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LOSSLESSNESS AND STABILITY OF LDI LADDERS AND FILTERS*

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Abstract. The stability of digital ladder filters close in form, via the lossless discrete integrator (LDI) transformation, to doubly terminated *LC* Cauer ladder low-pass filters is studied. The LDI transformation does not necessarily map stable analogue into stable digital filters. Necessary and sufficient conditions for these digital filter to be lossless and for a corresponding doubly terminated filter to be stable are given. LDIs are often used in the design of switched capacitor (SC) filters. For this case we provide a threshold sampling rate above which the SC LDI filter retains the stability of the analogue filter. In spite of the stability problem with the LDI mapping, it has been observed that when applied to simulate analogue filters the resulting filter is stable. It can now be argued that, in these cases, other factors in the determination of the sampling rate leads typically to a choice that is also sufficiently above the threshold rate for stability. The theory described is also useful to derive more general digital filters and to perform the design of stable LDI filters exclusively in the *Z*-plane without reference to analogue prototypes.

1. Introduction

This paper studies digital ladder configurations that resemble analogue *LC* Cauer ladder forms with analogue integrators replaced either by losseless digital integrators (LDI) or by related forward and backward digital integrators (FDI and BDI). We find that these filters are intimately related to stability formulation and the continued fraction expansions of [1]–[3] and append the formulation there with conditions under which the digital Cauer ladder is lossless and under conditions for stability of the digital filter that correspond to the analogue doubly terminated Cauer ladder filter.

The current work stems from our search for a Cauer-type circuit interpretation to the stability formulation obtained in [1]–[5]. Since the stability

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test for discrete systems is very close to the Routh test for continuous systems [6], and in analogue filters there exist well-known ties between the Routh test, the Cauer continued fractions, and corresponding *LC* ladder filters, the existence of a similar triangle in association with the above-mentioned formulation becomes inevitable. This paper considers in depth only the one filter configuration that generates the digital equivalent (in the full sense defined in [6]) to the low-pass doubly terminated *LC* Cauer ladder filter. However, it will become apparent that the approach and principles presented can also be used to derive other stable digital filters that simulate other doubly terminated lossless configurations and to design not only low-pass filters. The actual details of implementation for other filter forms need further study and may be less plain.

The digital filter that we address is similar to the design proposed in [7] and [8]. It simulates the doubly terminated Cauer ladder low-pass analogue filter [9], [10]. Similarity is obtained by formally replacing the analogue integrator associated with the *L* and *C* elements in the lossless Cauer ladder by LDIs. The LDI was proposed in the design of digital filters by Bruton who suggested in [7] simulating analogue integration by replacing the integration $1/s$ with $1/(z^{1/2} - z^{-1/2})$. We use the name LDI also for the underlying mapping

$$\frac{1}{s} \to \mathrm{LDI} := \frac{1}{(z^{1/2} - z^{-1/2})} \tag{1}$$

and call it the "LDI transformation." The LDI has become popular in the design of switched capacitor (SC) digital filters because each LDI can be realized by two switched capacitors and one amplifier in a manner that suits mass production with MOS technology [7], [8], [11], [12]. The LDI has been proven to provide a useful tool for the design of digital filters in configurations that are close to analogue prototypes. However, it may be of interest not only in conjunction with sampled data filters but also as exclusively Z-plane digital filter design methods with the prospect of passing to the digital filter desirable features (e.g., low sensitivity to parameter values) of a corresponding analogue filter.

In [1]–[5] we presented several stability tests for discrete systems that are close in form and character to the Routh test for continuous systems. In one of its possible versions (that in [2] and [3]) the new stability test is related to the continued fraction expansions that underly the Cauer *LC* ladder by the mapping

$$s = \tfrac{1}{2}(z^{1/2} - z^{-1/2}) \tag{2}$$

that also underlies the LDI. Unlike bilinear Routh tests [13], [14] that offer discrete-time stability tests related to the Routh test via the bilinear transform in a straightforward manner, the tests in [2] and [3] cannot be deduced from a simple application of the LDI mapping to the Routh procedure. The reason is that, unlike the bilinear transformation, the LDI transformation is not "stability preserving." The mapping properties of (2) were illustrated in [3]. For this reason (2) does not map a stable analogue filter necessarily into a stable digitial filter. Nevertheless, the LDI has proven useful in the design of digital filters, mostly to SC filters where it competes successfully with designs that are based on "stability preserving" but less easily realizable bilinear-transform-based designs (e.g., [15] and [16]).

In this paper we extend the stability formulation [1]–[5] to provide constructive tools for the design of stable LDI filters. Our approach is based on translating familiar connections between stability, losslessness, and passivity from the analogue to the discrete case. It is well known in classical circuit theory that passive filters are stable and that a passive filter can be realized by a resistively terminated lossless ladder. We first define and characterize the discrete equivalents to the concepts of losslessness and positive-realness (i.e., "passivity") and then we separate the design of stable digital filters based on the LDI into two steps. As a first step we obtain necessary and sufficient conditions for a Cauer ladder with the LDI replacing the analogue integrators to be lossless in the discrete sense extending results from [1]–[5]. In the next step we show that with a proper pair of terminations, a lossless digital Cauer ladder becomes a positive real and hence stable filter.

As already mentioned, the LDI transformation found its most wide application in the design of SC digital filters. In this field the LDI offers some advantages over designs based on bilinear transformation (that includes simpler implementation and lower frequency warping). It is interesting that designers of SC LDI filters often have not encountered its stability problem. In fact, this surprising phenomena is believed to be the cause why in some instances the mapping (2) has been assumed to map the left half s-plane inside the unit circle [17]–[19]. We examine, in some detail, and explain this "stability in practice" phenomena.

We derive a threshold frequency that is a function of the set of integrator values, such that the Cauer filter is stable if and only if the sampling rate is higher than this threshold. We then argue that other factors in the choice of the SC clock rate usually lead to a sampling rate that is higher than the threshold to guarantee the filter's stability. The fact that a high enough sampling rate is able to stabilize an LDI configuration that simulates a stable analogue filter can be understood also via a continuity argument of convergence to the stable analogue filter. A statement of this kind was first made already by Bruton in [7] who was well aware of the stability difficulty associated with his proposed design. Our contribution in this context is in putting the condition into precise mathematical terms. The currently threshold frequency can also be incorporated in the design of a digital filter without reference to an analogue prototype. It can be treated as one more parameter in the design that can be used to meet the stability requirement.

Our original interest in this work was to find the digital "circuit" aspect for the new stability formulation obtained in [1]–[5]. Since this formulation already contained a Routh-like test for discrete systems and corresponding continued fraction expansions, we were curious to complete the two to the trinity of intimately analogue concepts consisting of the Routh test for continuous systems, the Cauer CFE, and the Cauer LC ladder filters. We concentrate in this paper on only one digital filter—the filter that forms the discrete equivalent of the Cauer filters and reflects and satisfies our original interest. However, it is fairly emphasized that the principles and many of the results are extendible to additional lossless configurations and to the design of new forms of digital filters without reference to analogue filters.

This paper is the full version of a past conference presentation [20]. It contains the details and provides proofs that were omitted from [20]. The paper is written as a series of parallel results. Known S-plane properties and less known or new corresponding Z-plane properties are laid side by side and arranged in formats that make the analogy most transparent.

In Section 2 we give necessary and sufficient conditions for a digital ladder, based on the LC Cauer prototype, to be lossless. In Section 3 we use the concept of positive real functions to obtain the conditions for a corresponding low-pass digital filter, formed by its resistive termination, to be stable. In Section 4 we address the important case when this design is implemented as an SC filter with LDI elements. We derive a threshold frequency such that, for sampling rates higher than it, the filter will be stable. Then we argue that in SC filter design other considerations in the choice of the switching rate are concurrent with fulfilling this form of stability requirement.

A Comment on Terminology. In order to stress equivalences between parallel S-plane and Z-plane properties and yet distinguish them we use the same terminology for both but add an S- or Z-prefix for the S- and Z-plane term, respectively, except in some cases where other indications, such as a function's (s) or (z) argument, provide the distinction. Thus, for example, S-stability means that the analog filter has poles in the left half-(S-)plane while Z-stability corresponds to a digital filter having its poles in $|z| < 1$. We also refer to polynomials $H_n(s)$ and $H_n(z)$ as "stable" if the zeros of $H_n(z)$ reside in the left half-plane Re $s < 0$, and the zeros of $H_n(z)$ reside inside the unit circle.

2. Stability and the Cauer configuration

We associate with a real polynomial in the s-variable, $H_n(s)$, a rational function $\rho_n(s)$:

$$\rho_n(s) = \frac{H_n(s) - H_n(-s)}{H_n(s) + H_n(-s)}. \tag{3}$$

This function is sometimes called an (S-)*tangent function*. We associate with a polynomial in the z-variable, $H_n(z)$, another rational function, to become its *Z-tangent function*,

$$\rho_n(z) = \frac{H_n(z) - H_n^{\#}(z)}{H_n(z) + H_n^{\#}(z)}, \tag{4}$$

where $H_n^{\#}(z)$ denotes the reciprocal polynomial $z^n H_n(z^{-1})$. The name "tangent function" derives from the fact that in each respective case the tangent function relates to the spectral phase of the polynomial. More precisely, the tangent function forms, for values along $s = j\omega$ or $z = \exp(j\Omega)$ (the respective "frequency curves"), the ratio of the polynomials imaginary over real parts. These tangent functions will soon be recognized as the immittance of the lossless ladders in the low-pass filters considered.

The following two theorems summarize properties of Routh's test for continuous-time systems and its discrete-time equivalents as proposed earlier by Bistritz. We repeat these results to make them conveniently available for subsequent use and on this occasion display them in a representation that emphasizes their common characteristics.

