Generalized Bezoutians and Families of Efficient Zero-Location Procedures

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***Abstract*—The procedures of Routh–Hurwitz and Schur–Cohn for determining the zero-distribution of polynomials with respect to the imaginary axis and the unit circle, respectively, serve, at the same time, to efficiently evaluate the inertia of certain so-called *Bezoutian* matrices. These well-known procedures require $O(n^2)$ operations to determine the inertia of an $n \times n$ Bezoutian, in contrast to the $O(n^3)$ operations that would be required to determine the inertia of an arbitrary (Hermitian) matrix of the same size.**

We introduce generalized Bezoutians whose inertia specifies the zero-distribution with respect to arbitrary circles and straight lines in the complex plane. We recognize these Bezoutians as members in the family of matrices with (generalized) *displacement* structure, for which we have recently developed efficient $O(n^2)$ procedures for triangular factorization and, hence, inertia determination. Moreover, our formulation displays a large variety of $O(n^2)$ procedures that can be associated with a single (generalized) Bezoutian matrix. For Bezoutians on the imaginary axis and the unit circle, our formulation leads to (among other possibilities) the Routh–Hurwitz and Schur–Cohn tests.

I. Introduction

A PROBLEM that has been of interest for over a century is that of determining whether a given system is *stable*. Linear time-invariant systems with a rational transfer function are stable when their poles (i.e., the zeros of their denominator polynomial) are constrained to some domain of the complex plane. For continuous-time systems this domain is the left-half plane, while for discrete-time systems it is the inside of the unit circle.

An obvious extension of the stability problem is to determine the distribution of the zeros of a given polynomial with respect to some given simple curve Ω in the complex plane that partitions the complex plane into two disjoint domains whose common boundary is Ω. The problem is to find how many of the zeros of a given polynomial lie on the curve, and how many lie in each of the two domains defined by the curve. It is interesting to notice that while it is impossible to obtain a closed from algebraic expression for the zeros of a polynomial of degree higher than 4, the *distribution* of these zeros with respect to many simple curves can be determined with a finite number of algebraic operations.

Hermite [1] was the first to present a comprehensive solution of this problem for the real line R. He constructed a Hermitian matrix, which we shall denote by $\boldsymbol{B}_R$, whose inertia (i.e., the number of positive, zero, and negative eigenvalues) determines the distribution with respect to the real line of the zeros of a given polynomial $p(z)$. The relation between the coefficients of the polynomial in question and the elements of the matrix $\boldsymbol{B}_R$ is most conveniently described in terms of the *generating function* $B_R(z,w)$, which is a bivariate polynomial of degree $n-1$, viz.,

$$B_R(z,w) := [1 \quad z \quad z^2 \quad \cdots \quad z^{n-1}] \cdot \boldsymbol{B}_R[1 \quad w \quad w^2 \quad \cdots \quad w^{n-1}]^* \quad (1)$$

where n denotes the size of the square matrix $\boldsymbol{B}_R$, which is equal to the degree of the polynomial $p(z)$, and the asterisk $(*)$ denotes Hermitian transpose (complex conjugate for scalars). Hermite's matrix has a generating function of the form

$$B_R(z,w) = \frac{p(z)p^*(w) - p^*(z^*)p(w^*)}{j(z-w^*)} \quad (2a)$$

where $p^*(z)$ denotes conjugation of both the coefficients and the variable, viz.,

$$p^*(z) := [p(z)]^*. \quad (2b)$$

Hermite showed that, if $p(z)$ has η zeros that either lie on the real line, or are arranged in pairs symmetric to it, then $rank\, \boldsymbol{B}_R = n - \eta$. Moreover, if π of the remaining zeros lie in the upper-half plane and ν of them lie in the lower-half plane, then $In\ \boldsymbol{B}_R = \{\pi, \eta, \nu\}$. The functional-analytic interpretation enabled by the generating function formula (2) has been the basis for the work of many early contributors and, in particular, that of Cayley [2] and Darboux [3].

Hermite's rank result indicates that the rank deficiency of $\boldsymbol{B}_R$ is associated with the zeros of $p(z)$ that are shared by $p^*(z^*)$. This suggests that if $p^*(z^*)$ is replaced in (2a) by an arbitrary polynomial $q(z)$ of the same degree as $p(z)$, viz., if we define

$$B_R(z,w) := \frac{p(z)q(w^*) - q(z)p(w^*)}{j(z-w^*)} \quad (3)$$

then the rank of $\boldsymbol{B}_R$ should decrease by the number of zeros shared by $p(z)$ and $q(z)$, namely,

$$rank\, \boldsymbol{B}_R = n - \deg gcd\{p(z), q(z)\} \quad (4a)$$

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where

$$n := \deg p(z) = \deg q(z) \quad (4b)$$

and $gcd\{p(z), q(z)\}$ denotes the *greatest common divisor* of the polynomials $p(z), q(z)$. This result was first discovered by Sylvester [4], who independently obtained some of Hermite's zero-location results, though only for real polynomials. Sylvester coined the term *Bezoutian*[1] for the matrix $\boldsymbol{B}_R$ in honor of the French mathematician E. Bezout. As noted by Wimmer [29], Cayley [2] was apparently the first to introduce the generating function description (2) of the Bezoutian matrix. We shall call this matrix, and also its corresponding generating function, an R-Bezoutian, to emphasize its association with the real line R.

The dual problem of zero-location with respect to the unit circle T has received much less attention, no doubt because of the late (c. 1950) interest in discrete-time systems. Even though it was already known to Hermite (see [5]) that the solution to this problem could be obtained by a transformation of variables in the generating function $B_R(z,w)$, the first *efficient* procedure for unit circle zero-location was obtained only 60 years later by Cohn [6], extending the earlier work of Schur [7]. The *T-Bezoutian* of Schur and Cohn has the generating function

$$B_T(z,w) = \frac{p(z)p^*(w) - p^\#(z)[p^\#(w)]^*}{1 - zw^*} \quad (5a)$$

where $p^\#(z)$ denotes a reversal in the order of coefficients, i.e.,

$$p^\#(z) = z^{\deg p(z)}[p(1/z^*)]^*. \quad (5b)$$

It should be noted that the Schur–Cohn Bezoutian was obtained independently, and not via a transformation of previously known real-line results. Fujiwara was the first to point out the similarities between unit-circle and real-line results and to suggest a (partly) unified framework for their analysis [8]. In particular he observed that the numerators of $B_R(z,w)$ in (2) and of $B_T(z,w)$ in (5) have the same form once we adopt the notational convention that $p^\#(z) := p^*(z^*)$ for the real line R.

Hermite and Sylvester were not concerned with the efficiency of the computational procedures for evaluating the inertia of the R-Bezoutian $\boldsymbol{B}_R$. This problem was first addressed by Routh [9], and, independently, by Hurwitz [10], who derived an efficient procedure for determining root-distribution of polynomials with respect to the real line[2] without actual calculation of the roots (and thus, in effect, also for evaluating the inertia of $\boldsymbol{B}_R$ *without explicitly evaluating its elements*); both these authors were influenced by earlier work of Sturm [11]. The Routh–Hurwitz test is to this day the method of choice for zero-location with respect to the imaginary (and real) line. Similarly, the Schur–Cohn test, which was subsequently modified by Marden [12] and Jury [13], is the standard (efficient) technique for zero-location with respect to the unit circle. However, the Schur–Cohn test is not an exact analog of the Routh–Hurwitz test, and the relation between these tests has never been very clear.

[1] Sylvester preferred the spelling *Bezoutiant*.
[2] Actually, Routh and Hurwitz were concerned with zero-location with respect to the *imaginary axis*. Nevertheless, this problem differs only in minor details from the real-line problem considered by Hermite and Sylvester.

We note that both these rather different appearing procedures must be computing the inertia of the corresponding Bezoutians in a fast (i.e., $O(n^2)$) way. This observation aroused our interest in this issue because we recognized the Hermite–Sylvester and the Schur–Cohn Bezoutians as being members of a class of matrices possessing what we have called *displacement structure* [17], [18]. Displacement structure enables fast algorithms for matrix factorization, and hence for inertia computation (see Section II). When applied to the Bezoutian $\boldsymbol{B}_R$, our procedure leads us very directly to the Routh–Hurwitz test and, in fact, first to a different but completely equivalent test. For the Bezoutian $\boldsymbol{B}_T$, it leads us to a slight variation of the Jury–Marden test; in fact, our approach actually gives the original Schur–Cohn procedure [6], which has some interesting differences from the Jury–Marden procedure (see Example 2.2).

One feature of our approach, besides the unified form of derivation, is that it shows the classical tests as only some possibilities among a host of efficient algorithms, including the tests recently introduced by Bistritz [14], [35], Lepschy *et al.* [30], Reddy and Rajan [32], Schussler [33], and Steffen [34], which can all be systematically generated and compared. Moreover, it turns out that our discussion can be presented for zero-distribution with respect to arbitrary circles and straight lines in the complex plane, and not just for the unit circle and the real (or imaginary) axes. We develop a Bezoutian for such curves and recognize it as also having displacement structure.

To be more specific about the above remarks, let us note that we can characterize circles and lines in the complex plane by a *sesquilinear bivariate polynomial*, i.e., a polynomial of degree 1 in z and in w^*, viz.,

$$d_\Omega(z,w) := \alpha + \beta z + (\beta w)^* + \delta z w^*, \qquad |\beta|^2 - \alpha\delta > 0. \quad (6a)$$

The inequality

$$d_\Omega(z,z) > 0 \quad (6b)$$

defines a domain that we shall denote by Ω_+. When $\delta = 0$, this domain is a half plane, while for $\delta < 0$ (respectively, $\delta > 0$) it is the inside (respectively, outside) of a circle. The boundary curve of this domain, which is a circle or a straight line, will be denoted by Ω and the remaining part of the complex plane by Ω_-. The imaginary axis is obtained with the choices $\alpha = 0 = \delta$ and $\beta = 1$, the unit circle with $\alpha = 1 = -\delta$ and $\beta = 0$.

We define the *Ω-Bezoutian* corresponding to two polynomials $p(z), q(z)$ of equal degree[3] via the generating function

$$B_\Omega^{p,q}(z,w) := \frac{p(z)[q^\#(w)]^* - q(z)[p^\#(w)]^*}{d_\Omega(z,w)}. \quad (7a)$$

The choice $d_R(z,w) := j(z - w^*)$ and $p^\#(z) := p^*(z^*)$ yields the Hermite–Sylvester Bezoutian (3), while the Schur–Cohn Bezoutian (5) corresponds to $d_T(z,w) = 1 - zw^*$, $q(z) = p^\#(z)$, and $p^\#(z)$ as defined by (5b). In general, the *polynomial reflection* $p(z) \to p^\#(z)$ is defined in terms of the coefficients $\alpha, \beta, \gamma, \delta$ of the bivariate polynomial $d_\Omega(z,w)$, in such a way that the zeros of the transformed polynomial $p^\#(z)$

[3] When $\deg p(z) \neq \deg q(z)$ we can equalize their degrees by multiplying the polynomial with the smaller degree by a suitable power of z.

are the *point-reflections* with respect to Ω of the zeros of $p(z)$. This means that if ζ is a zero of the polynomial $p(z)$ then the corresponding zero of $p^{\#}(z)$ is the (unique) solution ζ^R of the equation

$$d(\zeta, \zeta^R) = 0 \tag{7b}$$

which provides an implicit definition of point reflection. Explicit expressions for point- and polynomial-reflection are given in Section III. Here it will suffice to mention two important properties of polynomial reflection, viz.,

$$\{p(z)q(z)\}^{\#} = p^{\#}(z)q^{\#}(z) \tag{8a}$$

$$|p^{\#}(z)| = |p(z)|, \qquad \text{for all } z \in \Omega. \tag{8b}$$

We shall show in Section III that $B_\Omega^{p,q}(z,w)$ is a bivariate *polynomial* of degree $n-1$ in z and in w^* where $n := \deg p(z) = \deg q(z)$. Then we show that the square matrix $\boldsymbol{B}_\Omega^{p,q}$, which consists of the coefficients of the generating function $B_\Omega^{p,q}(z,w)$, satisfies the following fundamental theorem.

Theorem 1 (Bezoutian Rank and Inertia):

i) The rank of $\boldsymbol{B}_\Omega^{p,q}$ is $n-\eta$ where n is the degree of $p(z)$ (and $q(z)$), and

$$\eta := \deg gcd\{p(z), q(z)\}.$$

ii) If $q(z) = p^{\#}(z)$, the Bezoutian matrix $\boldsymbol{B}_\Omega^{p,p^{\#}}$ is Hermitian and its inertia specifies the location of the zeros of $p(z)$, with respect to the curve Ω, as follows: if

$$In\,\boldsymbol{B}_\Omega^{p,p^{\#}} = \{\pi, \eta, \nu\}$$

then $p(z)$ shares η zeros with $p^{\#}(z)$ while π of the remaining zeros lie in Ω_- and ν of them lie in Ω_+. ■

Incidentally, we may note that in addition to generalizing the previously known real-line and unit-circle results for zero-location, our Theorem 1 also demonstrates the fact that evaluation of greatest common divisors can be done via $\boldsymbol{B}_\Omega^{p,q}$ on arbitrary circles and lines in the complex plane, whereas all previous schemes seemed to have focused on the particular choice $d_R(z,w) = j(z-w^*)$.

