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# REFLECTIONS ON SCHUR–COHN MATRICES AND JURY–MARDEN TABLES AND CLASSIFICATION OF RELATED UNIT CIRCLE ZERO LOCATION CRITERIA*

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**Abstract.** We use the so-called *reflection coefficients* (RCs) to examine, review, and classify the Schur–Cohn and Marden–Jury (SCMJ) class of tests for determining the zero location of a discrete-time system polynomial with respect to the unit circle. These parameters are taken as a platform to propose a partition of the SCMJ class into four useful types of schemes. The four types differ in the sequence of polynomials (the "table") they associate with the tested polynomials by scaling factors: (A) a sequence of monic polynomials, (B) a sequence of least arithmetic operations, (C) a sequence that produces the principal minors of the Schur–Cohn matrix, and (D) a sequence that avoids division arithmetic. A direct derivation of a zero location rule in terms of the RCs is first provided and then used to track a proper zero location rule in terms of the leading coefficients of the polynomials of the B, C, and D scheme prototypes. We review many of the published stability tests in the SCMJ class and show that each can be sorted into one of these four types. This process is instrumental in extending some of the tests from stability conditions to zero location, from real to complex polynomial, in providing a proof of tests stated without a proof, or in correcting some inaccuracies. Another interesting outcome of the current approach is that a byproduct of developing a zero location rule for the Type C test is one more proof for the relation between the zero location of a polynomial and the inertia of its Schur–Cohn matrix.

## 1. Introduction

The condition of stability for a linear discrete shift invariant system corresponds to necessary and sufficient conditions for the zeros of the system's polynomial (its characteristic equation) to lie inside the unit circle in the $z$-plane. Consider a polynomial of degree $n$ with complex coefficients

$$p(z) = \sum_{i=0}^{n} p_i z^i \tag{1}$$

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and call it *stable* if it has all its zeros inside the unit circle. Algebraic stability tests for discrete systems are methods that determine in some finite number of arithmetic operations whether or not a characteristic polynomial of a system is stable. Zero location tests extend this problem to also counting the number of zeros of the polynomial inside the unit circle (IUC), on the unit circle (UC), and outside the unit circle (OUC).

The first *direct* algebraic criterion for unit circle stability is due to Schur [30], and its extension to zero location to Cohn [19]. Assume a polynomial $f_m(z)$ of degree $m$, its conjugate, reverse, and reciprocal (conjugate-reverse) polynomials will be denoted by

$$\begin{aligned} f_m(z) &= \sum_{i=0}^{m} f_{m,i} z^i, & \overline{f}_m(z) &:= \sum_{i=0}^{m} f_{m,i}^* z^i, \\ f_m^r(z) &:= \sum_{i=0}^{m} f_{m,m-i} z^i, & f_m^\sharp(z) &:= \overline{f}_m^r(z), \end{aligned} \tag{2}$$

respectively ($*$ will denote complex conjugation). Cohn provided an algorithm to implement his zero location criterion. Given a polynomial $p(z)$ he proposed to construct a sequence of polynomials $\{f_m(z)\}$ of descending degrees $m = n$, $n-1, \ldots, 1, 0$, starting with $f_n(z) = p(z)$, then for $m = n, \ldots, 1$, applying the following recursion:

$$f_{m-1}(z) = \begin{cases} \frac{1}{z}\left[f_m(z) - \frac{f_{m,0}}{f_{m,m}^*} f_m^\sharp(z)\right] & \text{for } |f_{m,m}| > |f_{m,0}| \\ f_m(z) - \frac{f_{m,m}}{f_{m,0}^*} f_m^\sharp(z) & \text{for } |f_{m,m}| < |f_{m,0}| \end{cases}. \tag{3}$$

The information on the numbers of IUC and OUC zeros may be obtained from the changes between the two types of recursions [9] (a recent account and a new proof are available in [22, Example 2.2]).