Theorem 1. *Each of the next three sets of conditions are equivalent and give necessary and sufficient conditions for the real polynomial $H_n(s)$ to be S-stable*:

(i) *Define the sequence of Routh polynomials $\{R_k(s)\}_{k=0}^{n}$, $R_k(s) = \sum_{i=0}^{k} r_{ki} s^i$, as follows. The sequence starts with two polynomials of degrees $k = n, n-1$, given by*

$$R_n(s) = H_n(s) + (-1)^n H_n(s), \tag{5a}$$

$$R_{n-1}(s) = \frac{H_n(s) - (-1)^n H_n(s)}{s}. \tag{5b}$$

The rest of the sequence is obtained recursively for $k = n, n-1, \ldots, 1$, by

$$\gamma_k = \frac{r_{k,k}}{r_{k-1,k-1}}, \tag{5c}$$

$$R_k(s) = \gamma_k s R_{k-1}(s) + R_{k-2}(s). \tag{5d}$$

The polynomial $H_n(s)$ is stable if and only if the leading terms $\{r_{k,k}\}$ are of the same sign for all $k = 0, 1, \ldots, n$. Note that the recursion (5) *implies that $R_k(-s) = (-1)^k R_k(s)$, namely that $R_k(s)$ is even/odd for even/odd k.* (*Hence, $r_{2q,2i+1} = 0$ and $r_{2q-1,2i} = 0$ for $i = 0, \ldots, q$ and $q = 1, 2, \ldots$.*)

(ii) *The polynomial $H_n(s)$ is stable if and only if the S-plane tangent function $\rho_n(s)$, defined in* (3), *can be written in the form*

$$\rho(s) = \frac{Ks \prod_{i=1}^{l} (s^2 + \omega_{2i}^2)}{\prod_{i=1}^{m} (s^2 + \omega_{2i-1}^2)}, \qquad K > 0, \tag{6a}$$

and with critical frequencies ω_i *that satisfy*

$$0 < \omega_1^2 < \omega_2^2 < \cdots < \omega_{n-1}^2, \tag{6b}$$

where $n = m + l + 1$, $l = m - 1$ *or* $l = m$.

(iii) *The polynomial* $H_n(s)$ *is stable if and only if* $\rho_n(s)$ *has the following* continued fraction expansion (CFE):

$$\rho_{2m}(s) \text{ or } \rho_{2m+1}^{-1}(s) = \frac{1}{\gamma_n s} + \frac{1}{\gamma_{n-1} s} + \cdots + \frac{1}{\gamma_1 s}, \tag{7}$$

and the coefficients γ_k *are positive for all* $k = 1, \ldots, n$.

Theorem 2. *Each of the next three sets of conditions are equivalent and give necessary and sufficient conditions for the real polynomial* $H_n(z)$ *to be Z-stable*:

(i) *Define the sequence of polynomials* $\{T_k(z)\}_{k=0}^{n}$, $T_k(z) = \sum_{i=0}^{k} t_{ki} z^i$, *as follows. The sequence starts with two polynomials of degrees* $k = n$, $n - 1$, *given by*

$$T_n(z) = H_n(z) + (-1)^n H_n^{\#}(z), \tag{8a}$$

$$T_{n-1}(z) = \frac{H_n(z) - (-1)^n H_n^{\#}(z)}{z+1}. \tag{8b}$$

The rest of the sequence is obtained recursively for $k = n, n-1, \ldots, 1$, *by*

$$\delta_k = \frac{t_{k,k}}{t_{k-1,k-1}}, \tag{8c}$$

$$T_k(z) = \delta_k (z - 1) T_{k-1}(z) + z T_{k-2}(z). \tag{8d}$$

The polynomial $H_n(z)$ *is stable if and only if the sequence of numbers* $\{\hat{\sigma}_k\}_{k=0}^{n}$ *defined by* $\hat{\sigma}_k := (-1)^k T_k(-1) = \sum_{i=0}^{k} (-1)^i t_{k,i}$ *has the same signs for all* $k = 0, 1, \ldots, n$. *Note that the recursion* (8) *implies that* $T_k^{\#}(z) = (-1)^k T_k(z)$, *namely that* $T_k(z)$ *is symmetric/antisymmetric for even/odd* k. (*Hence* $t_{ki} = (-1)^k t_{k,k-i}$ *for* $i = 0, \ldots, k$.)

(ii) *The polynomial* $H_n(z)$ *is stable if and only if the Z-plane tangent function* $\rho(z)$ *defined in* (4) *can be written for* $n = 2m + 1$ *and* $n = 2m$, *respectively, in the form*

$$\rho_{2m+1}(z) = \frac{K(z-1) \prod_{i=1}^{m} (z^2 - 2zX_{2i} + 1)}{(z+1) \prod_{i=1}^{m} (z^2 - 2zX_{2i-1} + 1)}, \qquad K > 0, \tag{9a}$$

$$\rho_{2m}(z) = \frac{K(z-1)(z+1) \prod_{i=1}^{m-1} (z^2 - 2zX_{2i} + 1)}{\prod_{i=1}^{m} (z^2 - 2zX_{2i-1} + 1)}, \qquad K > 0, \tag{9b}$$

where

$$-1 < X_{n-1} < X_{n-2} < \cdots < X_2 < X_1 < 1. \tag{9c}$$

(iii) *The polynomial $H_n(z)$ is stable if and only if the tangent function $\rho_n(z)$ has the following CFEs for $n = 2m + 1$ and $2m$, respectively*:

$$(z+1)^{-1}\rho_{2m+1}^{-1}(z) = \frac{1}{\delta_{2m+1}(z-1)} + \frac{1}{\delta_{2m}(1-z^{-1})} + \cdots + \frac{1}{\delta_1(z-1)}, \tag{10a}$$

$$(z+1)^{-1}\rho_{2m}(z) = \frac{1}{\delta_{2m}(z-1)} + \frac{1}{\delta_{2m-1}(1-z^{-1})} + \cdots + \frac{1}{\delta_1(1-z^{-1})}, \tag{10b}$$

and the coefficients δ_i are such that the sequence of numbers σ_k, $k = 1, \ldots, n$, that they define by the recursion

$$\sigma_{-1} = 0, \qquad \sigma_0 = 1, \qquad \sigma_k = 2\delta_k\sigma_{k-1} - \sigma_{k-2}, \quad k = 1, \ldots, n, \tag{11a}$$

are all positive. (These are the $\hat{\sigma}_k$ of condition (i) *rescaled $\sigma_k = \hat{\sigma}_k/\hat{\sigma}_0$.)*

An equivalent requirement is that the coefficients δ_k are such that the tridiagonal maxtrix

$$\Delta_n = \begin{bmatrix} 2\delta_1 & -1 & & 0 & \\ -1 & 2\delta_2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2\delta_{n-1} & -1 \\ & 0 & & -1 & 2\delta_n \end{bmatrix} \tag{11b}$$

is positive definite. (To verify the equivalence of the two conditions observe that σ_k is the determinant of Δ_k, the $k \times k$ upper-left submatrix of Δ_n.)

The content of Theorem 1 is well known and can be found in many classical texts on circuit theory, e.g., [9]. The CFEs in (10) can be obtained from (7) by substitution of the mapping (2) redefining $\delta_k := \gamma_k/2$, and dividing out the square root elements $z^{\pm 1/2}$. It is relatively easy to show that if $H_n(z)$ is stable, then the structure in (9) maps into the structure in (6) because the transformation (1) maps the unit circle bijectively onto the imaginary axis interval

$$J = \{s \mid s = j\omega, \omega \in [-1, 1]\} \to C = \{z \mid z = e^{j\Omega}, \omega \in [-\pi, \pi]\} \tag{12a}$$

by

$$\omega = \sin\left(\frac{\Omega}{2}\right). \tag{12b}$$

Thus, stable $H_n(z)$ maps into a stable $H_z(s)$ and therefore has positive δ_k's. A similar proof was indicated in [17] to show that another rational function related to the tangent function has CFE similar to (10) with positive coefficients if $H_n(z)$ is stable. However, the converse of this is not true. Namely, the positivity of the coefficients δ_i does not imply that $H_n(z)$ is stable. The derivation of conditions which are both necessary and sufficient for Z-stability associated with this transform is more difficult because each value of s, other than those that belong to the interval J, is mapped into both the inside and the outside of the unit circle. Consequently, zeros of Re $s < 0$ are not necessarily mapped into the unit disk or vice versa. More details and proofs that the proper necessary and sufficient conditions are as stated in Theorem 2 are contained in the references. (References [2] and [3] are closest to the current context because they employ LDI mapping. Other proofs can be found in [4], [22], and [23].) The fact that necessary and sufficient conditions for Z-stability impose on the δ_k's conditions that are tighter than those that S-stability imposes on the γ_k's is best seen via (11b).

In classical circuit theory the various conditions of Theorem 1 are related to the LC ladder configuration in Figure 1 known as the Cauer ladder [9]. The immittance of this ladder is given by $\rho_n(s)$ as can be shown by comparison of Figure 1 with the CFE (7). Thus the three equivalent conditions of Theorem 1 can be appended by a fourth equivalent condition: $H_n(s)$ is stable if and only if its $\rho_n(s)$ can be presented in a Cauer ladder with $L_i > 0$ and $C_i > 0$ elements. Inspection on the way the ladder in Figure 1 relates to the CFE of Theorem 1 suggests that the CFE in Theorem 2 can be similarly associated with the realization depicted by Figure 2. In this formal comparison forward and backward differences, $z - 1$ and $1 - z^{-1}$, replace L's and C's. The ladder in Figure 2 could be given a physical interpretation but that would call for the discrete-time counterparts of field variables such as voltage

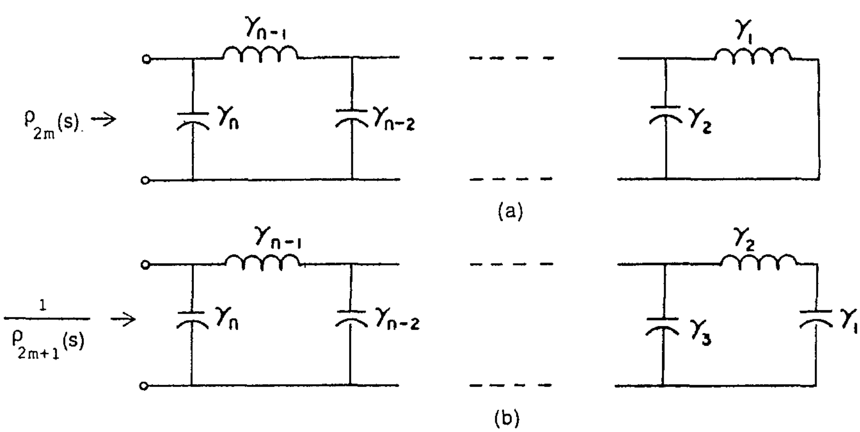

Figure 1. *LC* ladder (a) for n even and (b) for n odd.