Next we recall that the definition (7) of $B_\Omega^{p,q}(z,w)$ implies that the Ω-Bezoutian matrix $\boldsymbol{B}_\Omega^{p,q}$ has what we have called displacement structure [17], [18]. An infinite matrix

$$\boldsymbol{R} = \{r_{i,j};\, 0 \leqslant i,j < \infty\} \tag{9a}$$

is said to have displacement structure if its generating function $R(z,w)$, viz.,

$$R(z,w) := [1 \quad z \quad z^2 \quad \cdots]\boldsymbol{R}[1 \quad w \quad w^2 \quad \cdots]^* \tag{9b}$$

can be expressed in the form

$$R(z,w) = \frac{G(z)JH^*(w)}{d(z,w)} \tag{10a}$$

where the *displacement kernel* $d(z,w)$ is a bivariate Hermitian power series, i.e.,

$$d(z,w) = \sum_{i,j=0}^{\infty} d_{i,j} z^i (w^*)^j, \qquad d_{i,j} = d_{j,i}^* \tag{10b}$$

the constant matrix J is Hermitian and nonsingular, i.e.,

$$J = J^*, \qquad \det J \neq 0 \tag{10c}$$

and $G(z), H(z)$ are (vector) power series, i.e.,

$$G(z) := \sum_{i=0}^{\infty} g_i z^i$$
$$H(z) := \sum_{i=0}^{\infty} h_i z^i \tag{10d}$$

where g_i, h_i are row vectors. Thus the Bezoutian characterization (7) is a particular case of the displacement characterization (10a) with $d(z,w) = d_\Omega(z,w)$, $J = \text{diag}\{1,-1\}$ and $G(z) = [p(z) \quad q(z)]$, $H(z) = [q^{\#}(z) \quad p^{\#}(z)]$.

We have recently derived fast algorithms for LDL^* factorization of matrices that have a displacement structure with a displacement kernel $d(z,w)$ that satisfies the additional constraint [18]

$$d(z,w) = d_1(z)d_1^*(w) - d_2(z)d_2^*(w). \tag{10e}$$

Since this constraint is satisfied, in particular, by the sesquilinear function $d_\Omega(z,w) = \alpha + \beta z + (\beta w)^* + \delta z w^*$ (see Appendix A), our factorization procedure can also be used to efficiently evaluate the rank and inertia[4] of Ω-Bezoutians. The following theorem summarizes the specialization of this procedure to Ω-Bezoutians.

Theorem 2 (Efficient Evaluation of Bezoutian Inertia):

The inertia of a *strongly regular* Bezoutian $\boldsymbol{B}_\Omega^{p,p^{\#}}$ coincides with the inertia of the diagonal matrix $\text{diag}\{d_i; 0 \leqslant i \leqslant n-1\}$, where $d_i := B_i(\zeta_i, \zeta_i)$, and the *extraction points* ζ_i can be chosen anywhere in the complex plane (they need not be distinct). Here $B_i(z,w)$ is a sequence of Ω-Bezoutians, viz.,

$$B_i(z,w) = \frac{G_i(z)JG_i^*(w)}{d_\Omega(z,w)} \tag{11a}$$

where

$$J := T^{-1}\,\text{diag}\{1,-1\}T^{-*} \tag{11b}$$

T is an arbitrary nonsingular matrix, and $G_i(z)$ is a row vector polynomial of degree $n-i$. The vector polynomials $G_i(z)$ are propagated by a two-term recursion:

$$(z-\zeta_i)G_{i+1}(z) = G_i(z)\Theta_i(z), \qquad \text{for } i = 0,1,2,\cdots n-1 \tag{11c}$$

where $G_0(z) = [p(z) \quad p^{\#}(z)]T$, $\Theta_i(z)$ is the 2×2 matrix function

$$\Theta_i(z) := \left[I - \lambda_i(z)JG_i^*(\zeta_i)d_i^{-1}G_i(\zeta_i)\right]U_i \tag{11d}$$

with

$$\lambda_i(z) := \frac{d_\Omega(z,\tau_i)}{d_\Omega(z,\zeta_i)d_\Omega(\zeta_i,\tau_i)}. \tag{11e}$$

Here $\{\tau_i\}$ are arbitrary points on the curve Ω, i.e.,

$$d_\Omega(\tau_i,\tau_i) = 0 \tag{11f}$$

and the (nonsingular) matrices U_i are J-unitary, i.e.,

$$U_i J U_i^* = J. \tag{11g}$$

The vector function $G_i(z)$ is a polynomial of degree $n-i$, even though $\Theta_i(z)$ itself is a rational matrix function of degree 1. Moreover, the flexibility in selecting U_i can be used

[4]By a famous law of Sylvester the matrices D and LDL^* have the same inertia.

to ensure that for all i,

$$G_i(z) = [\,p_i(z) \quad p_i^{\#}(z)\,]T, \qquad \deg p_i(z) \leqslant \deg p_0(z) - i. \tag{11h}$$

■

The undetermined parameters in $\Theta_i(z)$ can be used to obtain further simplification in the recursions (11), which will be described in Section II. The most significant undetermined parameters in (11) are the extraction points $\{\zeta_i\}$ and the matrix T, which determines the *type* of the recursion. For the unit-circle problem (i.e., $d(z,w) = 1 - zw^*$) it is customary to choose all $\zeta_i = 0$, and a *scattering-type* formulation (i.e., $T = I$), which reduces (11) to a recursion identical to the one originally published by Cohn [6] (see Example 2.2 for more details). For the imaginary-axis problem (i.e., $d(z,w) = z + w^*$) the same choice for ζ_i combined with an *immittance-type* formulation (i.e., $T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$) reduces (11) to a slightly modified but computationally equivalent form of the Routh (–Hurwitz) recursions (see Example 2.1 for more details). This terminology is justified by the observation that when $p_0(z) \equiv p(z)$ has no roots within the unit circle then a scattering-type recursion propagates $G_i(z) = [\,p_i(z) \quad p_i^{\#}(z)\,]$ and the ratio $p_i^{\#}(z)/p_i(z)$ is a scattering function (i.e., it is analytic and bounded by unity within the unit circle). Similarly, an immittance-type recursion propagates $G_i(z) = [\,p_i(z) + p_i^{\#}(z) \quad p_i(z) - p_i^{\#}(z)\,]$, and the ratio $[\,p_i(z) - p_i^{\#}(z)]/[\,p_i(z) + p_i^{\#}(z)]$ is an immittance function (i.e., it is analytic and has a positive real part within the unit circle). The only other computationally reasonable choices for ζ_i are ± 1, because we have to evaluate the polynomials $p_i(z)$ and $p_i^{\#}(z)$ (or $p_i(z) \pm p_i^{\#}(z)$) at these points. Recently, Reddy and Rajan [32] and subsequently Lepschy *et al.* [30] presented an alternative to the Routh procedure that is based on the choice $\zeta_i = 1$. We shall describe in Section II how our formulation leads to a procedure that is equivalent (and almost identical) to these algorithms. We shall also present an alternative to the Schur–Cohn procedure that is based on the same choice (i.e., $\zeta_i = 1$). This new procedure turns out to be equivalent (in a sense defined in [36]) to the bilinear Routh tests of [33]–[35] although no bilinear transformation is involved in our derivation.

We also remark that the efficient procedure (11) can be extended also to structured *non-Hermitian* matrices with generating functions of the form

$$R(z,w) = \frac{G(z)JH^*(w)}{d(z,w)}$$

and therefore, in particular, to the non-Hermitian Ω-Bezoutian $B_\Omega^{p,q}(z,w)$. This makes it possible to evaluate the gcd of two polynomials in a variety of ways, corresponding to various choices of the curve Ω.

It should be noted that here we consider only the so-called *strongly regular* cases, where all the leading minors of the Bezoutian are nonzero. The singular cases need somewhat more complex algorithms, as already known, for example, for the Routh–Hurwitz test. Generalizations of the fast factorization procedures for nonregular Bezoutians have been recently developed, and will be discussed separately (see, e.g., Pal and Kailath [19], [20]).

To close this introduction, we note that there is a close relationship with the Lyapunov equation approach to the zero-location problem. We show in Appendix C that the generating function expression (7) can be manipulated into a Lyapunov-type equation, which relates the inertia of a certain matrix $\tilde{\boldsymbol{B}}$ to the location of eigenvalues of a given matrix $\boldsymbol{A}$. A similar result has been obtained by Gutman and Jury [21]. Moreover, their results hold for a broader class of curves Ω than the one discussed in our paper. Thus the zero-distribution of $p(z)$ can, in principle, be determined by choosing $\boldsymbol{A}$ as a companion matrix associated with the polynomial $p(z)$, solving the corresponding Lyapunov equation for $\tilde{\boldsymbol{B}}$, and finally determining the inertia of $\tilde{\boldsymbol{B}}$ (see also [22]). However, this requires $O(n^3)$ operations (for a polynomial of degree n), in contrast to the fast algorithms that we presented in Theorem 2, which require only $O(n^2)$ operations to evaluate the inertia of $B_\Omega^{p,p^{\#}}$. Moreover, our fast algorithms do not involve an explicit evaluation of the elements of the Bezoutian matrix. For this reason we shall not consider Lyapunov-type equations in further detail in this paper.

In the same context, we should also mention the extensive studies of Pták and Young (see, e.g., [23], [24]) on Lyapunov-type equations and generalized Bezoutian matrices. While these authors provide explicit expressions for their generalized Bezoutian matrices, they do not present $O(n^2)$ procedures for determining their inertia.

II. Efficient Algorithms for Inertia Computation

One way to determine the inertia of a Hermitian matrix $\boldsymbol{R}$ is to compute its triangular factorization $\boldsymbol{R} = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^*$ where $\boldsymbol{L}$ is lower triangular with unity diagonal elements, and $\boldsymbol{D}$ is diagonal. By Sylvester's law that congruence preserves inertia, we see that the inertia of $\boldsymbol{R}$ is given by the signs of the diagonal elements of the matrix $\boldsymbol{D}$. By observing that the Hermitian Ω-Bezoutian $\boldsymbol{B}_\Omega^{p,p^{\#}}$ of (7) has a (generalized) displacement structure in the sense of [18], we know that its inertia can be determined via the efficient factorization procedure of [18] in $O(n^2)$ operations and *without explicitly evaluating the elements of the Bezoutian matrix*.

We now present a brief summary of the derivation of the factorization procedure. First, we notice that the conventional LDU factorization procedure can be expressed in the form

$$\boldsymbol{R}_{i+1} := \boldsymbol{R}_i - d_i \boldsymbol{l}_i \boldsymbol{l}_i^*, \qquad \boldsymbol{R}_0 := \boldsymbol{R}$$

where $\{\boldsymbol{l}_i; i = 0,1,2,\cdots\}$ denote the columns of the matrix $\boldsymbol{L}$ and $\{d_i; i = 0,1,2,\cdots\}$ denote the diagonal elements of the matrix $\boldsymbol{D}$. Also,

$$d_i := \boldsymbol{e}_i \boldsymbol{R}_i \boldsymbol{e}_i^*$$
$$\boldsymbol{l}_i := \boldsymbol{R}_i \boldsymbol{e}_i^* d_i^{-1}$$

where $\boldsymbol{e}_i$ is the unit vector

$$\boldsymbol{e}_i := [\,\underbrace{0 \ \cdots \ 0}_{i} \ 1 \ 0 \ \cdots\,].$$

This recursion can be compactly expressed in terms of generating functions by defining

$$(zw^*)^i R_i(z,w) := [1 \ z \ z^2 \ \cdots\,]\boldsymbol{R}_i[1 \ w \ w^2 \cdots\,]^*$$

and

$$z^i l_i(z) := [1 \ z \ z^2 \ \cdots\,]\boldsymbol{l}_i$$

which results in

$$(zw^*)R_{i+1}(z,w) = R_i(z,w) - R_i(z,0)R_i^{-1}(0,0)R_i(0,w). \tag{12a}$$

The components of the triangular factorization of $\boldsymbol{R}$ are obtained via the expressions

$$d_i = R_i(0,0)$$
$$l_i(z) = R_i(z,0)d_i^{-1}. \tag{12b}$$

This formulation of the factorization procedure assumes that the matrix $\boldsymbol{R}$ is *strongly regular*, i.e., that all its leading principal minors are nonsingular; it cannot be applied to matrices with singular minors. Singularities may, of course, arise in practical applications; the extensions necessary to factor imaginary-axis and unit-circle Bezoutians with arbitrary rank profiles are described elsewhere [19], [20] (see also [14], [27]).

