent anywhere in the sequence of* $n+1$ *polynomials, the number of OUC zeros is* ν *(or "Theorem 6 holds"). If a first-type singularity does occur then, regardless of second-type singularity procedures, the number of UC, OUC and RP zeros is still given by Theorem 7.*

Remark 4. Each of the two types of singularities may occur more than once. The first type will recur if (and only if) $D_n(z)$ has UC or RP zeros in multiplicity higher than one (the derivation in (26) lowers UC and RP multiplicities by one). Singularities of the second type may occur and recur haphazardly. A remarkable feature of the replacement (27) is that it leaves UC and RP zeros unchanged. Since the flow of such zeros through the recursion is unaffected, if $D_n(z)$ has such zeros, their factor in $D_n(z)$ is correctly identified and they are correctly counted even in the presence of second-type singularities.

Appendix

Proof of Theorem 3. Write $P_n(z)$ as

$$P_n(z)=\sum_{i=0}^{n} p_i z^i, \quad p_i = a_i + \mathrm{j}b_i. \tag{A.1}$$

If $P_n(z)$ is symmetric, $p_{n-i}=\bar{p}_i$, and the balanced polynomial becomes for $z=e^{j\psi}$,

$$\hat{P}_n(e^{j\psi})=(a_0-jb_0)\,e^{jn\psi/2}+(a_0+jb_0)\,e^{-jn\psi/2}$$
$$+(a_1-jb_1)\,e^{j(n-2)\psi/2}+(a_1+jb_1)\,e^{-j(n-2)\psi/2}+\cdots$$
$$=2a_0\cos\tfrac{1}{2}n\psi=2b_0\sin\tfrac{1}{2}n\psi+2a_1\cos\tfrac{1}{2}(n-2)\psi+2b_1\sin\tfrac{1}{2}(n-2)\psi+\cdots$$

which is real.

If $P_n(z)$ is antisymmetric, $p_{n-i}=-\bar{p}_i$, and for $z\in C$ the balanced polynomial becomes

$$\hat{P}_n(e^{j\psi})=(-a_0+jb_0)\,e^{jn\psi/2}+(a_0+jb_0)\,e^{-jn\psi/2}$$
$$+(-a_1+jb_1)\,e^{j(n-2)\psi/2}+(a_1+jb_1)\,e^{-j(n-2)\psi/2}+\cdots$$
$$=j\{2b_0\cos\tfrac{1}{2}n\psi+2a_0\sin\tfrac{1}{2}n\psi+2b_1\cos\tfrac{1}{2}(n-2)\psi+2a_1\sin\tfrac{1}{2}(n-2)\psi+\cdots\},$$

a purely imaginary expression.

To prove the converse, the balanced polynomial of a polynomial $P_n(z)$ that is given by (A.1) takes on C the form

$$\hat{P}_n(e^{j\psi})=\{(p_0+p_n)\cos\tfrac{1}{2}n\psi+(p_1+p_{n-1})\cos\tfrac{1}{2}(n-2)\psi+\cdots\}$$
$$+j\{(-p_0+p_n)\sin\tfrac{1}{2}n\psi+(-p_1+p_{n-1})\sin\tfrac{1}{2}(n-2)\psi+\cdots\}.$$

If $P_n(z)$ satisfies (8a) then

$$\hat{P}_n(e^{j\psi})-\bar{\hat{P}}_n(e^{j\psi})=2\,\mathrm{Im}\,\hat{P}_n(e^{j\psi})=0$$

with

$$\bar{\hat{P}}_n(e^{j\psi})=\{(\bar{p}_0+\bar{p}_n)\cos\tfrac{1}{2}n\psi+(\bar{p}_1+\bar{p}_{n-1})\cos\tfrac{1}{2}(n-2)\psi+\cdots\}$$
$$+j\{(-\bar{p}_0+\bar{p}_n)\sin\tfrac{1}{2}n\psi+(-\bar{p}_1+\bar{p}_{n-1})\sin\tfrac{1}{2}(n-2)\psi+\cdots\}.$$

Therefore

$$0=\hat{P}_n(e^{j\psi})-\bar{\hat{P}}_n(e^{j\psi})$$
$$=\{[p_0+p_n-\bar{p}_0-\bar{p}_n]\cos\tfrac{1}{2}n\psi+[p_1+p_{n-1}-\bar{p}_1-\bar{p}_{n-1}]\cos\tfrac{1}{2}(n-2)\psi+\cdots\}$$
$$+j\{[-p_0+p_n-\bar{p}_0+\bar{p}_n]\sin\tfrac{1}{2}n\psi+[-p_1+p_{n-1}-\bar{p}_1+\bar{p}_{n-1}]\sin\tfrac{1}{2}(n-2)\psi+\cdots\}.$$

The last equality holds for any $\psi\in[0,2\pi]$ if and only if $p_i=\bar{p}_{n-i}$, $i=1,\ldots,n$, or iff $P_n(z)$ is symmetric.

Assume next that (8b) holds for $P_n(z)$. Then

$$\hat{P}_n(e^{j\psi})-\bar{\hat{P}}_n(e^{j\psi})=2\,\mathrm{Re}\,\hat{P}_n(e^{j\psi})=0$$

implies

$$0=\{[p_0+p_n+\bar{p}_0+\bar{p}_n]\cos\tfrac{1}{2}n\psi+[p_1+p_{n-1}+\bar{p}_1+\bar{p}_{n-1}]\cos\tfrac{1}{2}(n-2)\psi+\cdots\}$$
$$+j\{[-p_0+p_n+\bar{p}_0-\bar{p}_n]\sin\tfrac{1}{2}n\psi+[-p_1+p_{n-1}+\bar{p}_1-\bar{p}_{n-1}]\sin\tfrac{1}{2}(n-2)\psi+\cdots\}$$

which holds for all ψ iff $p_i=-\bar{p}_{n-i}$, $i=1,\ldots,n$, that is, iff $P_n(z)$ is antisymmetric.

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