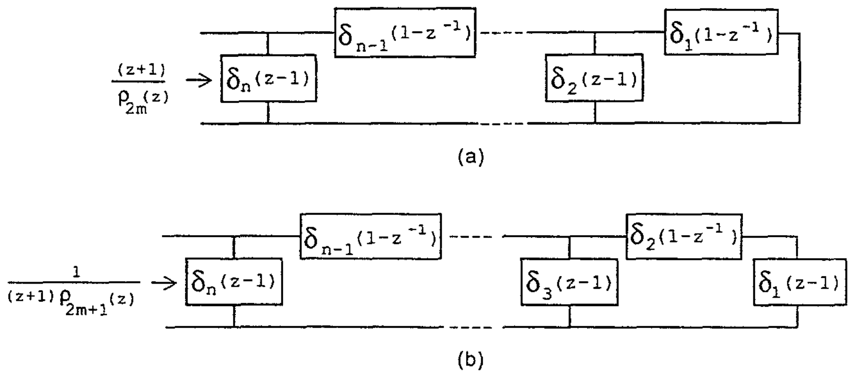

Figure 2. FD–BD digital ladder (a) for n even and (b) for n odd.

and current. Field variables do not constitute concepts as widely used in discrete-time systems modeling as they are in continuous-time systems. Alternatively, state-variables and signal flow graphs do constitute concepts commonly accepted in both the S- and the Z-planes and therefore may better qualify to draw similarities between the LC ladder and its Z-plane equivalent. Figure 3 gives a configuration that consists of a cascade of n integrators I_i. By proper interpretation of I_i this configuration adequately realizes the CFEs in both Theorems 1 and 2. For the S-plane case, the integrators I_i of the ladder (or lattice, a more proper name for cascades that do not have the ladder-like structure of the previous figures) in Figure 3 represent integrators $1/\gamma_i s$ where γ_i receives alternatingly the value of the ith inductor L_i or the ith capacitor C_i, while for the Z-plane case they may present $1/\delta_i(z^{1/2} - z^{-1/2})$, the LDI equivalent (1) of the analogue integrators. Alternatively, the CFEs (10) suggest that the I_i may represent (alternatingly, again) forward difference integrators (FDI) and the backward difference integrators (BDI) defined by

$$\text{FDI} := \frac{1}{\delta_i(z-1)}, \qquad \text{BDI} := \frac{1}{\delta_i(1-z^{-1})}, \tag{13}$$

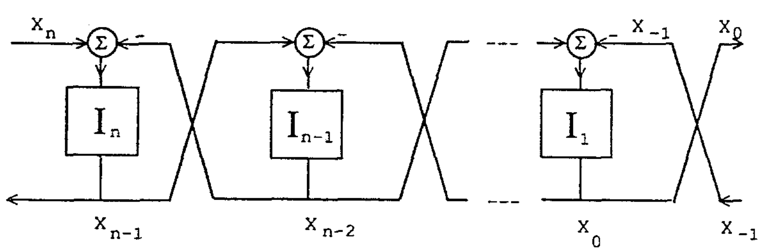

Figure 3. All integrator ladder as cascade of two ports.

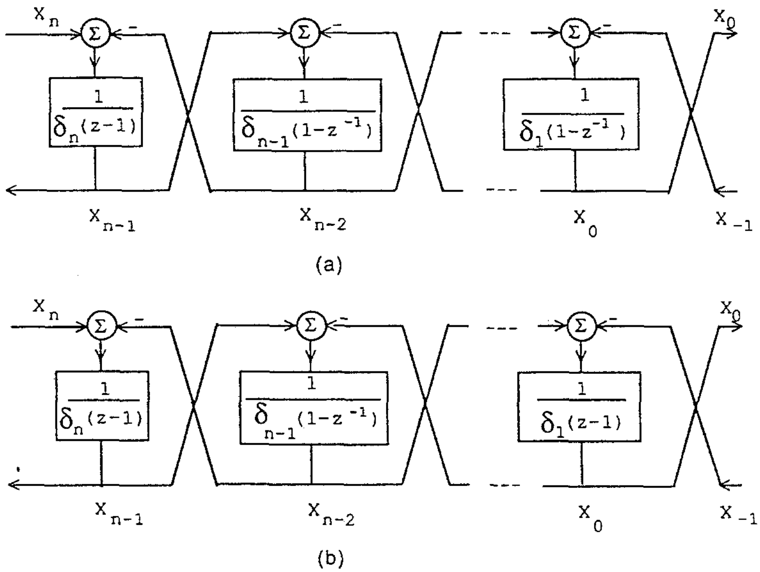

Figure 4. FDI–BDI digital ladder (a) for n even and (b) for n odd.

resulting in the ladder detailed in Figure 4. The forthcoming sections base on Theorem 2 the derivation of further results pertinent to the design of stable low-pass digital filters that simulate, via the LDI transform, the resistively terminated Cauer ladder [7], [8].

3. Lossless ladders and stable filters

In this section we focus on the lattice of Figure 2. We would like to construct a (Z-)stable low-pass filter based on this lattice form. We proceed by defining digital counterparts of analogue passive networks by extending the concepts of positive real functions and lossless positive real functions to digital circuits.

Definition 1 (Positive Realness (PR)).

(S) A function $f(s)$ is S-PR if it is analytic in $\text{Re}\{s\} > 0$ and $\text{Re}\{f(s)\} \geq 0$ for $\text{Re}\{s\} > 0$.

(Z) A function $f(z)$ is Z-PR if it is analytic in $|z| > 1$ and $\text{Re}\{f(z)\} \geq 0$ for $|z| > 1$.

Remark 1. Clearly, if a function is PR, then so is its inverse.

Remark 2. Physical passive element networks are characterized by rational function driving point immittances. For the class of rational functions, analyticity may be dropped from the definition because it is implied by the positivity of its real part.

Remark 3. Brune's well-known criteria asserts that a necessary and sufficient condition for a rational function to represent the immittance of a physical network made of passive elements is that it is PR. Due to this association with physical passive elements it has become customary to interpret the adjective *real* in the definition of *positive real* functions as a restriction to real functions (that is, all polynomial coefficients of the rational function are real). In view of today's newer digital and complex filter applications it may be of advantage to interpret *positive real* functions in a more direct and less restrictive manner. Namely, let a positive real function literally stand for function with positive real part in a desirable domain with no exclusion of complex functions. Most of the mathematical features of PR functions can be shown to hold or extend in an expected way to complex functions.

For rational functions, analyticity amounts to showing that the numerator and denominator polynomials are "stable" polynomials. This task can be carried out with ease by an appropriate stability test. An unstable polynomial already indicates a non-PR function. This still does not eliminate the awkward task of testing whether the real part of a rational function is positive in an indicated "vast" domain. This more difficult task can be elevated by the following lemma.

Lemma 1 (PR Lemma).

(*S*) *A function $f(s)$ is PR if and only if:*
  (i) *It is analytic in* $\mathrm{Re}\{s\} > 0$.
  (ii) *Poles of $f(s)$ on the imaginary axis* $\mathrm{Re}\{s\} = 0$ (*if any*) *are simple and associated with positive residue.*
  (iii) $\mathrm{Re}\{f(s)\} \geq 0\}$ *for* $\mathrm{Re}\{s\} = 0$ *that are not poles.*

(*Z*) *A function $f(z)$ is PR if and only if:*
  (i) *It is analytic in* $|z| > 1$.
  (ii) *Poles of $f(z)$ on the unit circle* $|z| = 1$ (*if any*) *are simple and each such pole $e^{j\Omega_0}$ has (complex) residue with corresponding $e^{j\Omega_0}$ phase.*
  (iii) $\mathrm{Re}\{f(z)\} \geq 0$ *for* $|z| = 1$ *that are not poles.*

Proof. We consider just the second (*Z*) part of the lemma because this part may have not yet been discussed in the literature and because the proof for $f(s)$ follows afterward as a simpler case. Suppose the three conditions of the lemma hold for $f(z)$. Consider the function $m(z) = e^{-f(z)}$. It is analytic in $|z| > 1$ and the maximum value of its modulus, $|m(z)| = e^{-\mathrm{Re}\{f(z)\}}$, corresponds to the minimum value of $\mathrm{Re}\{f(z)\}$. Since by the maximum modulus principle the maximum value of $|m(z)|$ occurs on the boundary of the domain $|z| > 1$, the minimum of $\text{Re}\{f(z)\}$ occurs on $|z| = 1$ where it is given to be nonnegative. Therefore $\text{Re}\{f(z)\} \geq 0$ for $|z| > 1$. By a similar argument the function $g(z) := 1/f(z)$ also has a nonnegative real part in $|z| > 1$. So it follows by the same argument that neither $\text{Re}\{f(z)\}$ nor $\text{Re}\{g(z)\}$ cannot become zero in $|z| > 1$ except only at the boundary. Since a pole of $f(z)$ corresponds to a zero of $g(z)$, $f(z)$ is analytic in $|z| > 1$ and the proof that the three conditions of the lemma imply the condition in Definition 1 is complete. Note that in fact we obtained that Definition 1 could consist of the statement of analyticity followed by the more explicit split conditions: (a) $\text{Re}\{f(z)\} \geq 0$ for $|z| \geq 1$ with (b) $\text{Re}\{f(z)\} > 0$ for $|z| > 1$ (see Remark 4 below). So far, in using the maximum principle, the boundary of $|z| > 1$ was tacitly assumed devoid of singularities (say, poles on $|z| = 1$ are detoured by infinitesimal arcs). Next, it is required to show that the condition $\text{Re}\{f(z)\} > 0$ for $|z| > 1$ implies that if there exist poles on $|z| = 1$ they are simple and have residue with phase as in condition (iii). The proof is carried out by expressing the residue $|r|e^{j\phi}$ at a pole on $|z| = 1$ in terms of $z - e^{j\Omega_0} = \varepsilon e^{\varphi}$ with $0 < \varepsilon \to 0$. Then, in a small enough vicinity of the pole, $f(z)$ may be approximated by $|r|e^{j\Omega_0}/(\varepsilon e^{j\varphi})^m$ where m is the multiplicity of the pole. Therefore in this vicinity $\text{Re}\{f(z)\} \sim |r|\varepsilon^{-m} \cos(\Omega_0 - \varphi m) > 0$. First, it is observed that multiplicity $m \geq 2$ contradicts $\text{Re}\{f(z)\} > 0$ for $|z| > 1$ and for the case $m = 1$ to be consistent with the condition request $|\varphi - \Omega_0| \leq \pi/2$, that holds for $|\varphi| \leq \pi/2$ if and only if $\varphi = \Omega_0$. The outlined proof of this point is essentially the same as that used in [13] for lossless PR functions (defined below) where [13] also illustrates the proof with figures. Finally, using the maximum modulus principle, it becomes evident that the definition of PR for $f(z)$ implies in fact $\text{Re}\{f(z)\} > 0$ for $|z| > 1$ (i.e., the real part may vanish only on the unit circle). With this the proof that PR as defined implies conditions (i)–(iii) in Lemma 1 is also complete. □