When the recursion (12) is applied to matrices with a displacement structure (see (10)), i.e., if we assume that for $0 \leqslant i \leqslant n$

$$R_i(z,w) = \frac{G_i(z)JG_i^*(w)}{d(z,w)} \tag{13a}$$

we find that

$$zw^*R_{n+1}(z,w) = \frac{G_n(z)\left\{J - \dfrac{d(z,w)}{d(z,0)d(0,w)}JM_nJ\right\}G_n^*(w)}{d(z,w)}$$

where

$$M_n := G_n^*(0)R_n^{-1}(0,0)G_n(0).$$

If we can find a matrix function $\Theta_i(z)$ that satisfies, for all i, the identity

$$\Theta_i(z)J\Theta_i^*(w) = J - \frac{d(z,w)}{d(z,0)d(0,w)}JM_iJ \tag{13b}$$

then, in particular,

$$R_{n+1}(z,w) = \frac{G_{n+1}(z)JG_{n+1}^*(w)}{d(z,w)}$$

which proves, by induction, that (13a) holds for all i. Thus we need only to proceed with the recursion

$$zG_{i+1}(z) = G_i(z)\Theta_i(z).$$

Moreover, both $l_i(z)$ and d_i can be determined directly from $G_i(z)$ without the need to explicitly evaluate $R_i(z,w)$, viz.,

$$d_i = \lim_{z\to 0} \frac{G_i(z)JG_i^*(z)}{d(z,z)}$$
$$l_i(z) = \frac{G_i(z)JG_i^*(0)}{d(z,0)}d_i^{-1}.$$

Since $G_i(z)$ have $O(n)$ coefficients, as compared to the $O(n^2)$ coefficients of $R_i(z,w)$ (see Lemma 1 below), it is clear how we can get a reduction in computational effort by propagating $G_i(z)$ instead of $R_i(z,w)$.

We have shown in [18], [28] that the fundamental equation (13b) has a solution if, and only if, the displacement kernel $d(z,w)$ satisfies the constraint (10e), which is always satisfied by the sesquilinear kernels $d_\Omega(z,w) = \alpha + \beta z + (\beta w)^* + \delta z w^*$ (see Appendix A). An explicit form for $\Theta_i(z)$ can be deduced from the following two observations:

(i) The solution is nonunique: if $\Theta(z)$ satisfies (13b), so does $\Theta(z)U$, where U is any J-unitary matrix, i.e., $UJU^* = J$.

(ii) The matrix $\Theta(\tau)$ is J-unitary for every $\tau \in \Omega$.

Consequently, if $\Theta(z)$ satisfies (13b) so does $\Theta(z)\Theta^{-1}(\tau)$ for every $\tau \in \Omega$. In other words, (13b) always has a specific solution $\Theta(z)$ with the property $\Theta(\tau) = I$. Therefore, by setting $w = \tau$ in (13b) we obtain the explicit expressions (11d–g).

The foregoing discussion establishes Theorem 2 with the exception of (11h). To derive this we first establish the polynomial nature of $G_i(z)$ (see Appendix B for proof).

Lemma 1 ($G_i(z)$ is Polynomial): When $G_0(z) = [p(z)\ \ p^\#(z)]T$, where $p(z)$ is a polynomial of degree n, then the vector function $G_i(z)$ that is propagated by the recursion (11) is a polynomial of degree $n-i$ (or less), viz.,

$$G_i(z) = [\,p_i(z) \quad q_i(z)\,]T,$$
$$\max\{\deg p_i(z), \deg q_i(z)\} \leqslant n-i. \qquad \blacksquare$$

The lemma implies that a single step of the procedure (11) requires $O(n)$ operations. Thus the complete procedure requires $O(n^2)$ operations, in contrast to conventional techniques for evaluating inertia of matrices without displacement structure (such as LDU decomposition), which require $O(n^3)$ operations. More specifically, the exact number of computations required by our procedure is always of the form $a_0n^2 + a_1n + a_2$ with the coefficients a_0, a_1, a_2 determined by the particular choice of T, ζ_i, τ_i, U_i in the factorization procedure (11). Clever choices of the free parameters may, of course, reduce the actual computational counts.

Finally, the proof of Theorem 2 will be complete when we show that the polynomial $q_i(z)$ of Lemma 1 is, in fact, equal to $p_i^\#(z)$. However, before turning to do so it will be instructive to consider in some detail the specific choices for $\{\zeta_i, \tau_i, U_i\}$ involved in the classical Routh–Hurwitz and Schur–Cohn tests. We also decide, for the sake of concreteness and simplicity, that we shall consider in the sequel only the *scattering-type formulation* of the recursions (11), i.e., we set

$$T = I, \qquad J = \mathrm{diag}\{1, -1\}.$$

Other recursion types (including conversion to three-term recursions) can be obtained by a systematic transformation, as described in [15], [16], and [31].

Example 2.1: Tests of the Routh–Hurwitz Type

The Bezoutian with respect to the imaginary axis is obtained from (7) by setting

$$d_\Omega(z,w) = z + w^*$$

for which $p^*(z) = [p(-z^*)]^*$. For simplicity we shall only consider the problem of determining the root distribution of a polynomial with *real coefficients*, so that

$$p^\#(z) = p(-z).$$

In addition, since we begin with $T = I$, we have the classical form

$$B_0(z,w) = \frac{p(z)p^*(w) - p(-z)[p(-w)]^*}{z+w^*}.$$

For definiteness let us also begin with the simple choice $\zeta_i = 0$. Then the fundamental recursion (11) becomes

$$zG_{i+1}(z) = G_i(z)\Theta_i(z)$$

where

$$\Theta_i(z) = \left[I - \left(\frac{1}{z} + \frac{1}{\tau_i^*}\right)JM_i\right]U_i.$$

We see that the choice $\tau_i = j\infty$ (recall that τ_i must be on the $j\omega$-axis) results in a further simplification of $\Theta_i(z)$.

We can now prove by induction on the index i that the matrix U_i can be chosen in such a way that $G_i(z)$ always has the form $G_i(z) = [p_i(z) \quad p_i(-z)]$. Assuming this to be true for $1, 2, \cdots, i$ (it certainly has this form for $i = 0$) we obtain

$$G_i(0) = p_i(0)[1 \quad 1]$$

$$d_i = \lim_{z \to 0} B_i(z,z) = 2p_i(0)p_i'(0)$$

and

$$M_i = \rho_i \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\rho_i := \frac{p_i(0)}{2p_i'(0)}.$$

Therefore,

$$\Theta_i(z) = \begin{pmatrix} 1 - \rho_i z^{-1} & \rho_i z^{-1} \\ -\rho_i z^{-1} & 1 + \rho_i z^{-1} \end{pmatrix} U_i$$

where U_i can be any J-unitary matrix. The simple choice $U_i = J = \operatorname{diag}\{1, -1\}$ results in

$$G_{i+1}(z) = [p_{i+1}(z) \quad p_{i+1}(-z)],$$

which establishes (11h) for all i, so that the recursion can be continued. The fundamental recursion (11) becomes

$$zp_{i+1}(z) = p_i(z) - \rho_i z^{-1}[p_i(z) - p_i(-z)] \tag{14a}$$

$$zp_{i+1}(-z) = \rho_i z^{-1}[p_i(z) - p_i(-z)] - p_i(-z). \tag{14b}$$

Notice that the second recursion in the pair can be obtained from the first one by replacing z with $-z$, which confirms our claim about the form of $G_i(z)$. The inertia of $\boldsymbol{B}_0$ is determined by the signs of $\{d_i\}$ which are the same as the signs of the coefficients $\{\rho_i\}$, since $d_i\rho_i = |p_i(0)|^2$.

This seemingly new recursion is (slightly) different in form from but completely equivalent in the amount of computation to the well-known Routh recursion, which utilizes the even and odd parts of a given polynomial. In fact, the Routh test can be obtained from this recursion by adding and subtracting (14a) and (14b), which produces the so-called *immittance-type form* of the Bezoutian. Alternatively, it can be directly obtained by starting with $T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Either way, the immittance-type recursions are

$$f_{i+1}(z) = g_i(z)$$

$$z^2 g_{i+1}(z) = f_i(z) - 2\rho_i g_i(z)$$

where

$$f_i(z) := p_i(z) + p_i(-z)$$

$$zg_i(z) := p_i(z) - p_i(-z).$$

This can be rewritten in a three-term form, viz.,

$$z^2 f_{i+1}(z) = f_{i-1}(z) - 2\rho_{i-1} f_i(z) \tag{15}$$

which is precisely the well-known Routh algorithm [9].

Example 2.2: Tests of the Schur–Cohn Type

The Bezoutian with respect to the unit circle is obtained from (7) by setting

$$d_\Omega(z,w) = 1 - zw^*$$

for which $p^\#(z) = z^{\deg p(z)}[p(1/z^*)]^*$, the conjugate reversal of the polynomial coefficients. Since we begin with $T = I$, we have the classical form

$$B_0(z,w) = \frac{p(z)p^*(w) - p^\#(z)[p^\#(w)]^*}{1 - zw^*}.$$

Furthermore, let us again start with the simple choice $\zeta_i = 0$. The fundamental recursion (11) becomes

$$zG_{i+1}(z) = G_i(z)\Theta_i(z)$$

where

$$\Theta_i(z) = [I - (1 - z\tau_i^*)J\, M_i]U_i.$$

The choice $\tau_i = 1$ (recall that τ_i must be on the unit circle) results in a further simplification of $\Theta_i(z)$.

We can now prove by induction on the index i that the matrix U_i can be chosen in such a way that $G_i(z)$ always has the form $G_i(z) = [p_i(z) \quad p_i^\#(z)]$. Assuming this to be true for $1, 2, \cdots, i$ (it certainly has this form for $i = 0$) we obtain, in particular, $G_i(0) = [p_i(0) \quad p_i^\#(0)]$. To continue, we need to distinguish between the two cases[5] $\epsilon_i = \pm 1$ where

$$\epsilon_i := \operatorname{sgn}\{G_i(0)JG_i^*(0)\} = \operatorname{sgn}\{|p_i(0)|^2 - |p_i^\#(0)|^2\}.$$

Thus for $\epsilon_i = 1$ we observe that

$$G_i(0) = p_i(0)[1 \quad k_i]$$

$$d_i = |p_i(0)|^2(1 - |k_i|^2)$$

where $k_i := p_i^\#(0)/p_i(0)$. It follows that

$$M_i = G_i^*(0)d_i^{-1}G_i(0) = \frac{1}{1 - |k_i|^2}\begin{pmatrix} 1 & k_i \\ k_i^* & |k_i|^2 \end{pmatrix}$$

and therefore the choice

$$U_i := \frac{1}{\sqrt{1 - |k_i|^2}}\begin{pmatrix} 1 & -k_i \\ -k_i^* & 1 \end{pmatrix}$$

which satisfies the requirement $U_iJU_i^* = J$, simplifies the matrix $\Theta_i(z)$ to the form

$$\Theta_i(z) = U_i\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}.$$

This makes it possible to establish that

$$G_{i+1}(z) = [p_{i+1}(z) \quad p_{i+1}^\#(z)],$$

[5]The case $\epsilon_i = 0$ is excluded by the assumption that the Bezoutian is strongly regular.

which establishes (11h) for all i. The fundamental recursion (11) becomes

$$p_{i+1}(z)=\left(1-|k_i|^2\right)^{-1/2}\left[p_i(z)-k_i^*p_i^\#(z)\right]$$
$$zp_{i+1}^\#(z)=\left(1-|k_i|^2\right)^{-1/2}\left[p_i^\#(z)-k_ip_i(z)\right]$$

namely, a normalized form of the well-known Schur algorithm. The conventional (i.e., unnormalized) form is obtained by scaling $p_i(z)$ so as to eliminate the square-root term $(1-|k_i|^2)^{-1/2}$. This amounts to rescaling the Bezoutian $B_i(z,w)$ by a *positive* constant, which leaves the inertia of the Bezoutian unaltered. For simplicity, we denote the rescaled version of the polynomial $p_i(z)$ also by $p_i(z)$. It satisfies the recursion

$$p_{i+1}(z)=\left[p_i(z)-k_i^*p_i^\#(z)\right]$$
$$zp_{i+1}^\#(z)=\left[p_i^\#(z)-k_ip_i(z)\right].$$

Notice that the second recursion in the pair can be obtained from the first one by a conjugate reversal. Also, since $p_i^\#(z)$ is easily obtained from $p_i(z)$, only one of these recursions has to be propagated.

When $\epsilon_i=-1$ we choose $k_i:=[p_i(0)/p_i^\#(0)]^*$ (notice that $|k_i|<1$ again) and the same form for U_i as before. Since now

$$G_i(0)=p_i^\#(0)[k_i^*\quad 1]$$
$$d_i=|p_i^\#(0)|^2\left(|k_i|^2-1\right)$$

it follows that

$$M_i=G_i^*(0)d_i^{-1}G_i(0)=\frac{1}{|k_i|^2-1}\begin{pmatrix}|k_i|^2 & k_i\\ k_i^* & 1\end{pmatrix}$$

and therefore

$$\Theta_i(z)=U_i\begin{pmatrix}1 & 0\\ 0 & z\end{pmatrix}.$$

Again it follows, by induction, that $G_i(z)=[p_i(z)\quad p_i^\#(z)]$, resulting in the recursion

$$zp_{i+1}(z)=\left(1-|k_i|^2\right)^{-1/2}\left[p_i(z)-k_i^*p_i^\#(z)\right]$$
$$p_{i+1}^\#(z)=\left(1-|k_i|^2\right)^{-1/2}\left[p_i^\#(z)-k_ip_i(z)\right].$$

This recursion can also be rescaled to eliminate the square-root term.