The Schur–Cohn test in the form in which it is more familiar today is the growth of a modification devised to Cohn's setting by Marden [25], [26]. Subsequently, Marden's approach has been advanced in several stability table forms by Jury and other system theory researchers [2], [7], [8], [12]–[20], [27], [29]. Marden's scheme avoids switching between two modes as in (3) when the polynomial has both IUC and OUC zeros. Instead the recursion retains a *uniform* structure, viz.,

$$f_{m-1}(z) = f_{m,0}^* f_m(z) - f_{m,m} f_m^\sharp(z), \qquad f_n(z) = p(z).$$

The information on the numbers of IUC and OUC zeros is obtained from the relative magnitudes of $f_{m,m}$ and $f_{m,0}$ in a manner that will be shortly detailed.

In digital signal processing applications, stability testing is better known by an approach that comes from its own yard and terminology. There, a polynomial is determined as stable if (and only if) its "reflection coefficients" are all with moduli less than unity. The "reflection coefficients" are parameters that are used in several digital signal processing models [28] that may be associated with solving a Toeplitz set of equations by the Levinson algorithm [24].

It will turn out that most of the tests available today that stem from the works of Schur, Cohn, and Marden can be regarded as associating the tested polynomial $p(z)$ by a sequence $\{f_m(z)\}$ formed by a recursion obeys the following form:

$$zf_{m-1}(z) = \psi_m\{f_m(z) + k_m f_m^\sharp(z)\}, \qquad k_m = -\frac{f_{m,0}}{f_{m,m}^*}. \tag{4}$$

Here the $k_m$ will be shortly identified as the aforementioned *reflection coefficients* (RCs) and the $\psi_m$'s represent some nonzero (real or complex) numbers. Different tests may be the result of the choice of $\psi_m$ or of operating a reversion or conjugation (or both) at either or both sides of (4). Further differences may be caused by the initiation of the recursions. Namely, $f_n(z)$ may be taken to be $p(z)$, or $p^r(z)$, or $\overline{p}(z)$, or $p^\sharp(z)$ (where a demand to begin with a monic $p(z)/p_n$ may also sometimes be imposed). We shall refer to the class of tests that may be described by (4) within all above listed variations as the Schur–Cohn Marden–Jury (SCMJ) class. A typical stability or zero location from a subset of $n$ or $n+1$ *distinguished entries* from the array $f_{i,j}$ of coefficients. The distinguished entries can be conveniently arranged to be *real* even for complex polynomials. A long-standing tradition has been to lay out the entries $f_{i,j}$ in a tabular array. We shall therefore interchangeably refer to a member of the SCMJ class also as a (stability-) "table." The zero location rules depend on the signs of the distinguished entries. The number of IUC or OUC zeros is usually given by either the count of the number of negative or positive entries or by the number of sign variations or sign consistencies in an ordered sequence of distinguished entries. The scaling factors $\psi_m$ and the remaining possible variations that differentiate one test in the SCMJ class from another often have an intricate effect on the proper form of an accompanying stability and zero location rule for each individual test. A valid zero location rule turns out to be in particular prone to error because it is more sensitive to the accumulating effects of sign changes.

We would like to draw attention to a peculiar outcome of our definition of the SCMJ class by which Cohn's scheme does not actually belong to the SCMJ class (unless one's interest is restricted only to stability conditions). The main difference between Cohn's original scheme and the SCMJ class we defined may be described as follows. The "reflection coefficients" in Cohn's recursions (momentarily referring so to the fractions that appear in (3)) are always with moduli less than unity, while the recursion is switching between the indicated two modes. In contrast, in the SCMJ recursions (4), the recursions retain a uniform structure and instead the $k_m$'s are allowed to take both $|k_m| < 1$ and $|k_m| > 1$ values. The information on distribution of zeros with respect to the unit circle in the SCMJ class is held in the moduli of the $k_m$'s rather than in Cohn's variations of the structure of the recursion.

It is worthwhile to clarify that other tests to determine zero location with respect to the unit circle are available today that do not conform to the SCMJ class. The recent class of "immittance" tests [3], [4], [6] is notable as an alternative approach that also offers a lower count of arithmetic operations than the lowest count of operations that is achievable within the SCMJ class (the Type B tests) by approximately a factor of two.