Remark 4. Note that if the PR function is assumed to be rational, then the three conditions in the lemma are implied even without including analyticity as part of the definition of PR functions. In other words, for the class of rational function PR could be defined more concisely by:

(S) A rational function $f(s)$ is PR if $\text{Re}\{f(s)\} > 0$ for $\text{Re}\{s\} > 0$.
(Z) A rational function $f(z)$ is PR if $\text{Re}\{f(z)\} > 0$ for $|z| > 1$.

A special and important class of PR functions arises when the function is rational and has all its poles and zeros on the boundary.

Definition 2 (Lossless Positive Realness (LPR)).

(S) An S-PR function is said to be S-LPR if it is positive real and its zeros and poles are all on $\text{Re}\{s\} = 0$.

(Z) A Z-PR function is said to be Z-LPR, if it is positive real and its zeros and poles are all on $|z| = 1$.

Lemma 2 (LPR Lemma).

(S) *A rational function $f(s)$ is S-LPR if and only if its poles and zeros reside on* $\operatorname{Re} s = 0$, *are simple and interlace, and there exist some point s_0 in* $\operatorname{Re} s > 0$ *for which the real part of $f(s_0)$ is positive.*

(Z) *A rational function $f(z)$ is Z-LPR if and only if its poles and zeros reside on $|z| = 1$, are simple and interlace, and there exists some point z_0 in $|z| > 1$ for which the real part of $f(z_0)$ is positive.*

Proof Outline. The simplicity of poles and zeros follows from Lemma 1 since both the function and its inverse are PR. Individual proofs for the (S) and (Z) cases follow by showing that the characterization of residues of the function and its inverse (as in Lemma 1) is possible if and only if the zeros and poles interlace. See proof for Z-LPR properties [13] and for S-LPR properties, e.g., [9] (treated there in both cases as tangent functions of polynomials, see Corollary 3 below). If the (S) property is taken to be established, then the (Z) part of the lemma can also be deduced by bilinear mapping the $s = j\omega$ onto the unit circle [23]. □

The above properties impose restrictions on the structure of rational function candidates for being PR. An S-PR or S-LPR rational function can have numerator and denominator polynomials of degrees that are equal or differ by one. Furthermore, if such a rational function is *real* the two polynomials must be of different degrees so that the function has a pole or zero at $s = 0$. Z-PR or Z-LPR rational functions must have numerator and denominator polynomials of equal degree. In addition, if the Z-PR or Z-LPR function is *real* then it must have a zero or a pole at $z = 1$ and another zero or pole at $z = -1$, depending on the parity of n, must appear to balance the numerator and denominator polynomials to equal degrees.

Using the LPR terminology we can now rephrase the respective conditions (ii) in Theorems 1 and 2 as follows.

Corollary 3.

(S) *The tangent function $\rho_n(s)$ of the polynomial $H_n(s)$ is S-LPR if and only if $H_n(s)$ is stable.*

(Z) *The tangent function $\rho_n(z)$ of the polynomial $H_n(z)$ is Z-LPR if and only if its $H_n(z)$ is stable.*

The next theorem and the following corollary give necessary and sufficient conditions for losslessness preservation by the LDI transformation.

Theorem 4. *An SLPR function $\rho_n^s(s)$ is mapped into a Z-LPR function $\rho_n^z(z)$ by*

$$\rho_n^z(z) = (z^{1/2} + z^{-1/2})^{\pm 1} \rho_n^s\left(\frac{\omega_L(z^{1/2} - z^{-1/2})}{2}\right), \tag{14}$$

where the $\pm$ is $+/-$ for n even/odd, if and only if ω_L is greater than the largest critical frequency, ω_{n-1} (defined in (6b)*), of $\rho_n^s(s)$.*

Proof. Assume that $\rho_n(s)$ is S-LPR. That means that it is characterized by the (S) part of the Lemma 2, or, equivalently, that it has a structure given by (6). Regardless of whether or not the inequality $\omega_{n-1} < \omega_L$ holds, it is not difficult to show that the right-hand side of (14) has for $n = 2m$ and for $n = 2m + 1$, respectively, the forms

$$\rho_{2m+1}(z) = \frac{(z-1)T_{2m}(z)}{(z+1)T'_{2m}(z)}, \qquad \rho_{2m}(z) = \frac{(z-1)(z+1)T_{2m-2}(z)}{T_{2m}(z)},$$

where $T_{2m}(z)$, $T'_{2m}(z)$, and $T_{2m-2}(z)$ are some symmetric polynomials each of its indicated degree. In fact, a typical factor in (6) maps as follows:

$$(s^2 + \omega_k^2) = \frac{\omega_L^2}{4}[z^2 - 2zX_k + 1]z^{-1},$$

therefore their product sums up to a symmetric function. Assume first that $\omega_{n-1} < \omega_L$, then all the critical frequencies are contained in the interval

$$J = \{s \mid s = j\omega,\ \omega \in [-\omega_L, \omega_L]\}$$

and mapped one-to-one and onto the unit circle

$$C = \{z \mid z = e^{j\Omega},\ \omega \in [-\pi, \pi]\}$$

by

$$\omega = \omega_L \sin\left(\frac{\Omega T}{2}\right).$$

So that $\rho_n(z)$ has factors $(z^2 - 2zX_k + 1)$ with $X_k = \cos(\Omega_k T)$ ordered as requested in (9c) and the function Z-LPR.

To prove necessity, assume $\rho_n(z)$ resulting from (14) is Z-LPR. Then, since (9) holds, the zeros and poles of $\rho_n(z)$ are on the unit circle and ordered interlacingly. The ordering (9c), with $X_k = 2 - [\sin(\Omega_k/2)]^2$, induces, via $\omega_k/\omega_L = \sin(\Omega_k T/2)$, a corresponding ordering on the critical frequencies, $(\omega_1/\omega_L)^2 < \cdots < (\omega_{n-1}/\omega_L)^2 < 1$, and $\omega_{n-1} < \omega_L$ is implied. The necessity part could also be shown by logical negation. Assume $\omega_{n-1} > \omega_L$, then the mapping (1) maps a pair of imaginary zeros or poles $\pm j\omega_k$ of $\rho_n(s)$ that satisfies $\omega_k > \omega_L$ (and there is at least ω_{n-1} in this subset) into a pair of (real and negative) reciprocal zeros, one inside and the other outside the unit circle, implying that $\rho_n(z)$ is not Z-LPR by contradiction to Z-LPR features in Lemma 2 (equally, by contradiction to (9c) as $\rho_n(z)$ now contains factors $(z^2 - 2zX_k + 1)$ with $X_k > 1$). □

The obvious consequence of Theorem 6 on the Cauer ladder in Figure 6 is stated in the following.

Corollary 5. *A Z-Cauer ladder of Figure* 4 *with values for FDI and BDI* (13) *obtained from the S-Cauer lattice by the substitution of* γ_i *with corresponding* $\delta_i = \frac{1}{2}\omega_L\gamma_i$, *is lossless if and only if the highest critical frequency in* (6b) *satisfies* $\omega_{n-1} < \omega_L$.

The Cauer ladder in Figure 3, when the integrators are $I_i = 1/\gamma_i s$, is composed of a cascade of two ports characterized by

$$\begin{bmatrix} x_i(s) \\ x_{i-1}(s) \end{bmatrix} = \theta_i \begin{bmatrix} x_{i-1}(s) \\ x_{i-2}(s) \end{bmatrix}, \qquad \theta_i = \begin{bmatrix} \gamma_i s & 1 \\ 1 & 0 \end{bmatrix}. \tag{15}$$

The complete Cauer ladder is then characterized by the following 2×2 chain matrix that relates its terminal ports:

$$\begin{bmatrix} x_n(s) \\ x_{n-1}(s) \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} x_0(s) \\ x_{-1}(s) \end{bmatrix}. \tag{16}$$

The elements of this chain matrix can be obtained as

$$A(s) = R_n^{(1)}(s), \qquad B(s) = R_{n-1}^{(2)}(s), \tag{17}$$
$$C(s) = R_{n-1}^{(1)}(s), \qquad D(s) = R_{n-2}^{(2)}(s),$$

through recursions similar to (5d) as follows:

$$R_i^{(\mu)}(s) = \gamma_{\mu+i-1} s R_{i-1}^{(\mu)}(s) + R_{i-2}^{(\mu)}(s), \tag{18}$$

$i = 1, \ldots, k$, with $R_0^{(\mu)} = 1$ and $R_{-1}^{(\mu)} = 0$ ($\mu = 1, 2$). Note that $R_k^{(\mu)}(s)$ is a kth degree polynomial (even/odd for k even/odd) and that the two sequences differ in the $R_k^{(\mu)}$ is built on the parameters (integrator values) $\gamma_\mu, \ldots, \gamma_{k-1+\mu}$ for $\mu = 1, 2$, respectively.