In summary, the combined recursion is

$$p_i(z)-k_i^*p_i^\#(z)=\begin{cases}p_{i+1}(z), & \epsilon_i=1\\ zp_{i+1}(z), & \epsilon_i=-1\end{cases}\tag{16a}$$

where ϵ_i is as defined above and

$$k_i:=\begin{cases}p_i^\#(0)/p_i(0), & \epsilon_i=1\\ \left[p_i(0)/p_i^\#(0)\right]^*, & \epsilon_i=-1.\end{cases}\tag{16b}$$

The recursion for $p_i^\#(z)$ can be obtained by a conjugate reversal of (16a). The inertia of $\boldsymbol{B}_0$ is determined by ϵ_i, the signs of d_i. The knowledgeable reader will recognize that our formulation has led us to a recursion that is slightly different from the one introduced by Marden [12] and Jury [13]; the latter uses only the $\epsilon_i=1$ part of (16), but allows the magnitude of k_i to exceed unity. Consequently, their method for determining the inertia of the Bezoutian matrix is different from ours. In contrast, Cohn's original formulation [6] of the Schur–Cohn test is identical to our (16). The Cohn test has the feature that whenever $\epsilon_i=-1$, we can say that there is a root inside the unit circle; in the Jury–Marden test, we have to keep track of the variation of signs of a certain sequence. ■

As promised earlier, we now turn to establishing the property (11h) (with $T=I$) for all $d_\Omega(z,w)$. In view of Lemma 1, we only need to show that $q_i(z)=p_i^\#(z)$. We know from (11) that

$$(z-\zeta_i)\left[p_{i+1}(z)\quad q_{i+1}(z)\right]=\left[p_i(z)\quad q_i(z)\right]\Theta_i(z)$$

which can also be rearranged (via polynomial reflection) in the equivalent form

$$(z-\zeta_i)^\#\left[q_{i+1}^\#(z)\quad p_{i+1}^\#(z)\right]=\left[q_i^\#(z)\quad p_i^\#(z)\right]\left[\Theta_i(z)\right]_\#^\natural$$

where $[\Theta_i(z)]_\#$ denotes the *parareflection* of the matrix function $\Theta_i(z)$, viz.,

$$[\Theta(z)]_\#:=\left[\Theta(z^R)\right]^*\tag{17a}$$

and the superscript $\natural$ denotes transposition with respect to the antidiagonal, viz.,

$$A^\natural:=\tilde{I}A^T\tilde{I}.\tag{17b}$$

Here the superscript T denotes ordinary (nonconjugated) transposition with respect to the main diagonal, and $\tilde{I}$ denotes the matrix with unity elements along the antidiagonal and zeros elsewhere, viz.,

$$\tilde{I}:=\begin{pmatrix}0 & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0\end{pmatrix}.$$

Thus (11h) can be established by induction (recall that it holds for $i=0$) if we can ensure that for all i and for all z,

$$[\Theta_i(z)]^\natural=b_i(z)[\Theta_i(z)]_\#\tag{18a}$$

where

$$b_i(z):=\frac{z-\zeta_i}{(z-\zeta_i)^\#}\tag{18b}$$

is a (scalar) *generalized Blaschke factor* (see Section III for further discussion of its properties). In order to establish (18) we first summarize in Lemma 2 the symmetry properties of the constant matrix U_i (see Appendix B for proof).

Lemma 2 (J-Unitary Matrices): Every J-unitary matrix U (where $J:=\text{diag}\{1,-1\}$) can be decomposed in the form

$$U=\Phi\Theta\Psi$$

where

$$\Phi:=\text{diag}\{\phi,\phi^*\},\qquad |\phi|=1$$
$$\Theta:=\frac{1}{\sqrt{1-|k|^2}}\begin{pmatrix}1 & -k\\ -k^* & 1\end{pmatrix},\qquad |k|<1$$

and

$$\Psi:=\text{either diag}\{1,\psi\}\text{ or diag}\{\psi,1\},\qquad |\psi|=1.$$

Conversely, every matrix of this form is J-unitary. Moreover, the matrices Φ, Θ, Ψ satisfy the symmetry relations

$$\Phi^{\natural} = \Phi^*$$
$$\Theta^{\natural} = \Theta^*$$
$$\Psi^{\natural} = \psi \Psi^*. \qquad \blacksquare$$

We can now incorporate Lemma 2 into the characterization of the recursions (11), viz.,

$$\Theta_i(z) := [I - \lambda_i(z) J\, M_i] \Phi_i \Theta_i \Psi_i = \Phi_i \Theta_i \Lambda_i(z) \Psi_i \quad (19a)$$

where

$$\Lambda_i(z) := I - \lambda_i(z) J \xi_i^* d_i^{-1} \xi_i \quad (19b)$$

and

$$\xi_i := G_i(\zeta_i) \Phi_i \Theta_i. \quad (19c)$$

With this expression we can finally turn to establish the symmetry properties of $\Theta_i(z)$ (see Appendix B for proof).

Lemma 3 (Symmetry of $\Theta_i(z)$): The matrix function $\Theta_i(z)$ of (19) satisfies the symmetry constraint

$$[\Theta_i(z)]^{\natural} = \psi_i b_i^*(\tau_i) b_i(z) [\Theta_i(z)]_{\#}. \qquad \blacksquare$$

Thus the symmetry constraint (18) on $\Theta_i(z)$ can be satisfied by setting $\psi_i = [b_i^*(\tau_i)]^{-1} = b_i(\tau_i)$ (notice that (7d) implies that $b_i(\tau_i)$ has unit modulus). This completes the proof of (11h).

Once the extraction points ζ_i have been selected, the remaining undetermined parameters in the recursion are $\{\phi_i, k_i, \tau_i\}$. We now turn to derive some simplified forms, similar to the Routh and the Schur–Cohn recursions, by making suitable choices for these parameters.

Simplified Scattering-Type Recursions: The simplest (but not necessarily most efficient) choice is $\Phi_i = I = \Theta_i$ and $\Psi_i = \mathrm{diag}\{1, \psi_i\}$. This results in a recursion for $p_i(z)$ with two nontrivial coefficients, viz.,

$$(z - \zeta_i) p_{i+1}(z) = p_i(z) - \rho_i \lambda_i(z) [p_i(z) - k_i^* p_i^{\#}(z)] \quad (20a)$$

with $p_0(z) := p(z)$, and

$$\mathrm{sgn}\, d_i = \mathrm{sgn}\, \rho_i \quad (20b)$$

where

$$\rho_i := \frac{|p_i(\zeta_i)|^2}{d_i}$$
$$k_i := \frac{p_i^{\#}(\zeta_i)}{p_i(\zeta_i)}. \quad (20c)$$

A similar recursion holds for $p_i^{\#}(z)$; however, it is usually simpler to determine $p_i^{\#}(z)$ by applying a polynomial reflection directly to $p_i(z)$.

It appears that when $d_\Omega(\zeta_i, \zeta_i) = 0$, the computational complexity of this recursion *cannot be significantly reduced* (except by changing the type of the recursion, which we shall not consider here). In particular, if $p_i(z)$ has real coefficients and if ζ_i is real, then $k_i = \pm 1$ (under the assumption that $d_\Omega(\zeta_i, \zeta_i) = 0$) and there is only one nontrivial coefficient (i.e., ρ_i) in the recursion (20a), so that further improvement is impossible.

On the other hand, when $d_\Omega(\zeta_i, \zeta_i) \neq 0$ we can obtain a single nontrivial coefficient in the recursion even for complex polynomials by choosing $\Phi_i = I$, and

$$k_i := \begin{cases} p_i^{\#}(\zeta_i)/p_i(\zeta_i), & \epsilon_i = 1 \\ [p_i(\zeta_i)/p_i^{\#}(\zeta_i)]^*, & \epsilon_i = -1 \end{cases} \quad (21a)$$

where

$$\epsilon_i := \mathrm{sgn}\{G_i(\zeta_i) J G_i^*(\zeta_i)\} = \mathrm{sgn}\{|p_i(\zeta_i)|^2 - |p_i^{\#}(\zeta_i)|^2\}. \quad (21b)$$

A direct calculation shows that in (19b)

$$\Lambda_i(z) = \begin{cases} \mathrm{diag}\{b_i(z) b_i^*(\tau_i), 1\}, & \epsilon_i = 1 \\ \mathrm{diag}\{1, b_i(z) b_i^*(\tau_i)\}, & \epsilon_i = -1 \end{cases}$$

so that a suitably matched choice of Ψ_i eliminates the $b_i^*(\tau_i)$ term, resulting in

$$\Theta_i(z) = \begin{cases} \Theta_i \begin{pmatrix} b_i(z) & 0 \\ 0 & 1 \end{pmatrix}, & \epsilon_i = 1 \\ \Theta_i \begin{pmatrix} 1 & 0 \\ 0 & b_i(z) \end{pmatrix}, & \epsilon_i = -1. \end{cases}$$

Finally, we can remove the square-root factor $(1 - |k_i|^2)^{-1/2}$ from Θ_i, which establishes the unnormalized form

$$p_{i+1}(z) = \begin{cases} \dfrac{p_i(z) - k_i^* p_i^{\#}(z)}{(z - \zeta_i)^{\#}}, & \epsilon_i = 1 \\ \dfrac{p_i(z) - k_i^* p_i^{\#}(z)}{z - \zeta_i}, & \epsilon_i = -1 \end{cases} \quad (21c)$$

with $p_0(z) := p(z)$. Also, since $d_i = G_i(\zeta_i) J G_i^*(\zeta_i) / d_\Omega(\zeta_i, \zeta_i)$ it follows that

$$\mathrm{sgn}\, d_i = \epsilon_i\, \mathrm{sgn}\{d_\Omega(\zeta_i, \zeta_i)\}. \quad (21d)$$

Notice that the undetermined parameter τ_i has been completely eliminated from the recursions. The recursion for $p_i^{\#}(z)$ can be obtained (when necessary) by a polynomial reflection of (21c).

We demonstrate the utility of these generalized recursions by specializing them to obtain *new procedures for zero-location* with respect to the unit circle and the imaginary axis. First, we present a unit circle procedure that bears a much closer resemblance to the Routh test than either the Schur–Cohn or the Bistritz tests. It is equivalent (in a sense defined in [36]) to the bilinear-Routh procedures presented in [33]–[35].

Example 2.3 (A New Unit-Circle Algorithm)

This new algorithm is obtained from (20) by setting

$$d_\Omega(z, w) = 1 - z w^*$$
$$\zeta_i = 1$$
$$\tau_i = -1.$$

Consequently, $p^{\#}(z) = z^{\deg p(z)} [p(1/z^*)]^*$, which implies that for polynomials with *real coefficients*

$$k_i = p_i^{\#}(1)/p_i(1) = 1$$

so that (20) reduces to

$$(z - 1) p_{i+1}(z) = p_i(z) - \rho_i \frac{1 + z}{2(1 - z)} [p_i(z) - p_i^{\#}(z)]. \quad (22a)$$

A conjugate reversal produces the dual recursion

$$(1-z)p_{i+1}^{\#}(z) = p_i^{\#}(z) - \rho_i \frac{1+z}{2(1-z)}\left[p_i(z) - p_i^{\#}(z)\right]. \tag{22b}$$

As in Example 2.1 (the Routh algorithm) we may take advantage of the fact that $k_i = 1$ to introduce *immittance-type* variables, viz.,

$$f_i(z) := p_i(z) + p_i^{\#}(z)$$
$$g_i(z) := \frac{p_i(z) - p_i^{\#}(z)}{z-1}.$$

Notice that $g_i(z)$ is a *polynomial* (because $p_i(1) = p_i^{\#}(1)$), and that both $f_i(z)$ and $g_i(z)$ are *symmetric with respect to polynomial reflection*, namely,

$$f_i^{\#}(z) = f_i(z)$$
$$g_i^{\#}(z) = g_i(z).$$

We can now combine (22a and b) into a (two-term) recursion for the immittance-type variables, viz.,

$$f_{i+1}(z) = g_i(z)$$
$$(z-1)^2 g_{i+1}(z) = f_i(z) - \rho_i \frac{z+1}{2} g_i(z). \tag{23a}$$

or, into a three-term recursion for $f_i(z)$, viz.,

$$(z-1)^2 f_{i+1}(z) = f_{i-1}(z) - \rho_{i-1}\frac{z+1}{2} f_i(z)$$
$$\rho_{i-1} = \frac{f_{i-1}(1)}{f_i(1)}. \tag{23b}$$

Finally, the inertia of the Bezoutian is determined by (20b), viz.,

$$\operatorname{sgn} d_i = \operatorname{sgn} \rho_i. \tag{23c}$$

While these recursions resemble very closely the Routh algorithm, there are two important differences: i) they involve a nontrivial division by $(z-1)^2$ at each step of the recursion, and ii) the polynomials $f_i(z)$ are not "even" (i.e., none of their coefficients vanish). The division by $(z-1)^2$ can be accomplished with $2(n-i)$ additions per recursion step (and no multiplications) so that the entire procedure still requires $O(n^2)$ multiplications and $O(n^2)$ additions. Notice also that since $f_i(z)$ are symmetric polynomials, only half of their coefficients need to be propagated.