When the integrators in Figure 3 are assumed to be LDI, $I_i = 1/\delta_i(z^{1/2} - z^{-1/2})$, we have, instead of (15),

$$\begin{bmatrix} \hat{x}_i(z) \\ \hat{x}_{i-1}(z) \end{bmatrix} = \theta_i^{\mathrm{L}} \begin{bmatrix} \hat{x}_{i-1}(z) \\ \hat{x}_{i-2}(z) \end{bmatrix}, \qquad \theta_i^{\mathrm{L}} = \begin{bmatrix} \delta_i(z^{1/2} - z^{-1/2}) & 1 \\ 1 & 0 \end{bmatrix}. \tag{19}$$

The terminal-to-terminal relations for the cascade of n LDI elements can again be characterized by an overall chain matrix, say,

$$\begin{bmatrix} \hat{x}_n(z) \\ \hat{x}_{n-1}(z) \end{bmatrix} = \begin{bmatrix} \hat{A}(z) & \hat{B}(z) \\ \hat{C}(z) & \hat{D}(z) \end{bmatrix} \begin{bmatrix} \hat{x}_0(z) \\ \hat{x}_{-1}(z) \end{bmatrix}, \qquad \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = \theta_n^{\mathrm{L}} \ldots \theta_1^{\mathrm{L}}, \tag{20}$$

in which $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ represent polynomials in $z^{1/2}$ and $z^{-1/2}$ that rather than define them directly we now define immediately by relations to single variable polynomials associated with the recursion (8d). Indeed, it is possible to scale out the $z^{\pm 1/2}$ factors by replacing pairs of LDI sections by FDI and BDI sections defined by the elementary matrices

$$\theta_i^{\mathrm{F}} = \begin{bmatrix} \delta_i(z-1) & 1 \\ 1 & 0 \end{bmatrix}, \qquad \theta_i^{\mathrm{B}} = \begin{bmatrix} \delta_i(1-z^{-1}) & 1 \\ 1 & 0 \end{bmatrix}. \tag{21}$$

The obvious relations that hold between the three elementary transition matrices is

$$\theta_i^{\mathrm{L}} = \mathrm{diag}[z^{-1/2}, 1]\theta_i^{\mathrm{F}}\ \mathrm{diag}[1, z^{1/2}]$$

and

$$\theta_i^{\mathrm{L}} = \mathrm{diag}[1, z^{-1/2}]\theta_i^{\mathrm{B}}\ \mathrm{diag}[z^{1/2}, 1]. \tag{22}$$

In this case it is possible to replace a realization with n LDI by a cascade of n elements of FDI and BDI arranged alternatingly as in Figure 4. We demonstrate this possibility for even degree (the odd degree case follows a similar derivation and leads to the same result). We have, for $n = 2m$,

$$\begin{bmatrix} \hat{A}(z) & \hat{B}(z) \\ \hat{C}(z) & \hat{D}(z) \end{bmatrix} = \begin{bmatrix} z^{-1/2} & 0 \\ 0 & 1 \end{bmatrix} \theta_{2m}^{\mathrm{F}} \theta_{2m-1}^{\mathrm{B}} \cdots \theta_1^{\mathrm{B}} \begin{bmatrix} z^{1/2} & 0 \\ 0 & 1 \end{bmatrix}.$$

After substitution into (21) and for the following redefinition of state variables,

$$\begin{bmatrix} x_{2i}(z) \\ x_{2i-1}(z) \end{bmatrix} = z^i \begin{bmatrix} z^{1/2}\hat{x}_{2i}(z) \\ \hat{x}_{2i-1}(z) \end{bmatrix},$$

we obtain

$$\begin{bmatrix} x_n(z) \\ x_{n-1}(z) \end{bmatrix} = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} \begin{bmatrix} x_0(z) \\ x_{-1}(z) \end{bmatrix}, \tag{23a}$$

where

$$\begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} = \begin{bmatrix} \delta_n(z-1) & z \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} \delta_2(z-1) & z \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_1(z-1) & 1 \\ 1 & 0 \end{bmatrix}. \tag{23b}$$

The relations (23) hold for both even and odd n. The four polynomials in the last chain matrix can be obtained by recursions similar to (8d). They are given by

$$A(z) = T_n^{(1)}(z), \qquad B(z) = T_{n-1}^{(2)}(z), \tag{24}$$
$$C(z) = T_{n-1}^{(1)}(z), \qquad D(z) = T_{n-2}^{(2)}(z),$$

where $T_k^{(\mu)}(z)$ is a kth degree polynomial (symmetric/antisymmetric for k even/odd) and $\{T_k^{(\mu)}(z)\}$, $\mu = 1, 2$, are two sequences that follow the recursion

$$T_i^{(\mu)}(z) = \delta_{i+1-\mu}(z-1)T_{i-1}^{(\mu)}(z) + zT_{i-2}^{(\mu)}(z), \tag{25}$$

$i = 1, \ldots, n-1$, starting with $T_0^{(\mu)} = 1$, $T_{-1}^{(\mu)} = 0$ and differ only in that $T_k^{(\mu)}(z)$ is determined from the set of parameters (integrator values) $\delta_\mu, \ldots, \delta_{k+1-\mu}$ for $\mu = 1, 2$, respectively.

The elements in the chain matrix (20) can now be regarded as the *balanced polynomials* version of the corresponding elements in the chain matrix (23). The balanced polynomial $\hat{Q}_n(z)$ is a polynomial of degree $[n/2]$ in the two variables $z^{1/2}$ and $z^{-1/2}$ defined in relation to a corresponding polynomial $Q_n(z)$ (of degree n in just the z variable) by

$$\hat{Q}_n(z) = z^{-n/2}Q_n(z).$$

Balanced polynomials are interesting in association to symmetric and anti-symmetric polynomials because then they associate the latter with sine and cosine polynomials. (They are put to use in Appendix A.)

The Cauer filter is obtained from the Cauer ladder by adding terminators at its terminal ports as illustrated in Figure 5. The analogue Cauer filter is obtained when the Cauer lattice is resistively terminated. In this case the Cauer filter in Figure 5 is composed from the S-Cauer ladder and the terminators are

$$X_l = k_l \quad \text{and} \quad X_s = k_s, \tag{26}$$

where k_l and k_s are some nonnegative numbers. The terminal ports of the S-Cauer ladder are constrained by

$$x_{-1}(s) = k_l x_0(s), \qquad u(s) = x_n(s) + k_s x_{n-1}(s). \tag{27}$$

Thus the transfer function $x_0(s)/u(s) = 1/Q_n(s)$ is determined by the polynomial

$$Q_n(s) = A(s) + k_l B(s) + k_s[C(s) + k_l D(s)], \tag{28}$$

with $A(s)$, $B(s)$, $C(s)$, and $D(s)$ given by the polynomials (17).

It is well known that $Q_n(s)$ is stable whenever $H_n(s)$ is stable. A typical proof that uses a jargon of established synonymous physical and mathematical mixture of terms could run as follows: $H_n(s)$ is stable therefore $\rho_n^s(s)$ has the CFE (7) with positive coefficients which admits the lattice realization by the two port (lossless) section of L's and C's whose termination with resistors

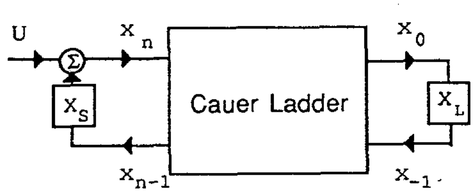

Figure 5. The doubly terminated Cauer filter configuration.

form a passive network. The input immittance to this filter $u(s)/x_n(s)$, given by $Q_n(s)/[A(s) + r_l B(s)]$, is therefore a positive real function. It follows that its numerator and denominator polynomials are stable. In particular, $Q_n(s)$ is stable. We show that our study so far leads the way for a similar flow of reasoning to make sense.

We return to Figure 5 where the ladder is now taken to be the Z-Cauer ladder of Figure 4, and choose the terminators as

$$X_l = (z + 1)k_l \quad \text{and} \quad X_s = (z + 1)k_s, \tag{29}$$

where the necessity of the $(z + 1)$ dependency will soon become apparent. The constraints on the terminal ports of the Z-Cauer ladder are

$$x_{-1}(z) = k_l(z + 1)x_0(z), \qquad u(z) = x_n(z) + k_s(z + 1)x_{n-1}(z). \tag{30}$$

Consequently, $x_0(z)/u(z) = 1/Q_n(z)$ is an all-pole transfer function with denonator

$$Q_n(z) = A(z) + k_l(z + 1)B(z) + k_s(z + 1)[C(z) + k_l(z + 1)D(z)], \tag{31}$$

where the polynomials $A(z)$, $B(z)$, $C(z)$, and $D(z)$ are given by (24).

The next theorem establishes the anticipated close similarity between the S- and Z-Cauer filters, featuring for the Z-Cauer filter.

Theorem 6. *Let $H_n(z) = T_n(z) + (z + 1)T_{n-1}(z)$ be the polynomial with the decomposition in* (8) *that determines $\delta_1, \ldots, \delta_n$ (via* (8)) *or is determined by them (via* (25)), *and let $M_{n-1}(z)$ and $M_{n-2}(z)$ be the polynomials determined* (*via* (25)) *by the subset $\delta_2, \ldots, \delta_n$. (In other words, we currently split the notation $T_m^{(\mu)}(z)$ of* (25) *into $T_m(z) = T_m^{(1)}(z)$ and $M_m(z) = T_m^{(2)}(z)$).*

(1) *The necessary and sufficient condition for the polynomial*

$$Q_n(z) = T_n(z) + k_s(z + 1)T_{n-1}(z) + k_l(z + 1)M_{n-1}(z) + k_l k_s(z + 1)^2 M_{n-2}(z)$$

(31) *to be stable for any all $k_s \geq 0$ and $k_l \geq 0$ $(k_l + k_s \neq 0)$ is that $H_n(z)$ be stable.*

(2) *Moreover, if the Cauer filter in Figure 5 realizes this stable transfer function $x_0(z)/u(z) = 1/Q_n(z)$, then all rational functions that simulate the S-Cauer filter's input or output immittance functions are Z-PR, and all transfer functions that simulate S-stable functions are Z-stable.*

Proof. The proof for Theorem 6 is given in Appendix A. □

Corollary 7. *The Z-Cauer filter of Figure 5 with the ladder of Figure 4, having integrator values δ_i related to the S-Cauer integrator values γ_i by $\delta_i = \frac{1}{2}\omega_L \gamma_i$, is stable if and only if*

$$\omega_L > \omega_{n-1}, \tag{32}$$

where ω_{n-1} is the highest critical condition in (6b). *The following alternative but equivalent view is also useful. Let*

$$\delta_i = \tfrac{1}{2}\omega_L \gamma_i, \tag{33a}$$

then the Cauer filter is stable if and only if

$$\Delta_n = \begin{bmatrix} \omega_L\gamma_1 & -1 & & & 0 \\ -1 & \omega_L\gamma_2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & \omega_L\gamma_{n-1} & -1 \\ 0 & & & -1 & \omega_L\gamma_n \end{bmatrix} \tag{33b}$$

is positive definite.

Proof. Condition (32) combines Corollary 5 with Theorem 6. The condition in (33) follows from the equivalence, when (33a) holds, of (33b) and (11b). □

Remark 5. Since ω_L can be interpreted as the sampling rate, this corollary sustains and puts in precise terms an assertion already made in the original paper by Bruton [7]. Namely, based on convergence to the stable analogue case, it is expected that for high enough ω_L the filter will become stable.

Remark 6. The present condition is useful for designing stable LDI filters without reference to analogue prototypes. It means that for any set of positive integrator values δ_i there exists a minimal positive scaling factor $\bar{\omega}$ (greater or less than 1) such that any scaled sequences $\{\omega\delta_i\}$ of integrators will realize a stable filter for any $\omega > \bar{\omega}$ and suggest a practical procedure to determine this $\bar{\omega}$.

Remark 7. The conditions set for losslessness and stability via the LDI transformation do not depend on the specifics of Cauer filter configuration. These can therefore also be used for more general digital configurations that implement with the lossless digital integrators various documented analog filters.

4. Sampled filters

Consider now the case where the digital filter is related to its analog prototype in a sampled data mode. Typically, the sampling rate is chosen high enough to handle a band-limited input signal, say in the frequency range $[0, \omega_c]$. The z-variable represents the *Z-transform* so that if $v(t)$ is the sampled signal and $v(nT) \leftrightarrow V(z)$, then $v(nT - T) \leftrightarrow z^{-1}V(z)$. In SC filters sampling is determined by its clock period τ such that the input and output sampled data signals change values at the switching instants nT and hold over the period interval T which is equal to $\tau/2$ (in LDI, or τ in the FDI/BDI configuration that admits a lower clock rate; the actual details of switching policy are not essential for the following argument). Denote frequencies of the analogue and digital filter by $\omega = 2\pi f$ and $\Omega = 2\pi F$, respectively. Then the mapping used in (14)

$$s = \frac{\omega_L(z^{1/2} - z^{-1/2})}{2} \qquad \text{implies} \quad \omega = \omega_L \sin\left(\frac{\Omega T}{2}\right) \tag{34}$$

by the substitutions $z = e^{j\Omega T}$ and $s = j\omega$. The admissible range of frequencies in this mapping are $f \in [0, f_L]$ and $F \in [0, 1/2T]$. The scaling factor $\omega_L = 2\pi f_L$ that was used in Theorem 4 now becomes related to the sampling rate $f_s = 1/T$ by $f_L = \frac{1}{2}f_s$. The meaningful spectra that the digital filter handles has to fall in the range $[0, f_L]$. In practice, since the mapping by the sine function becomes less linear the closer it gets to f_L, the bandpass is usually chosen well within the lower half of this admissible frequency range. In the rest of this section we substantiate the assertion that the last critical frequency ω_{n-1} associated with the $H_n(s)$ by Theorem 1, see (6), is a relevant estimate of the low-pass filter's cutoff frequency ω_c. Whenever this assertion holds, practical considerations that set ω_c to less than half ω_L guarantee that condition (32) is satisfied, hence the filter will be stable even though ignorant of the stability difficulty inherent with LDIs.

The next theorem shows that the highest critical frequency of the S-tangent function associated with a stable $H_n(s)$ bounds the moduli of almost all the zeros of $H_n(s)$.

Theorem 8. *At least $n - 1$ zeros of an S-stable polynomial $H_n(s)$ are bounded by the rectangle*

$$B = \{s | s = \sigma + j\omega, \ -\omega_{n-1} < \sigma < 0, \ |\omega| < \omega_{n-1}\}, \tag{35}$$

where ω_{n-1} is the largest critical frequency of $\rho_n(s)$, the tangent function associated with $H_n(s)$. The nth *and last zero of $H_n(s)$ may be a real and negative zero of magnitude larger than ω_{n-1}.*

Proof. The proof of Theorem 8 amounts to showing that the root locus of (using notation from Theorem 1)

$$H_n(s) = R_n(s) + KsR_{n-1}(s), \tag{36}$$

as K varies from 0 to ∞, has the form illustrated in Figure 6 for $n = 4$ and $n = 5$. Some crucial details, of the proof are given in Appendix B. □

The theorem asserts that the last critical frequency ω_{n-1} is much higher than the bandpass range of the all-pole low-pass filter $1/H_n(s)$. Our filter of interest is, however, not $1/H_n(s)$ but $1/Q_n(s)$ that is related to $H_n(s)$ via (28)

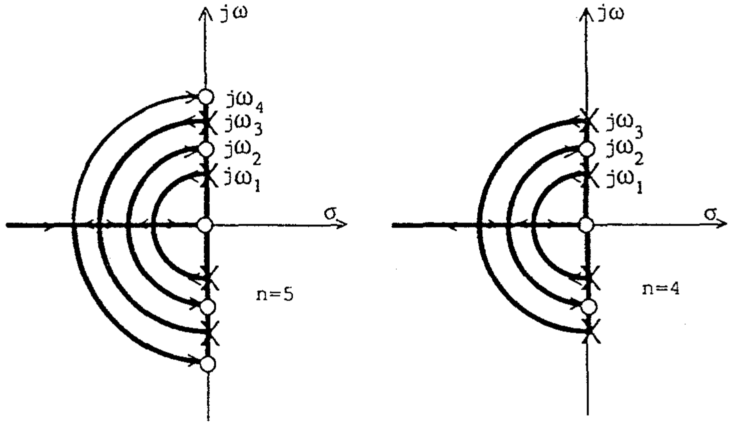

Figure 6. Root location for $H_n(s)$ in relation to critical frequencies of $\rho_n(s)$.

(because we want to see whether a practical design having chosen the sampling rate in accordance to the bandwidth of the filter it wanted to simulate could still be assumed inattentively to meet the stability requirement (32)). Indeed, $Q_n(s)$ is different (unless $k_l = 0$) from $H_n(s)$. However, if k_l, k_s, and γ_i have a close range of values, which is typically the case in practice, ω_{n-1} can be argued to stay in a close approximation also for ω_c, the passband edge of the filter $1/Q_n(s)$. In typical LDI SC designs the sampling rate is chosen many times higher than the edge of the analogue filter's frequency range that the SC filter is to simulate without significant warping. Therefore, these designs have still implicitly satisfied condition (32) and therefore were found to be stable. Thus to an explanation why LDI designs were often found stable in practice. The theory in this paper can of course be used to design the Cauer filter exclusively in the Z-plane. One possible procedure could be to test a primary desirable filter (using Theorem 2) and if an unstable filter is indicated, then apply Theorem 7 to stabilize it by extracting the highest critical frequence of $Q_n(s)$ and use it as an estimate of the extent of upscale of the integrator values that is needed in order to make the matrix M_n of (33) positive definite.

5. Concluding remarks

This paper laid side by side continuous system concepts like Routh's stability test, Cauer ladders and losslessness, Cauer filter and passivity, and their discrete system equivalents. Our concentration on the Cauer filter alone, that stems from our original interest in demonstrating the most immediate digital filter facet of our recently developed stability formulation, should by no means restrict the broadness of further applications of some of the derived results or the generality of the concepts behind our approach.

First we remark that, clearly, Theorem 2 is applicable to test the stability of any discrete system polynomial, and is apparently the most efficient test currently available. (Although the versions in [4] or [5] have a more pleasant set-up for when the link to the imaginary axis interval, or LDI, is not requested.) The design of a digital filter using the concepts of Z-LPR and Z-PR functions is also of a broad enough scope to attain stable filters which is the reason why we supplied definitions and features for them somewhat beyond the immediate needs of the application considered.

Next it is remarked that filters in other configurations may also be designed, based on the LDI, to which the theory here may still apply quite directly. The LDI may still be applicable in configurations that do not admit pairing LDIs so as to render the preferable use of FDI/BDI pairs.

The relations between sampling rates and filter stability described in this paper may also be useful to other configurations based on analogue filters. As long as a set of δ_i represents the *effective* integrator values of a filter, Corollary 7 indicates a neat way to stabilize the filter by increasing the sampling rate, in the case of SC filter design, or by up-scaling the integrator values in other cases of *true* (i.e., not sampled) digital filter designs.