The similarity to the Routh algorithm becomes even more apparent if we choose to expand the polynomials $f_i(z)$ in the form

$$f_i(z) = \sum_{k=0}^{n-i} f_{i,k}(z-1)^k\left(\frac{z+1}{2}\right)^{-k+\deg f_i(z)}. \tag{24a}$$

The odd numbered coefficients in this expansion vanish, so $f_i(z)$ can be considered an "even" polynomial. The corresponding table form recursion is

$$f_{i+1,k-2} = f_{i-1,k} + 2\rho_{i-1} f_{i,k}, \qquad i \geqslant 1,\ k \geqslant 2 \tag{24b}$$

which is identical to the table form of the Routh algorithm (here also only the even coefficients are propagated). However, while the computational complexity of (24a) is comparable to that of (23b), the need to expand $p(z)$ in the form (24a) and to determine the expansion coefficients of $f_0(z)$ and of $f_1(z) = g_0(z)$ makes the overall computational requirements of (24) significantly higher than those associated with propagating (23) in terms of the conventional expansion of the polynomials $f_i(z)$. ∎

Next we turn to present an imaginary axis procedure that resembles the Schur–Cohn test and is equivalent to the tests presented by Reddy and Rajan [32] and by Lepschy *et al.* [30].

Example 2.4 (A New Imaginary-Axis Algorithm)

This new algorithm is obtained from (21) by setting

$$d_\Omega(z,w) = z + w^*$$
$$\zeta_i = 1.$$

Consequently, $p^{\#}(z) = [p(-z^*)]^*$, (as in Example 2.1) so that (21) reduces to

$$p_{i+1}(z) = -\frac{p_i(z) - k_i^* p_i^{\#}(z)}{1+\epsilon_i z}$$

where k_i is given by (21a). We can eliminate the minus sign without affecting the validity of the procedure (this corresponds to choosing $\Phi_i = -I$ instead of $\Phi_i = I$ in the derivation of the simplified recursions (21)). Taking into consideration the definition of $p^{\#}(z)$, this results in

$$p_{i+1}(z) = \frac{p_i(z) - k_i^*[p_i(-z^*)]^*}{1+\epsilon_i z} \tag{25a}$$

where

$$\epsilon_i := \operatorname{sgn}\{|p_i(1)|^2 - |p_i(-1)|^2\} \tag{25b}$$

and

$$k_i := \begin{cases} p_i(-1)/p_i(1), & \epsilon_i = 1 \\ [p_i(1)/p_i(-1)]^*, & \epsilon_i = -1 \end{cases}. \tag{25c}$$

The inertia of the associated Bezoutian is determined via (21d), which simplifies to

$$\operatorname{sgn} d_i = \epsilon_i. \tag{25d}$$

A necessary and sufficient condition for stability is that $\epsilon_i = 1$ for all i.

This new procedure differs from the one presented by Lepschy *et al.* [30] in the same manner that our version of the Schur–Cohn procedure (Example 2.2) differs from the Marden–Jury algorithm:

i) Our k_i are always bounded by unity.
ii) We determine inertia via the signs of ϵ_i, whereas Marden–Jury and Lepschy *et al.* determine it via the signs of the products $\prod_{j=0}^{i}(1-|k_j|^2)$.

When used as a stability test our procedure is carried out only so long as $\epsilon_i = 1$; in this case it completely coincides with the test presented by Reddy and Rajan [32].

III. Polynomial Reflection and Generalized Bezoutians

The properties of an Ω-Bezoutian, as defined in the introduction by (7) are determined by the polynomial pair $G(z) := [p(z) \quad q(z)]$ and by the denominator function

$d_\Omega(z,w)$ of (6). We shall first analyze some properties of $d_\Omega(z,w)$ that are fundamental to the proof of Theorem 1.

As stated in the introduction, the inequality $d_\Omega(z,z)>0$ characterizes a domain Ω_+ in the complex plane whose boundary Ω is either a circle or a straight line. We summarize in Appendix A several well-known facts from the so called Hermitian theory of circles (see, e.g. Schwerdtfeger [25]). The most important of these is the notion of *point reflection in a circle* (or in a straight line). The point reflection of z, which we denote by z^R, has been defined as the solution to the equation

$$d_\Omega(z,z^R)=0. \tag{26a}$$

Notice that point reflection is an involution, i.e., $(z^R)^R=z$, and that $z^R=z$ if, and only if, $z\in\Omega$. The explicit expression for z^R follows from (26a), viz.,

$$z^R=-\left(\frac{\alpha+\beta z}{\beta^*+\delta z}\right)^*. \tag{26b}$$

Polynomial reflection, which we denote by a "sharp" (#), is a transformation that maps the zeros of a polynomial $p(z)$ into their reflections, i.e., $p^\#(z)$ is a polynomial of the same degree as $p(z)$ and

$$p(z)=0\Leftrightarrow p^\#(z^R)=0.$$

This characterization determines $p^\#(z)$ uniquely up to a scaling operation,

$$p^\#(z)\sim(\beta^*+\delta z)^{\deg p(z)}[p(z^R)]^*.$$

Therefore, we *define* the polynomial reflection $p^\#(z)$ via the following explicit expression.

$$p^\#(z):=\left(\chi\frac{\beta^*+\delta z}{\sqrt{|\beta|^2-\alpha\delta}}\right)^{\deg p(z)}[p(z^2)]^* \tag{27a}$$

where χ is an arbitrary unit modulus constant, viz.,

$$|\chi|=1. \tag{27b}$$

For instance, the unit circle is characterized by $d_T(z,w)=1-zw^*$ and the corresponding polynomial reflection is

$$p^\#(z):=z^{\deg p(z)}[p(1/z^*)]^*$$

which requires choosing $\chi=-1$ in (27a). On the other hand, polynomial reflection with respect to the real line, which is characterized by $d_R(z,w)=j(z-w^*)$, is given by

$$p^\#(z)=[p(z^*)]^*$$

and requires choosing $\chi=j$ in (27a).

The following properties of polynomial reflection follow from the definition (27) and the identities (A.6) and (A.7):

$$[p^\#(z)]^\#=p(z) \tag{28a}$$

$$[\alpha p(z)]^\#=\alpha^*p^\#(z) \tag{28b}$$

$$[p(z)q(z)]^\#=p^\#(z)q^\#(z) \tag{28c}$$

$$|p^\#(z)|=|p(z)|,\qquad \text{for all } z\in\Omega. \tag{28d}$$

Notice that $z^\#\neq(z^R)^*$; while $z^\#$ is a polynomial of degree 1, the conjugated point-reflection $(z^R)^*$ is a ratio of two polynomials of degree 1. A polynomial $p(z)$ is called (conjugate) *symmetric* if $p(z)=p^\#(z)$. This can happen if, and only if, the set of its zeros (and their multiplicities) is invariant under point reflection. This implies that zeros are either on the boundary Ω or can be arranged in pairs that are symmetric with respect to Ω (in the sense of point-reflection).

It follows from (28d) that the ratio $p(z)/p^\#(z)$ has unit modulus for all $z\in\Omega$. For $p(z)=z-\zeta$ this ratio is a *generalized Blaschke factor*, viz.,

$$b_\zeta(z):=\frac{z-\zeta}{(z-\zeta)^\#}. \tag{29a}$$

When $d(z,w)=1-zw^*$ we recognize this function as the conventional Blaschke factor with respect to the unit circle; similarly, $d_\Omega(z,w)=z+w^*$ produces the conventional Blaschke factor with respect to the imaginary axis. Since the Blaschke factor is a rational function of degree 1, it maps circles into circles. More specifically, for $\zeta\in\Omega_+$ it maps Ω_+ onto the inside of the unit circle, and Ω_- onto the outside of the unit circle, while the boundary Ω is mapped onto the unit circle itself.

This property of the Blaschke factor follows from the fundamental identity

$$\frac{d_\Omega(z,w)d_\Omega(\zeta,\zeta)}{d_\Omega(z,\zeta)d_\Omega(\zeta,w)}=1-b_\zeta(z)b_\zeta^*(w) \tag{29b}$$

which implies, in particular, that

$$1-|b_\zeta(z)|^2=\frac{d_\Omega(z,z)d_\Omega(\zeta,\zeta)}{|d_\Omega(z,\zeta)|^2}.$$

Therefore, for every $\zeta\in\Omega_+$,

$$\begin{aligned}|b_\zeta(z)|&<1, &\quad z&\in\Omega_+\\ |b_\zeta(z)|&=1, & z&\in\Omega\\ |b_\zeta(z)|&>1, & z&\in\Omega_-\end{aligned} \tag{29c}$$

while the reverse inequalities hold for $\zeta\in\Omega_-$. The identity (29b) is established by direct calculation using the intermediate (and directly verifiable) result

$$d_\Omega(z,\zeta)b_\zeta(z)=-\chi^*(z-\zeta)\sqrt{|\beta|^2-\alpha\delta}. \tag{29d}$$

Also notice that when $\zeta\in\Omega$ the rational function $b_\zeta(z)$ becomes independent of z. This is a direct consequence of the fact that $b_\zeta(z)$ is analytic for every $z\neq\zeta^R$, while on the other hand $|b_\zeta(z)|=1$ for all z.

It follows from the definition of point- and polynomial-reflection that the numerator of $B_\Omega^{p,q}(z,w)$ of (7) vanishes for $w=z^R$. This is so because

$$[p^\#(z^R)]^*=\left[\left(\chi\frac{\beta^*+\delta z^R}{\sqrt{|\beta|^2-\alpha\delta}}\right)^{\deg p(z)}\right]^*p(z)$$

and, therefore,

$$\begin{aligned}p(z)[q^\#(z^R)]^*-q(z)[p^\#(z^R)]^*\\ \sim p(z)q(z)-q(z)p(z)=0.\end{aligned}$$

Since also $d_\Omega(z,z^R)=0$ it follows that $d_\Omega(z,w)$ divides the numerator of $B_\Omega^{p,q}(z,w)$ and, therefore, that $B_\Omega^{p,q}(z,w)$ is a *polynomial in z and in* w^*.

We now turn to exploring the role of the *gcd* of $p(z),q(z)$ in (7). First we establish the notion of *congruence* for generating functions.

Lemma 4 (Congruence): Let $R(z,w)$ be the generating function corresponding to a (possibly infinite) matrix $\boldsymbol{R}$, viz.,

$$R(z,w)=[1\ z\ z^2\ \cdots]\boldsymbol{R}[1\ w\ w^2\ \cdots]^*$$

and let $f(z), g(z)$ be arbitrary power series, viz., $f(z)=\sum_{i=0}^{\infty} f_i z^i$, $g(z)=\sum_{i=0}^{\infty} g_i z^i$. Then

$$f(z)R(z,w)g^*(w)=[1\ z\ z^2\ \cdots]\cdot \boldsymbol{L}(f)\boldsymbol{R}\boldsymbol{L}^*(g)[1\ w\ w^2\ \cdots]^* \quad (30a)$$

where $\boldsymbol{L}(f)$ denotes a lower-triangular Toeplitz matrix whose first column is determined by the coefficients of $f(z)$, viz.,

$$\boldsymbol{L}(f)=\begin{pmatrix} f_0 & & & 0\\ f_1 & f_0 & & \\ f_2 & f_1 & f_0 & \\ \vdots & & & \ddots \end{pmatrix}. \quad (30b)$$

Proof: The result follows directly from the observation that

$$[1\ z\ z^2\ \cdots]\boldsymbol{L}(f)=f(z)[1\ z\ z^2\ \cdots]$$

and similarly for $\boldsymbol{L}(g)$. ∎

The lemma implies that if $p(z), q(z)$ have a common divisor $r(z)$, viz.,

$$p(z)=\tilde{p}(z)r(z)$$
$$q(z)=\tilde{q}(z)r(z)$$

then the corresponding Bezoutian $\boldsymbol{B}_{\Omega}^{p,q}$ is congruent (in the sense of Lemma 4) to the Bezoutian $\boldsymbol{B}_{\Omega}^{\tilde{p},\tilde{q}}$ of $\tilde{p}(z), \tilde{q}(z)$. This is so because

$$B_{\Omega}^{p,q}(z,w)=r(z)B_{\Omega}^{\tilde{p},\tilde{q}}(z,w)[r^{\#}(w)]^*$$

which implies that

$$\boldsymbol{B}_{\Omega}^{p,q}=\boldsymbol{L}(r)\boldsymbol{B}_{\Omega}^{\tilde{p},\tilde{q}}\boldsymbol{L}^*(r^{\#}) \quad (31a)$$

and therefore

$$rank\,\boldsymbol{B}_{\Omega}^{p,q}=rank\,\boldsymbol{B}_{\Omega}^{\tilde{p},\tilde{q}}\leqslant \deg\tilde{p}(z)=\deg\tilde{q}(z). \quad (31b)$$

In order to establish part i) of Theorem 1 we need to show that $\boldsymbol{B}_{\Omega}^{\tilde{p},\tilde{q}}$ indeed has full rank. To this end we first establish an additive decomposition of $B_{\Omega}^{p,q}(z,w)$ as follows. Since $p(z)$ and $q(z)$ in (7) have the same degree n we can factor each one into a product of n polynomials of degree 1, say,

$$p(z)=p_1(z)p_2(z)\cdots p_n(z), \qquad \deg p_i(z)=1 \quad (32a)$$
$$q(z)=q_1(z)q_2(z)\cdots q_n(z), \qquad \deg q_i(z)=1. \quad (32b)$$

The following result establishes an additive decomposition of $B_{\Omega}^{p,q}(z,w)$ in terms of the elementary Bezoutians $B_{\Omega}^{p_i,q_i}(z,w)$.