Appendix A. Proof of Theorem 6

Proof. (Necessity) If $Q_n(z)$ is stable for all nonnegative k_l and k_s, then we obtain that $H_n(z)$ is stable because $Q_n(z)$ reduces to it for the choices $k_l = 0$, $k_s = 1$. Next, it has to be shown that stability of $H_n(z)$ implies stable $Q_n(z)$ for all nonnegative k_l and k_s. If $H_n(z)$ is stable, then all the following polynomials are stable for all $K > 0$:

(i) $$E_n^{(1)}(z; K) := T_n(z) + K(z+1)T_{n-1}(z), \qquad \text{(A1a)}$$

(ii) $$E_{n-1}^{(2)}(z; K) := M_{n-1}(z) + K(z+1)M_{n-2}(z), \qquad \text{(A1b)}$$

(iii) $$E_n^{(3)}(z; K) := T_n(z) + K(z+1)M_{n-1}(z), \qquad \text{(A1c)}$$

(iv) $$E_{n-1}^{(4)}(z; K) := T_{n-1}(z) + K(z+1)M_{n-2}(z), \qquad \text{(A1d)}$$

$H_n(z) = E_n^{(1)}(z; 1)$ and it can be deduced from Theorem 2 (condition (ii)) that whenever we have such a decomposition of a polynomial into its symmetric and antisymmetric polynomial then $E_n^{(1)}(z; 1)$ is stable if and only if $E_n^{(1)}(z; K)$ is stable for all $K > 0$. If $H_n(z)$ is stable, then $E_{n-1}^{(2)}(z; 1)$ featured by the subset $\delta_2, \ldots, \delta_n$ of $\delta_1, \ldots, \delta_n$ is stable too (think of positive definiteness of $\Delta_n > 0$ and its submatrices in condition (iii) of Theorem 2). Therefore $E_{n-1}^{(2)}(z; K)$ is also stable. Stability of $E_n^{(3)}(z; K)$ and $E_{n-1}^{(4)}(z; K)$ will be deduced in the following proof that stable $H_n(z)$ is sufficient for stability $Q_n(z)$. It is seen that a stable $Q_n(z)$ implies the stability of the above four polynomials as the special cases (i) $k_l = 0$, $k_s = K$, (ii) $k_l \gg k_s = K$, (iii) $k_s = 0$, $k_l = K$, and (iv) $k_s \gg k_l = K$.

(Sufficiency) To assume $H_n(z)$ is stable it has to be shown that $Q_n(z)$ is stable. Write $Q_n(z)$ as

$$Q_n(z) = E_n^{(1)}(z; k_s) + k_l(z + 1)E_{n-1}^{(2)}(z; k_s). \tag{A2}$$

If $H_n(z)$ is stable, then $E_n^{(1)}(z; k_s)$ and $E_{n-1}^{(3)}(z; k_s)$ are stable. Therefore the zero locations of $Q_n(z)$ as k_l varies from zero and infinity, consists of n branches that move from n inside-the-unit-circle (IUC) zeros to $n - 1$ IUC zeros and a zero at $z = -1$. To show that this root locus remains inside the unit circle we show that

$$r_n^{(1)}(z) := \frac{Q_n(z)}{E_n^{(1)}(z; k_s)} = 1 + k_l r_n^{(2)}(z), \qquad \text{with} \quad r_n^{(2)}(z) := \frac{(z + 1)E_{n-1}^{(2)}(z; k_s)}{E_n^{(1)}(z; k_s)} \tag{A3}$$

is not vanishing for all $|z| = 1$.

Multiply the numerator and denominator of

$$r_n^{(2)}(z) = \frac{(z + 1)M_{n-1}(z) + k_s(z + 1)^2 M_{n-2}(z)}{T_n(z) + k_s(z + 1)T_{n-1}(z)} \tag{A4}$$

by $z^{-n/2}$ to express it by balanced polynomials

$$r_n^{(2)}(z) = \frac{(z^{1/2} + z^{-1/2})\hat{M}_{n-1}(z) + k_s(z^{1/2} + z^{-1/2})^2 \hat{M}_{n-2}(z)}{\hat{T}_n(z) + k_s(z^{1/2} + z^{-1/2})\hat{T}_{n-1}(z)}. \tag{A5}$$

The balanced polynomials of symmetric and antisymmetric polynomials are, respectively, purely real and imaginary on $|z| = 1$ [5]. More specifically for our case here, the balanced polynomials of the symmetric real polynomials $T_{2i}(z)$ and $M_{2i}(z)$ become, for $z = e^{j2\varphi}$, real polynomials in $\cos\varphi$, while, for antisymmetric polynomials, $T_{2i+1}(z)$ and $M_{2i+1}(z)$ become imaginary polynomials in $\sin\varphi$, namely

$$\begin{aligned} \tau_{2i}(\varphi) &:= \hat{T}_{2i}(e^{j2\varphi}), & \mu_{2i}(\varphi) &:= \hat{M}_{2i}(e^{j2\varphi}), \\ j\tau_{2i+1}(\varphi) &:= \hat{T}_{2i+1}(e^{j2\varphi}), & j\mu_{2i+1}(\varphi) &:= \hat{M}_{2i+1}(e^{j2\varphi}), \end{aligned} \tag{A6}$$

where φ is used for simplicity instead of the explicit dependencies on $\cos\varphi$ and $\sin\varphi$. For $z = e^{j2\varphi}$, $r_n^{(2)}(z)$ becomes either

$$r_{2m}^{(2)}(e^{j2\varphi}) = \frac{j2\cos\varphi\,\mu_{2m-1}(\varphi) + 4k_s(\cos\varphi)^2\mu_{2m-2}(\varphi)}{\tau_{2m}(\varphi) + j2k_l\cos\varphi\,\tau_{2m-1}(\varphi)}$$

for $n = 2m$ or, for $n = 2m + 1$,

$$r_{2m+1}^{(2)}(e^{j2\varphi}) = \frac{2k_s\cos\varphi\,\mu_{2m}(\varphi) + j4k_s(\cos\varphi)^2\tau_{2m-1}(\varphi)}{j\tau_{2m+1}(\varphi) + 2k_s\cos\varphi\,\tau_{2m}(\varphi)}.$$

Thus for either parity of n we have

$$\mathrm{Re}\{r_n^{(2)}(e^{j2\varphi})\} = \frac{4k_s(\cos\varphi)^2[\tau_n(\varphi)\mu_{n-2}(\varphi) + \tau_{n-1}(\varphi)\mu_{n-1}(\varphi)]}{\tau_n^2(\varphi) + 4k_s^2(\cos\varphi)^2\mu_{n-1}^2(\varphi)}. \tag{A7}$$

Using (20b) and (19a) we have the auxiliary relation

$$\det\begin{bmatrix} \hat{A}(z) & \hat{B}(z) \\ \hat{C}(z) & \hat{D}(z) \end{bmatrix} = \hat{T}_n(z)\hat{M}_{n-2}(z) - \hat{M}_{n-1}(z)\hat{T}_{n-1}(z) = (-1)^n,$$

which for $z = e^{j2\varphi}$ becomes

$$\tau_n(\varphi)\mu_{n-2}(\varphi) + \tau_{n-1}(\varphi)\mu_{n-1}(\varphi) = 1. \tag{A8}$$

Setting this in (A7) we obtain

$$\mathrm{Re}\{r_n^{(2)}(e^{j2\varphi})\} = \frac{4k_l(\cos\varphi)^2}{\tau_n^2(\varphi) + 4k_l^2(\cos\varphi)^2\mu_{n-1}^2(\varphi)}. \tag{A9}$$

Clearly, this expression is nonnegative for all φ. Thus, from (A.3), the real part of $r_n^{(1)}(z)$ is positive for all $|z| = 1$. Therefore the root locus of (A2) does not cross the unit circle for any $0 \le k_l < \infty$, which completes the proof that $Q_n(z)$ is stable for all $0 \le k_l, k_s < \infty$.

Notice that we have in fact shown that $r_n^{(1)}(z)$ is PR because its real part is positive on the unit circle and it is analytic outside the unit circle. Furthermore, even $r_n^{(2)}(z)$ is PR because it is analytic outside the unit circle and (A9) is positive except the zero at $z = -1$. As a demonstration of the property of residues of poles on the unit circle of a PR function observe that a direct testing of whether $1/r_n^{(2)}(z)$ is PR would involve checking the residue of the pole at $z = -1$. It is easily seen to be given by $T_n(-1)/T_{n-1}(-1) < 0$ (recall the stability test in Theorem 2 to $H_n(z)$) which is in agreement with Lemma 1.

Here is a list of relevant rational functions that are PR if $H_n(z)$ is stable:

(i) $$r_n^{(1)}(z) := \frac{Q_n(z)}{E_n^{(1)}(z; k_s)}, \tag{A10a}$$

(ii) $$r_n^{(2)}(z) := \frac{(z+1)E_{n-1}^{(2)}(z; k_s)}{E_n^{(1)}(z; k_s)}, \tag{A10b}$$

(iii) $$r_n^{(3)}(z) := \frac{Q_n(z)}{E_n^{(3)}(z; k_l)}, \tag{A10c}$$

(iv) $$r_n^{(4)}(z) := \frac{(z+1)E_{n-1}^{(4)}(z; k_l)}{E_n^{(3)}(z; k_l)}. \tag{A10d}$$

Proofs that $r_n^{(3)}(z)$ and $r_n^{(4)}(z)$ are PR may follow closely the above proofs for $r_n^{(1)}(z)$ and $r_n^{(2)}(z)$ being PR. We have

$$r_n^{(3)}(z) = 1 + r_n^{(4)}(z) \tag{A11}$$

and after repeating steps similar to (A4)–(A9) it can be shown that

$$\mathrm{Re}[r_n^{(4)}(e^{j2\varphi})] = \frac{4k_s(\cos\varphi)^2}{\tau_n^2(\varphi) + 4k_s^2(\cos\varphi)^2\mu_{n-1}^2(\varphi)}. \tag{A12}$$

Thus the real part of $r_n^{(3)}(z)$ is positive on the unit circle. Combining this with $Q_n(z)$ being stable implies $r_n^{(3)}(z)$ is PR. It then follows that $E_n^{(3)}(z)$ is stable too, as was claimed at (A1c). Combine this with (A11) being nonnegative on the unit circle and we realize that $r_n^{(4)}$ is PR (it vanishes at $z = -1$ where, as before, its inverse has the necessarily negative residue $T_{n-1}(-1)/T_n(-1)$). It then follows that $E_n^{(4)}(z)$ is stable, as claimed at (A1d).

Clearly, stability of $H_n(z)$ implies stability of polynomials like (A1) and positive realness of rational functions like (A10) for other degrees m also, $m < n$. Finally, note that each of the rational functions (A10) holds significance as Z-equivalent of meaningful analogue driving point immittances. More specifically,

$$r_n^{(1)}(z) = \frac{u(z)/x_0(z)}{u(z)/x_0(z)_{|k_l\to\infty}}, \qquad r_n^{(2)}(z) = \frac{u(z)/x_0(z)_{|k_l=0}}{u(z)/x_0(z)_{|k_l\to\infty}},$$

$$r_n^{(3)}(z) = \frac{u(z)}{x_n(z)}, \qquad r_n^{(4)}(z) = \frac{(z+1)x_{n-1}(z)}{x_n(z)}. \qquad \square$$

Appendix B. Proof of Theorem 8

Consider the root locus of the equation

$$1 + K\rho_n(s) = 0 \tag{B1}$$

for $0 \le K < \infty$ where here we assume that $\rho_n(s)$ is given by (6) with $K = 1$. The roots of (B1) are the zeros of $H_n(s)$ (36). We prove Theorem 7 by showing that this root locus is of the pattern sketched in Figure 6. Namely all its branches are inside the rectangle B defined in (35) except one branch that goes to infinity along the negative real axis.

Since, clearly, $H_n(s)$ of (36) being stable for $K = 1$ is stable for all $K > 0$, so the root locus resides completely in the left half of the S-plane. Using standard root locus rules, the pattern in Figure 6 is understood in general. For example, the locus has to have n branches forming right angles to the $j\omega$ and at $n - 1$ breaking points along the negative real axis. Therefore the locus is clearly contained in the left half "horizontal" strip bounded by $\mathrm{Im}\{s\} = -j\omega_{n-1}$ from below and by $\mathrm{Im}\{s\} = j\omega_{n-1}$ from above. Apparently the only crucial fact that is not obvious is that, for all K, at least $n - 1$ roots have real parts in $[-\omega_{n-1}, 0]$. This feature is proven if we show that the leftmost breaking point, say $-\lambda_{n-1}$, is of modulus not greater than ω_{n-1}, i.e., if $\lambda_{n-1} \le \omega_{n-1}$.

As is well known the breaking points of the "closed-loop" root locus whose "open-loop" transfer function has zeros s_i^0, $i = 1, \dots, l$, and poles s_i^p, $i = 1, \dots, m$, are at solutions of

$$\sum_{i=0}^{l} \frac{1}{s + s_i^0} = \sum_{i=0}^{m} \frac{1}{s + s_i^p}. \tag{B2}$$

The breaking points in our case are therefore a solution of

$$L(s^2) = R(s^2), \tag{B3a}$$

where

$$L(s^2) := \frac{1}{s^2 + \omega_1} + \frac{2}{s^2 + \omega_3^2} + \cdots + \frac{2}{s^2 + \omega_{2m-1}^2}, \tag{B3b}$$

$$R(s^2) := \frac{1}{s^2} + \frac{2}{s^2 + \omega_2^2} + \cdots + \frac{2}{s^2 + \omega_{2m}^2}, \tag{B3c}$$

where $n = m + l + 1$, $l = m - 1$ or $l = m$. It suffices to show that for no real s such that $s < -\omega_{n-1}$ can equality $L(s^2) = R(s^2)$ hold. We proceed to prove that, for all real $\lambda^2 \geq \omega_{n-1}^2$, the following strict inequalities hold:

$$L(\lambda^2) - R(\lambda^2) > 0, \qquad n = 2m, \tag{B4a}$$

$$R(\lambda^2) - L(\lambda^2) > 0, \qquad n = 2m + 1. \tag{B4b}$$

Assume $n = 2m$,

$$L(\lambda^2) - R(\lambda^2) = \frac{2}{\lambda^2 + \omega_{n-1}^2} - \frac{1}{\lambda^2} + \left[\frac{2}{\lambda^2 + \omega_1^2} - \frac{2}{\lambda^2 + \omega_2^2}\right] + \cdots + \left[\frac{2}{\lambda^2 + \omega_{2l-1}^2} - \frac{2}{\lambda^2 + \omega_{2l}^2}\right]. \tag{B5}$$

The first two terms are positive for $\lambda^2 > \omega_{n-1}^2$ because

$$\frac{2}{\lambda^2 + \omega_{n-1}^2} - \frac{1}{\lambda^2} > \frac{2}{\lambda^2 + \lambda^2} - \frac{1}{\lambda^2} = 0.$$

The remaining terms are paired in brackets of positive contributions for any $\lambda^2 > 0$ because each time $\omega_{2i-1} < \omega_{2i}$, $i = 1, \dots, l$.

Assume next $n = 2m + 1$ and evaluate

$$R(\lambda^2) - L(\lambda^2) = \frac{1}{\lambda^2} + \frac{2}{\lambda^2 + \omega_{n-1}^2} - \frac{2}{\lambda^2 + \omega_1^2} + \left[\frac{2}{\lambda^2 + \omega_2^2} - \frac{2}{\lambda^2 + \omega_3^2}\right] + \cdots + \left[\frac{2}{\lambda^2 + \omega_{2m-2}^2} - \frac{2}{\lambda^2 + \omega_{2m-1}^2}\right]. \tag{B6}$$

Again the content of each pair of brackets is positive for all λ^2. We show that the first three terms are positive for $\lambda^2 > \omega_{n-1}^2$ as follows:

$$\frac{1}{\lambda^2} + \frac{2}{\lambda^2 + \omega_{n-1}^2} - \frac{2}{\lambda^2 + \omega_1^2} = \left[\frac{1}{\lambda^2} - \frac{1}{\lambda^2 + \omega_1^2}\right] + \frac{2}{\lambda^2 + \omega_{n-1}^2}$$

$$- \frac{1}{\lambda^2 + \omega_1} > \frac{2}{\lambda^2 + \omega_{n-1}^2} - \frac{1}{\lambda^2 + \omega_1^2}$$

$$= \frac{\lambda^2 + 2\omega_1^2 - \omega_{n-1}^2}{(\lambda^2 + \omega_{n-1}^2)(\lambda^2 + \omega_1^2)}$$

$$> \frac{2\omega_1^2}{(\lambda^2 + \omega_{n-1}^2)(\lambda^2 + \omega_1^2)} > 0.$$

The last inequality holds because $\lambda^2 > \omega_{n-1}^2$. Thus we showed that for both even and odd n values no break-in or break-out points are possible on the real axis interval $(-\infty, \omega_{n-1}]$. This completes the proof that all zeros of $H_n(s)$ are in the left half-plane with imaginary parts in the interval $[-j\omega_{n-1}, j\omega_{n-1}]$ and at least $n - 1$ zeros of $H_n(s)$ have real parts in the interval $[-\omega_{n-1}, 0]$. □

References

[1] Y. Bistritz, A stability new test for linear discrete systems in table form, *IEEE Trans. Circuits and Systems*, **30**, 917–919, 1983.

[2] Y. Bistritz, A new unit circle stability criterion, *Proceedings of the Symposium on Mathematical Theory of Networks and Systems*, MTNS-83, June 20–24, Israel, Lecture Notes in Control and Information Sciences, Vol. 58, Springer-Verlag, Berlin, 1984, pp. 69–87.

[3] Y. Bistritz, Z-domain continued fraction expansions for stable discrete systems polynomials, *IEEE Trans. Circuits and Systems*, **32**, 1162–1166, 1985.

[4] Y. Bistritz, Zero location with respect to the unit circle of discrete-time linear system polynomials, *Proc. IEEE*, **72**, 1131–1142, 1985.

[5] Y. Bistritz, A circular stability test of general polynomials, *Systems Control Lett.*, **7**, 89–97, April 1986.

[6] E. I. Jury and M. Mansour, On terminology relations between continuous and discrete systems criteria, *Proc. IEEE*, **73**, 884, 1985.

[7] L. T. Bruton, Low sensitivity digital ladder filters, *IEEE Trans. Circuits and Systems*, **22**, 168–176, 1975.

[8] S. O. Scanlan, Analysis and synthesis of switched-capacitor state-variable filters, *IEEE Trans. Circuits and Systems*, **28**, 85–93, 1981.

[9] E. A. Guillemin, *The Mathematics of Circuit Analysis*, MIT Press, Cambridge, MA, 1949.

[10] S. R. Parker, P. M. Chirlian, and E. Peskin, Continuants and the synthesis of low-pass resistively terminated *LC* ladder networks, *IEEE Trans. Circuit Theory*, **13**, 209–212, 1966.

[11] M. H. Lee and C. Chang, Low sensitivity switched-capacitor ladder filters, *IEEE Trans. Circuits and Systems*, **27**, 475–480, 1980.

[12] R. Gregorian, K. W. Martin, and G. C. Temes, Switched-capacitor circuit design, *Proc. IEEE*, **71**, 941–966, 1983.

[13] H. W. Schussler, A stability theorem for discrete systems, *IEEE Trans. Acoust. Speech Signal Process.*, **24**, 87–89, 1979.

[14] Y. Bistritz, Direct bilinear Routh stability criteria for discrete systems, *Systems Control Lett.*, **4**, 265–271, 1984.

[15] M. S. Lee, G. C. Temes, C. Chang, and M. B. Ghader, Bilinear switched capacitor ladder filters, *IEEE Trans. Circuits and Systems*, **28**, 811–822, 1981.

[16] J. S. Nossek and G. C. Gabor, Switched-capacitor filter design using bilinear element modeling, *IEEE Trans. Circuits and Systems*, **27**, 481–491, 1980.

[17] A. M. Davis, A new Z domain continued fraction expansion, *IEEE Trans. Circuits and Systems*, **29**, 658–662, 1982.

[18] M. S. Ghausi and K. R. Laker, *Modern Filter Design—Active RC and Switched Capacitors*, Prentice-Hall, Englewood Cliffs, NJ, 1981.

[19] R. Parthasarathy and S. Narayaniyer, Algorithms for the expansion of Davis's z-domain continued fractions, *IEEE Trans. Circuits and Systems*, **34**, 945–948, 1987.

[20] Y. Bistritz, Digital Cauer-type ladders for stable filters, *Proceedings of the 1986 International Symposium on Circuits. and Systems*, San Jose, CA, May 1986, pp. 686–689.

[21] P. Delsarte, Y. Genin, and Y. Kamp, Application of index theory of pseudo-lossless functions to the Bistritz stability test, *Philips J. Res.*, **39**, 226-241, 1984.

[22] Y. Bistritz, A Cauchy index approach for zero location of polynomials with respect to the unit circle, *Proceedings of the 24th IEEE Conference on Decision and Control*, Fort Lauderdale, FL, December 1985, pp. 1250–1251.

[23] Y. Bistritz, A discrete stability equation theorem and method for stable model reduction, *Systems Control Lett.*, **1**, 373–381, 1982.