Lemma 5 (Additive Decomposition—Polynomial Form): The Bezoutian $B_{\Omega}^{p,q}(z,w)$ satisfies the identity

$$B_{\Omega}^{p,q}(z,w)=\sum_{i=1}^{n}\phi_{n,i}(z)B_{\Omega}^{p_i,q_i}(z,w)\psi_{n,i}^*(w) \quad (33a)$$

where

$$\phi_{n,i}(z):=p_1(z)p_2(z)\cdots p_{i-1}(z)q_{i+1}(z)\cdots q_n(z) \quad (33b)$$
$$\psi_{n,i}(z):=q_1^{\#}(z)q_2^{\#}(z)\cdots q_{i-1}^{\#}(z)p_{i+1}^{\#}(z)\cdots p_n^{\#}(z). \quad (33c)$$

Proof: The proof given in Appendix B is based on a simple additive decomposition of the numerator of $B_{\Omega}^{p,q}(z,w)$. This approach is attributed to Liénard and Chipart [26] (see also section III in [5]) and has also been used by Pták and Young to obtain an extension of the Schur–Cohn test [23]. Its strength lies in the fact that it does not involve at all the denominator function of $B_{\Omega}^{p,q}(z,w)$, and therefore applies to all curves for which the definition (7) determines a finite Bezoutian matrix. ∎

We can now determine the rank of $\boldsymbol{B}_{\Omega}^{p,q}$ by considering the matrix form equivalent of Lemma 5 (see Appendix B for proof).

Lemma 6 (Additive Decomposition—Matrix Form): The Bezoutian matrix $\boldsymbol{B}_{\Omega}^{p,q}$ satisfies the identity

$$\boldsymbol{B}_{\Omega}^{p,q}=\sum_{i=1}^{n}\kappa_i\boldsymbol{\phi}_{n,i}\boldsymbol{\psi}_{n,i}^* \quad (34a)$$

where $\boldsymbol{\phi}_{n,i}$ (resp. $\boldsymbol{\psi}_{n,i}$) is the column vector consisting of the coefficients of the polynomial $\phi_{n,i}(z)$ (resp. $\psi_{n,i}(z)$), which is defined via (32) and (33). Also,

$$\kappa_i=-\frac{q_i(\zeta_i)[p_i^{\#}(\zeta_i)]^*}{d(\zeta_i,\zeta_i)} \quad (34b)$$

where ζ_i is the zero of $p_i(z)$. Moreover, if $gcd\{p(z),q(z)\}=1$, then the vectors $\{\boldsymbol{\phi}_{n,i}\}$ are linearly independent and so are the vectors $\{\boldsymbol{\psi}_{n,i}\}$. ∎

We are now ready for the proof of Theorem 1.

Proof of Theorem 1: Combining (31) with Lemma 6 we conclude that

$$\boldsymbol{B}_{\Omega}^{p,q}=\boldsymbol{L}(r)\left\{\sum_{i=1}^{n-\eta}\kappa_i\boldsymbol{\phi}_{n,i}\boldsymbol{\psi}_{n,i}^*\right\}\boldsymbol{L}^*(r^{\#})$$

where $n:=\max\{\deg p(z),\deg g(z)\}$, $r(z):=gcd\{p(z),q(z)\}$, and $\eta:=\deg r(z)$. Since by Lemma 6 the column vectors $\boldsymbol{\phi}_{n,i}$ (resp. $\boldsymbol{\psi}_{n,i}$) are linearly independent, it follows that $rank\,\boldsymbol{B}_{\Omega}^{p,q}=n-\eta$, which establishes part i) of Theorem 1.

When $q(z)=p^{\#}(z)$, the Bezoutian $\boldsymbol{B}_{\Omega}^{p,p^{\#}}$ is *Hermitian* and has real eigenvalues. Also

$$r^{\#}(z)=gcd\{p^{\#}(z),q^{\#}(z)\}=gcd\{p^{\#}(z),p(z)\}=r(z)$$

and $\phi_{n,i}(z)=\psi_{n,i}(z)$. Consequently,

$$\boldsymbol{B}_{\Omega}^{p,p^{\#}}=\boldsymbol{L}(r)\left\{\sum_{i=1}^{n-\eta}\kappa_i\boldsymbol{\phi}_{n,i}\boldsymbol{\phi}_{n,i}^*\right\}\boldsymbol{L}^*(r)$$

which proves that the number of positive and negative eigenvalues of $\boldsymbol{B}_{\Omega}^{p,p^{\#}}$ is determined by the signs of the scalar coefficients κ_i. Since (34b) reduces in this case to

$$\kappa_i=-\frac{|q_i(\zeta_i)|^2}{d_{\Omega}(\zeta_i,\zeta_i)}$$

it follows that $\operatorname{sgn}\kappa_i=-\operatorname{sgn}d_{\Omega}(\zeta_i,\zeta_i)$, which establishes part ii) of Theorem 1. ∎

An alternative proof of Theorem 1 can be based on the following interesting property of the function $d_{\Omega}(z,w)$ (see Appendix B for proof).

Lemma 7: Let $\{\zeta_1, \zeta_2, \cdots, \zeta_n\}$ be *distinct* points in the complex plane that satisfy the constraint $d_\Omega(\zeta_i, \zeta_j) \neq 0$ for all $1 \leqslant i, j \leqslant n$. Then

$$In\left\{\frac{1}{d_\Omega(\zeta_i,\zeta_j)}, \quad 1 \leqslant i \leqslant n\right\} = In \operatorname{diag}\{d_\Omega(\zeta_i,\zeta_i), 1 \leqslant i,j \leqslant n\}. \quad (35) \blacksquare$$

Part ii) of Theorem 1 can be easily deduced from Lemma 7, as follows. Let $p(z)$ be a polynomial whose zeros ζ_i satisfy the assumptions of Lemma 7, i.e., they are distinct and $d_\Omega(\zeta_i, \zeta_j) \neq 0$. As a consequence $p^\#(\zeta_i) \neq 0$ and, therefore,

$$B_\Omega^{p,p^\#}(\zeta_i,\zeta_j) = -\frac{p^\#(\zeta_i)[p^\#(\zeta_j)]^*}{d_\Omega(\zeta_i,\zeta_j)}.$$

Writing this in matrix form (for $1 \leqslant i, j \leqslant n$) we obtain

$$\begin{aligned}\left\{B_\Omega^{p,p^\#}(\zeta_i,\zeta_j), \quad 1 \leqslant i,j \leqslant n\right\} &= \boldsymbol{V}\boldsymbol{B}_\Omega^{p,p^\#}\boldsymbol{V}^* \\ &= -\boldsymbol{D}\left\{\frac{1}{d_\Omega(\zeta_i,\zeta_j)}, 1 \leqslant i,j \leqslant n\right\}\boldsymbol{D}^*\end{aligned}$$

where $\boldsymbol{D} := \operatorname{diag}\{p^\#(\zeta_i), 1 \leqslant i \leqslant n\}$ and $\boldsymbol{V}$ is a *Vandermonde matrix*, viz.,

$$\boldsymbol{V} := \begin{pmatrix} 1 & \zeta_1 & \zeta_1^2 & \cdots & \zeta_1^{n-1} \\ \vdots & & & & \\ 1 & \zeta_n & \zeta_n^2 & \cdots & \zeta_n^{n-1}\end{pmatrix}.$$

Since both $\boldsymbol{D}$ and $\boldsymbol{V}$ are nonsingular, it follows from Lemma 7 that $In\boldsymbol{B}_\Omega^{p,p^\#} = In \operatorname{diag}\{d_\Omega(\zeta_i,\zeta_i), 1 \leqslant i \leqslant n\}$, which establishes part ii) of Theorem 1. Part i) can be established in a similar manner. Also, this approach can be extended to the case when the zeros of $p(z)$ are not necessarily distinct. This involves the notion of divided-difference operators, which is beyond the scope of this paper (see, e.g., [17]).

IV. Concluding Remarks

We have presented in this paper a unified explicit expression for Bezoutians with respect to arbitrary circles and lines in the complex plane, and we have established their rank and inertia properties. We have also indicated that such generalized Bezoutian matrices satisfy a variety of Lyapunov-type equations, including those studied by Gutman and Jury [21].

By recognizing generalized Bezoutians as a particular instance of matrices with a *displacement structure*, we could apply the efficient factorization procedure of [18] to determine the rank and inertia of generalized Bezoutians. Thus the location of zeros of a given polynomial $p(z)$ of degree n with respect to a circle or a straight line Ω can be determined in $O(n^2)$ computations *without explicitly evaluating the elements of the Bezoutian matrix* $\boldsymbol{B}_\Omega^{p,p^\#}$.

Our formulation yields, in fact, a large variety of $O(n^2)$ procedures, one for each choice of certain unspecified parameters $\{T, \zeta_i, \tau_i, U_i\}$ in the factorization procedure (11). Special choices can make various improvements in the exact number of computations. The well-known classical procedures and several new variants thereof are also obtained by making specific choices (see Examples 2.1–2.4).

As we mentioned in Section II, our formulation (11) applies only to *strongly regular* Bezoutian matrices. Nevertheless, this formulation can be modified to accommodate the possible occurrence of singular minors in the Bezoutian. Generalizations of our fast factorization procedures for non-strongly regular Bezoutians on the unit circle and imaginary axis have been recently developed, and will be discussed separately (see Pal and Kailath [19], [20]).

Appendix A
The Hermitian Theory of Circles

We briefly review here some known mathematical results (see, e.g., [25]) that are relatively unknown in engineering circles.

The Partition $\{\Omega_+, \Omega, \Omega_-\}$

The equation $d_\Omega(z,z) > 0$ determines a domain Ω_+ whose boundary Ω is either a circle (with nonzero radius) or a straight line if, and only if, $\det J_d < 0$, i.e.,

$$d_\Omega(z,w) = [1 \quad z]J_d[1 \quad w]^*$$

$$J_d = \begin{pmatrix}\alpha & \beta^* \\ \beta & \delta\end{pmatrix}$$

$$InJ_d = \{1,1\}. \quad (A.1)$$

Notice that since J_d is Hermitian, its diagonal elements (i.e., α and δ) must be real.

Assuming that (A.1) holds, we obtain a partitioning of the complex plane into three mutually exclusive domains $\{\Omega_+, \Omega, \Omega_-\}$, with Ω being the common boundary of Ω_+ and Ω_-. Notice that if J_d determines a partition $\{\Omega_+, \Omega, \Omega_-\}$, then $-J_d$ determines the dual partition $\{\Omega_-, \Omega, \Omega_+\}$.

It will be helpful to rewrite $d_\Omega(z,w)$ in the form

$$d_\Omega(z,w) = \frac{(\beta^* + \delta z)(\beta^* + \delta w)^* - (|\beta|^2 - \alpha\delta)}{\delta}. \quad (A.2a)$$

Thus when $\delta \neq 0$, the curve Ω is a circle with center at $-\beta^*/\delta$ and radius $\sqrt{|\beta|^2 - \alpha\delta}\,/\,|\delta|$. The domain Ω_+ is the *inside* of this circle when $\delta < 0$, but it is the outside of the same circle when $\delta > 0$. When $\delta = 0$, the curve Ω is a straight line that cuts the axes at the points

$$\left(-\frac{\alpha}{2\operatorname{Re}\beta}, 0\right) \quad \text{and} \quad \left(0, \frac{\alpha}{2\operatorname{Im}\beta}\right).$$

The vector $-j\beta^*$ is orthogonal to the boundary line Ω, and points toward the domain Ω_+.

Since $InJ_d = \{1,1\}$ it follows that there exists a (nonunique) nonsingular matrix $T = \begin{pmatrix}a & b \\ c & d\end{pmatrix}$ such that $J_d = T\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}T^*$. This implies that

$$\begin{aligned}d_\Omega(z,w) &= [1 \quad z]T\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}T^*[1 \quad w]^* \\ &= (a + cz)(a + cw)^* - (b + dz)(b + dw)^*\end{aligned}$$

namely,

$$d_\Omega(z,w) = d_1(z)d_1^*(w) - d_2(z)d_2^*(w). \quad \text{(A.2b)}$$

In particular, when $\delta > 0$, it follows from (A.2a) that (A.2b) can be satisfied with $d_1(z) = (\sqrt{|\delta|})^{-1}(\beta^* + \delta z)$, $d_2(z) = \sqrt{|\beta|^2 - \alpha\delta}$ (and the other way around when $\delta < 0$). When $\delta = 0$ (but $\beta \neq 0$) we can satisfy (A.2b) by choosing $d_1(z) = \beta z + 1/2(\alpha+1)$ and $d_2(z) = \beta z + 1/2(\alpha - 1)$.

Point Reflection

The point reflection of z with respect to a circle (or a straight line) described by $d_\Omega(z,w) = \alpha + \beta z + (\beta w)^* + \delta z w^*$ is defined as (another) point z^R that satisfies the equation

$$d_\Omega(z, z^R) = 0. \quad \text{(A.3)}$$

The explicit solution to this equation is

$$z^R := -\left(\frac{\alpha + \beta z}{\delta z + \beta^*}\right)^* \quad \text{(A.4)}$$

which is a bilinear transformation followed by conjugation. It maps Ω_+ into Ω_- (and vice versa) and Ω into itself. Notice that point reflection is an involution, i.e.,

$$(z^R)^R = z. \quad \text{(A.5)}$$

For straight lines it becomes the usual reflection in a line.

The representation (A.2) implies two useful identities, as follows. Combining (A.2) and (A.3) we find that

$$(\beta^* + \delta z)(\beta^* + \delta z^R)^* = |\beta|^2 - \alpha\delta. \quad \text{(A.6)}$$

When $z \in \Omega$, this specializes to

$$|\beta^* + \delta z| = \sqrt{|\beta|^2 - \alpha\delta}, \qquad \text{for } z \in \Omega. \quad \text{(A.7)}$$

This identity is used in Section III in the scaling of the polynomial reflection operation.

Appendix B
Proofs of Lemmas

Proof of Lemma 1

Substituting the expression (11d) for $\Theta_i(z)$ in (13b), we obtain a corresponding identity for the scalar function $\lambda_i(z)$, viz.,

$$\lambda_i(z) + [\lambda_i(z)]_\# - d(\zeta_i,\zeta_i)\lambda_i(z)[\lambda_i(z)]_\# = 0. \quad \text{(B.1)}$$

Now, it follows from (13b) that for all i and for all z

$$G_i(z)J[G_i(z)]_\# = G_0(z)J[G_0(z)]_\# = 0$$

which implies that the denominator of the generating function

$$B_i(z,w) := \frac{G_i(z)JG_i^*(w)}{d_\Omega(z,w)}$$

divides the numerator. Consequently, *if we assume* that $G_i(z)$ is a polynomial of degree δ_i then it follows that $B_i(z,w)$ is a (bivariate) polynomial of degree $\delta_i - 1$. Moreover, since the recursion (11) for $G_i(z)$ can also be expressed in the form

$$(z - \zeta_i)G_{i+1}(z) = \left[G_i(z) - \frac{d_\Omega(z,\tau_i)}{d_\Omega(\zeta_i,\tau_i)}B_i(z,\zeta_i)G_i(\zeta_i)\right]U_i$$

it follows that, under the same assumption, $G_{i+1}(z)$ is a polynomial of degree $\delta_{i+1} \leqslant \delta_i - 1$. Since $G_0(z)$ is, indeed, a polynomial of degree $\delta_0 = n$ it follows, by induction, that $G_i(z)$ is a polynomial of degree $n - i$.

Proof of Lemma 2

Denote the elements of U by a, b, c, d, viz.,

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and observe that $UJU^* = J$ translates into

$$|a|^2 - |b|^2 = 1$$
$$|d|^2 - |c|^2 = 1$$
$$ac^* = bd^*.$$

Since this implies that $a \neq 0$ and $d \neq 0$, we can introduce

$$\rho := \frac{b}{a} = \left(\frac{c}{d}\right)^*$$

which, in turn, implies that $|\rho| < 1$ and that $|a| = |d| = (1 - |\rho|^2)^{-1/2}$. Thus the matrix U can be expressed in the form

$$U = \begin{pmatrix} \nu_a & 0 \\ 0 & \nu_d \end{pmatrix} \frac{1}{\sqrt{1 - |\rho|^2}} \begin{pmatrix} 1 & \rho \\ \rho^* & 1 \end{pmatrix}$$

where $\nu_a := a/|a|$ and $\nu_d := d/|d|$. This representation shows that U is determined by *four independent real parameters*: the real and imaginary parts of ρ and the phases of the unit modulus coefficients ν_a and ν_d. The expression given in Lemma 2, with $\Psi = \operatorname{diag}\{1, \psi\}$ now follows by letting $\phi := \nu_a$, $\psi := \nu_a\nu_d$, and $k := \rho(\nu_a\nu_d)^*$. The dual expression, with $\Psi = \operatorname{diag}\{\psi, 1\}$, is similarly obtained by letting $\phi := \nu_d^*$, $\psi := \nu_a\nu_d$, and $k := \rho\nu_a\nu_d$. The rest of the statements in the Lemma follows by observation.

Proof of Lemma 3

Taking into account the decomposition (19) and the symmetry properties listed in Lemma 2, we observe that

$$[\Theta(z)]^\natural = \Psi^\natural\Lambda^\natural(z)\Theta^\natural\Phi^\natural = \psi\Psi^*\Lambda^\natural(z)\Theta^*\Phi^*$$

where, for the sake of convenience, we have omitted everywhere the subscript i. Also, observe that

$$[\Theta(z)]_\# = \Psi^*[\Lambda(z)]_\#\Theta^*\Phi^*$$

so that Lemma 3 will be established when we show that

$$\Lambda^\natural(z) = b^*(\tau)b(z)[\Lambda(z)]_\#. \quad \text{(B.2)}$$

A direct calculation shows that $J^\natural = -J$ as well as

$$(\xi^*d^{-1}\xi)^\natural = \xi^*d^{-1}\xi - d(\zeta,\zeta)J$$

where we used the fact that, from (19c),

$$\xi J \xi^* = G(\zeta) J G^*(\zeta) = d(\zeta,\zeta) d.$$

Therefore, from (19b),

$$\begin{aligned}\Lambda^{\natural}(z) &= I - \lambda(z)(\xi^* d^{-1}\xi)^{\natural} J^{\natural} \\ &= I + \lambda(z)[\xi^* d^{-1}\xi - d(\zeta,\zeta) J] J \\ &= [1 - \lambda(z) d(\zeta,\zeta)] I + \lambda(z)\xi^* d^{-1}\xi J.\end{aligned}$$

On the other hand,

$$\begin{aligned}[\Lambda(z)]_{\#} &= I - [\lambda(z)]_{\#}\xi^* d^{-1}\xi J \\ &= [1 - \lambda(z) d(\zeta,\zeta)]^{-1} \\ &\quad \cdot \{[1 - \lambda(z) d(\zeta,\zeta)] I + \lambda(z)\xi^* d^{-1}\xi J\} \\ &= [1 - \lambda(z) d(\zeta,\zeta)]^{-1}\Lambda^{\natural}(z)\end{aligned}$$

where we have used the expression (B.1) for $[\lambda(z)]_{\#}$. Finally, by (29b),

$$1 - \lambda(z) d(\zeta,\zeta) = b(z) b^*(\tau)$$

which establishes (B.2) and, consequently, Lemma 3.

Proof of Lemma 5

First, observe that for $p(z) = p_1(z)p_2(z)$ and $q(z) = q_1(z)q_2(z)$ we have

$$B_{\Omega}^{p,q}(z,w) = q_2(z) B_{\Omega}^{p_1,q_1}(z,w)[p_2^{\#}(w)]^* + p_1(z) B_{\Omega}^{p_2,q_2}(z,w)[q_1^{\#}(w)]^*$$

regardless of the degrees of $p_i(z), q_i(z)$. In particular, this establishes (33) for $n = 2$. Assume it holds for some fixed n, and consider the Bezoutian $B_{\Omega}^{pp_{n+1},qq_{n+1}}(z,w)$, where $p(z), q(z)$ are both of degree n and $p_{n+1}(z), q_{n+1}(z)$ are both of degree 1. By our former observation we have

$$\begin{aligned}B_{\Omega}^{pp_{n+1},qq_{n+1}}(z,w) &= q_{n+1}(z) B_{\Omega}^{p,q}(z,w)[p_{n+1}^{\#}(w)]^* \\ &\quad + p(z) B_{\Omega}^{p_{n+1},q_{n+1}}(z,w)[q^{\#}(w)]^* \\ &= \sum_{i=1}^{n} [\phi_{n,i}(z) q_{n+1}(z)] B_{\Omega}^{p_i,q_i}(z,w) \\ &\quad \cdot [\psi_{n,i}(w) p_{n+1}^{\#}(w)]^* \\ &\quad + p(z) B_{\Omega}^{p_{n+1},q_{n+1}}(z,w)[q^{\#}(w)]^*.\end{aligned}$$

Observe that

$$\phi_{n+1,i}(z) = \phi_{n,i}(z) q_{n+1}(z)$$
$$\psi_{n+1,i}(z) = \psi_{n,i}(z) p_{n+1}^{\#}(z)$$

and

$$\phi_{n+1,n+1}(z) = p(z)$$
$$\psi_{n+1,n+1}(z) = q^{\#}(z)$$

which establishes (33a) for $n+1$ and, therefore, establishes the lemma by induction.

Proof of Lemma 6

The matrix identity (34a) is the coefficient-domain equivalent of (33a). More specifically, in view of Lemma 4, we obtain from (33a) the matrix identity

$$\boldsymbol{B}_{\Omega}^{p,q} = \sum_{i=1}^{n} \boldsymbol{L}(\phi_{n,i}) \boldsymbol{B}_{\Omega}^{p_i,q_i} \boldsymbol{L}^*(\psi_{n,i})$$

where $\boldsymbol{L}(\cdot)$ is as defined in (30b). Since $\deg B_{\Omega}^{p_i,q_i}(z,w) = 0$, it follows that

$$B_{\Omega}^{p_i,q_i}(z,w) = \kappa_i$$

for all z, w and, therefore, that $\boldsymbol{B}_{\Omega}^{p_i,q_i}$ is a matrix of the form

$$\boldsymbol{B}_{\Omega}^{p_i,q_i} = \kappa_i[1 \quad 0 \quad \cdots]^*[1 \quad 0 \quad \cdots].$$

Choosing $z = w = \zeta_i$, a zero of $p_i(z)$, we obtain (34b). Also notice that

$$\boldsymbol{L}(\phi_{n,i})[1 \quad 0 \quad \cdots]^* = \boldsymbol{\phi}_{n,i}$$

which establishes (34a).

To prove the linear independence of the vectors $\boldsymbol{\phi}_{n,i}$, we observe that $\Sigma\alpha_i\boldsymbol{\Phi}_{n,i}$ is the vector of coefficients of the polynomial $f(z) := \Sigma\alpha_i\phi_{n,i}(z)$. Therefore, we need to prove that $\Sigma_{i=1}^{n}\alpha_i\phi_{n,i}(z) \equiv 0$ cannot hold with nonzero α_i. Since we assume $gcd\{p(z), q(z)\} = 1$, it follows that $q_i(\zeta_j) \neq 0$ for all i, j and, consequently, that $\phi_{n,i}(\zeta_j) = 0$ for $j < i$ while $\phi_{n,i}(\zeta_j) \neq 0$ for $j \geq i$.

We now consider the possibility that $f(z) \equiv 0$. If this holds identically for all z then, in particular,

$$0 = f(\zeta_1) = \alpha_1\phi_{n,1}(\zeta_1) \to \alpha_1 = 0.$$

Consequently, $p_1(z)$ divides $f(z)$, viz.,

$$f_1(z) := \frac{f(z)}{p_1(z)} = \sum_{i=2}^{n} \alpha_i\phi_{n,i}^{(2)}(z)$$

where we define, for $r \leq i$,

$$\phi_{n,i}^{(r)}(z) := p_r(z) p_{r+1}(z) \cdots p_{i-1}(z) q_{i+1}(z) \cdots q_n(z).$$

Since $p_1(z)$ does not vanish identically we can have $f(z) \equiv 0$ only if $f_1(z) \equiv 0$, which, in turn, implies that

$$0 = f_1(\zeta_2) = \alpha_2\phi_{n,2}^{(2)}(\zeta_2) \to \alpha_2 = 0.$$

In this manner we establish that

$$f(z) \equiv 0 \to \alpha_i = 0, \qquad \text{for all } i$$

which establishes the lemma.

Proof of Lemma 7

The inertia of every strongly regular Hermitian matrix $\boldsymbol{R} := \{r_{i,j},\ 1 \leq i, j \leq n\}$ can be recursively determined via the transformation $\boldsymbol{R} \to \boldsymbol{R}_1 := \{r_{i,j}^{(1)},\ 2 \leq i, j \leq n\}$ (see, e.g., [18]), where,

$$r_{i,j}^{(1)} := r_{i,j} - r_{i,1} r_{1,1}^{-1} r_{1,j}, \qquad \text{for } 2 \leq i, j \leq n.$$

Since the matrix $\boldsymbol{R}_1$ is a Schur complement of the element $r_{1,1}$ in the matrix $\boldsymbol{R}$, it follows that

$$In\,\boldsymbol{R} = In\begin{pmatrix} r_{1,1} & 0 \\ 0 & \boldsymbol{R}_1 \end{pmatrix}.$$

In particular, if we apply the same transformation to

$$R = \Delta_{1:n} := \left\{ \frac{1}{d_\Omega(\zeta_i, \zeta_j)}, 1 \leq i, j \leq n \right\}$$

where $d_\Omega(z,w) = \alpha + \beta z + (\beta w)^* + \delta z w^*$, then we conclude that $r_{1,1} = d^{-1}(\zeta_1, \zeta_1)$ and $R_1 = (|\beta|^2 - \alpha\delta) D_1 \Delta_{2:n} D_1^*$, where

$$D_1 := \operatorname{diag}\left\{ \frac{\zeta_i - \zeta_1}{d_\Omega(\zeta_i, \zeta_1)}, 2 \leq i \leq n \right\}.$$

Since, by the assumptions of the lemma, $|\beta|^2 - \alpha\delta \neq 0$, $d_\Omega(\zeta_1, \zeta_1) \neq 0$, and D_1 is nonsingular, it follows that

$$In\, \Delta_{1:n} = In \begin{pmatrix} d_\Omega(\zeta_1, \zeta_1) & 0 \\ 0 & \Delta_{2:n} \end{pmatrix}$$

which establishes the lemma (by iteration).

Appendix C
Lyapunov Equations

Consider a matrix C in companion form, viz.,

$$C := \begin{pmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & & & \\ & \ddots & & \vdots \\ & & 1 & -a_{n-1} \end{pmatrix} \quad \text{(C.1a)}$$

and its associated monic polynomial

$$a(z) := a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \quad \text{(C.1b)}$$

and observe that

$$[1\ z\ z^2 \cdots z^{n-1}] C = z[1\ z\ z^2 \cdots z^{n-1}] - a(z)[1\ z\ z^2 \cdots z^{n-1}].$$

This suggests that we should ignore terms containing multiples of $a(z)$ in calculations involving companion matrices, viz.,

$$[1\ z\ z^2 \cdots z^{n-1}] C = z[1\ z\ z^2 \cdots z^{n-1}], \quad \bmod a(z) \quad \text{(C.2a)}$$

and, consequently, for all $i \geq 0$,

$$[1\ z\ z^2 \cdots z^{n-1}] C^i = z^i[1\ z\ z^2 \cdots z^{n-1}], \quad \bmod a(z). \quad \text{(C.2b)}$$

Now introduce the linear *displacement operator*

$$d(A,B)R = \sum_{i,j=0}^{\infty} d_{i,j} A^i R (B^*)^j \quad \text{(C.3)}$$

where A, B, R are arbitrary square matrices of the same size, and the coefficients $d_{i,j}$ come from the power series expansion (10b) of $d(z,w)$, viz.,

$$d(z,w) = \sum_{i,j=0}^{\infty} d_{i,j} z^i (w^*)^j.$$

It follows that

$$[1\ z\ z^2 \cdots z^{n-1}]\{d(C,C)R\}[1\ w\ w^2 \cdots w^{n-1}]^* = d(z,w) R(z,w), \quad \bmod a(z), \bmod a^*(w).$$

Consequently, if we let $d(z,w) = d_\Omega(z,w)$ and $R = B_\Omega^{p,q}$ then

$$[1\ z\ z^2 \cdots z^{n-1}]\{d_\Omega(C,C) B_\Omega^{p,q}\}[1\ w\ w^2 \cdots w^{n-1}]^* = p(z)[q^\#(w)]^* - q(z)[p^\#(w)]^*, \quad \bmod a(z), \bmod a^*(w) \quad \text{(C.4)}$$

where $a(z)$ is an arbitrary monic polynomial of degree n, and C is the companion matrix associated with $a(z)$ via (C.1). In particular, if $a(z)$ is the monic equivalent of $q(z) = \Sigma q_i z^i$, i.e., if

$$a(z) = \frac{q(z)}{q_n}$$

then

$$[1\ z\ z^2 \cdots z^{n-1}]\{d_\Omega(C,C) B_\Omega^{p,q}\} \cdot [1\ w\ w^2 \cdots w^{n-1}]^* = u(z) v^*(w)$$

where $u(z)$ (resp. $v(z)$) is the residue from the division of $p(z)$ (resp. $q^\#(z)$) by $a(z)$. In matrix form, this becomes

$$d_\Omega(C,C) B_\Omega^{p,q} = u v^* \quad \text{(C.5)}$$

where u (resp. v) is the vector of coefficients of the polynomial $u(z)$ (resp. $v(z)$) of degree $n-1$.

Finally, the Lyapunov-type equation (C.5) can be transformed into a variety of other forms via a suitable similarity transformation. A direct calculation shows that

$$d_\Omega(A,A)\tilde{B} = \tilde{u}\tilde{v}^* \quad \text{(C.6a)}$$

where

$$A := TCT^{-1}, \tilde{B} := T B_\Omega^{p,q} T^*,$$
$$\tilde{u} := Tu, \tilde{v} := Tv. \quad \text{(C.6b)}$$

Thus A can be any matrix similar to C, i.e., any matrix whose characteristic polynomial is $a(z)$. Observe that $\tilde{B}$ has the same inertia as the Bezoutian matrix $B_\Omega^{p,p^\#}$.

References

[1] C. Hermit, "Extrait d'une lettre de Mr. Ch. Hermite de Paris á Mr. Borchardt de Berlin, sur le nombre des racines d'une équation algébrique comprises entre des limits données," *Journal für die Reine und Angewandte Mathematik*, vol. 52, pp. 39–51, Berlin, 1856.

[2] A. Cayley, "Note sur la méthode d'élimination de Bezout," *Journal für die Reine und Angewandte Mathematik*, vol. 53, pp. 366–376, Berlin, 1857.

[3] G. Darboux, "Mémoire sur le théorème de Sturm," *Bull. Sci. Math.*, vol. 8, pp. 56–63, 91–112, 1875.

[4] J. Sylvester, "On a theory of syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm functions, and that of the greatest algebraical common measure," *Phil. Trans. Roy. Soc. London*, vol. 143, pp. 407–548, 1853.

[5] M. G. Krein and M. N. Naimark, "The method of symmetric and hermitian forms in the theory of the separation of roots of algebraic equations," *Linear and Multilinear Algebra*, vol. 10, pp. 265–308, 1981. (Originally in Russian, Kharkov 1936.)

[6] A. Cohn, "Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise," *Math. Zeit.*, vol. 14, pp. 110–148, 1922.

[7] I. Schur, "Uber Potenzreihen, die in Innern des Einheitskreises Beschränkt Sind," *Journal für die Reine und Angewandte Mathematik*, vol. 147, pp. 205–232, Berlin, 1917, and vol. 148, pp. 122–145, Berlin, 1918. (English translation in *I. Schur Methods in Operator Theory and Signal Processing, Operator Theory: Advances and Applications*, I. Gohberg (ed.), vol. 18, pp. 31–88. Basel, Switzerland: Birkhäuser Verlag, 1986.

[8] M. Fujiwara, "Über die algebraische Gleichungen, deren Wurzeln in einem Kreise order in einer Halbebene liegen," *Math. Zeit.*, vol. 24, pp. 161–169, 1926.

[9] E. Routh, *A Treatise on the Stability of a Given State of Motion*. London, U.K.: MacMillan, 1877.

[10] A. Hurwitz, "Über die Bedingungen unter welchen eine Gleichung nur Wurzeln mit negativen reellen Teilen besizst," *Math. Ann.*, vol. 46, pp. 273–284, 1895.

[11] C. Sturm, "Mémoire sur la Résolution des Équation Numérique," *Bul. de Ferussac*, vol. 11, 1829.

[12] M. Marden, *The Geometry of the Zeros of a Polynomial in the Complex Plane*. Amer. Math. Soc., 1949.

[13] E. I. Jury, *Theory and Applications of the Z-Transform Method*. New York: Wiley, 1964.

[14] Y. Bistritz, "Zero location with respect to the unit circle of discrete-time linear system polynomials," *Proc. IEEE*, vol. 72, pp. 1131–1142, Sept. 1984.

[15] Y. Bistritz, H. Lev-Ari, and T. Kailath, "Immittance-domain Levinson algorithms," *IEEE Trans. Inform. Theory*, vol. 35, pp. 675–682, May 1989. (See also *Proc. 1986 IEEE Int. Conf. Acoustics, Speech, and Signal Processing*, pp. 253–256, Tokyo, Japan, Apr. 1986.)

[16] P. Delsarte and Y. Genin, "The split Levinson algorithm," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-34, pp. 470–478, June 1986.

[17] H. Lev-Ari, "Nonstationary lattice-filter modeling," Ph.D. dissertation, Dep. Electrical Engineering, Stanford Univ., Stanford, CA, Dec. 1983.

[18] H. Lev-Ari and T. Kailath, "Triangular factorization of structured Hermitian matrices," in *I. Schur Methods in Operator Theory and Signal Processing, Operator Theory: Advances and Applications*, I. Gohberg (ed.), vol. 18, pp. 301–324. Basel, Switzerland: Birkhäuser Verlag, 1986.

[19] D. Pal and T. Kailath, "Inertia extraction of Bezoutians and root location of polynomials—Part I: Imaginary axis case," preprint, 1988.

[20] ____, "Inertia extraction of Bezoutians and root location of polynomials—Part II: Unit circle case," preprint, 1988.

[21] S. Gutman and E. I. Jury, "A general theory for matrix root clustering in subregions of the complex plane," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 853–863, Aug. 1981.

[22] E. I. Jury, *Inners and the Stability of Linear Systems*. New York: Wiley, 1982.

[23] V. Pták and N. J. Young, "A generalization of the zero location theorem of Schur and Cohn," *IEEE Trans. Automat Contr.*, vol. AC-25, pp. 978–980, Oct. 1980.

[24] V. Pták, "Lyapunov, Bezout and Hankel," *Lin. Alg. Appl.*, vol. 58, pp. 363–390, Apr. 1984.

[25] H. Schwerdtfeger, *Geometry of Complex Numbers*. New York: Dover, 1979.

[26] A. Liénard and M. Chipart, "Sur le Signe de la Partie Reélle des Racines d'Une Équation Algébrique," *J. Math. Pures Appl.*, vol. 10, Sér. 6, pp. 291–246, 1914.

[27] Y. Bistritz, "A circular stability test for general polynomials," *Syst. Contr. Lett.*, vol. 7, pp. 89–97, Apr. 1986.

[28] H. Lev-Ari and T. Kailath, "Lossless cascade networks: The crossroads of stochastic estimation, inverse scattering and filter synthesis," in *Proc. 1987 Int. Symp. Circuits and Systems*, pp. 1088–1091, Philadelphia, PA, May 1987.

[29] H. Wimmer, "On the history of the Bezoutian and the resultant matrix," *Lin. Alg. Appl.*, vol. 128, pp. 27–34, Jan. 1990.

[30] L. Lepschy, G. A. Mian, and U. Viaro, "A stability test for continuous systems," *Syst. Contr. Lett.*, vol. 10, No. 3, pp. 175–179, Mar. 1988.

[31] Y. Bistritz, H. Lev-Ari, and T. Kailath, "Immittance-type three-term Schur and Levinson algorithms for quasi-Toeplitz complex Hermitian matrices," to appear in *SIAM J. Matrix Analysis and Applications*, 1990.

[32] H. C. Reddy and P. K. Rajan, "All pass function based stability test for continuous time systems," in *Proc. 1985 Int. Symp. Circuits and Systems*, pp. 824–826, May 1985.

[33] H. W. Schüssler, "A stability theorem for discrete systems," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-24, pp. 87–89, 1976.

[34] P. Steffen, "An algorithm for testing stability of discrete systems," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-25, pp. 454–456, 1977.

[35] Y. Bistritz, "Direct bilinear Routh stability criteria for discrete systems," *Systems and Control Lett.*, vol. 4, pp. 265–271, July 1984.

[36] E. I. Jury and M. Mansour, "On terminology relations between continuous and discrete systems criteria," *Proc. IEEE*, vol. 73, p. 884, 1985.

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