The set of reflection coefficients of a polynomial is the central theme around which this paper revolves. This choice is motivated not only by the popularity of these coefficients in signal processing, but also by their following properties pertinent to our classification goals:

(1) The RCs of a given polynomial can be easily determined and they normally contain all (actually more than) the necessary information on the number of zeros of the polynomial inside and outside the unit circle.

(2) The RCs offer for the parametrization of the SCMJ class a set of parameters that is not affected by scaling and other possible twists that distinguish the many versions of tests in the SCMJ class.

It will soon become evident that for a polynomial's zero distribution the information of only the relative magnitudes of the RCs with respect to unity of a given *ordered* sequence of RCs is sufficient. The remaining information—knowing the actual numerical values of the RCs—represents the equivalent of knowing the exact numerical values of the zeros of the polynomial.

First we state and bring a direct proof for the rule to determine the numbers of IUC and OUC zeros of a polynomial from its given sequence of RCs. Then we focus on four choices of scaling parameters $\psi_m$ that represent interesting schemes in the SCMJ class, we define these as prototypes A, B, C, and D. We maintain a quite broad generality of treatment by always considering the zero location rule and not just stability conditions, and by always considering the polynomial to have complex coefficients. However, as the paper uses the reflection coefficients to classify the SCMJ class, the scope of zero location generality is limited by the assumption that the set of RCs is well defined. It should be said that it is also always possible to determine the distribution of zeros of any polynomial with respect to the unit circle in complementary cases that do not yield a well-defined and unique set of RCs. How this can be done has been shown already by Cohn, and some of the reviewed tests also describe treatment of these singular (or pseudosingular) cases. The assumption that the RCs are well defined will be referred to as *strong regularity* and will be characterized in several related equivalent conditions.

The four types of SCMJ zero location schemes that are defined are as follows. The first scheme associates the tested polynomial with a sequence $\{a_m(z)\}$ of monic polynomials and is labeled "Type A." The sequence $\{a_m(z)\}$ serves as a reference to all other sequences. It is also used to derive the relations between the RCs and the principal minors of the Schur–Cohn matrix. The second "Type B" schemes is presented by a sequence $\{b_m(z)\}$ that corresponds to choosing the scalars in (4) $\psi_m = 1$ for $m \leq n - 1$. This type represents the lowest arithmetic count algorithms in the SCMJ class. An example of a "Type B" test is Raible's test [29]. The third, "Type C" schemes generates $n$ distinguished entries that produce (or relate up to sign) to the Schur–Cohn determinants. In particular, the "Type C" prototype algorithm produces a sequence denoted by $\{c_m(z)\}$ whose leading coefficients are equal to the principal minors of the Schur–Cohn matrix. "Type C" schemes in the literature were advanced by Jury in several versions [12], [14], [18], [19]. We presented this part of the current paper in a recent symposium dedicated to Professor Jury [5]. "Type C" schemes are considered important in obtaining stability tests for two-dimensional systems and they perform better in finite precision arithmetic. The fourth and last "Type D" schemes, generate sequences $\{d_m(z)\}$ while avoiding the arithmetic operation of division. "Division-free" schemes of "Type D" available in the literature include the tests in [20], [27], and in [7], [8]. This form may be useful in applications that involve testing polynomials with coefficients that depend on parameters and avoiding divisions leaves it simpler to handle coefficients.

During a subsequent review of tests published in the literature; the uniform parametrization of the SCMJ class also turned out to be helpful in sometimes supporting theorems stated without proofs, or at times in providing missing zero location rules or in correcting zero location rules that were not stated properly. Because we consider polynomials with complex coefficients, a complex version for tests proposed only for real polynomials also becomes immediately apparent.

In the course of deriving the zero location rule for the Type C algorithm, we actually provide yet another proof of the celebrated Schur–Cohn theorem on the relation between the number of positive and negative eigenvalues of the Schur–Cohn matrix and the number of zeros of the polynomial inside and outside the unit circle. The specialty of this proof is that its starting point is the zero location rule proved first on the RCs. This is then used through relations between unit triangular (Cholesky) factorizations of the Schur–Cohn and Toeplitz matrices to achieve the sought proof (cf. [31]).

The outline of the paper is as follows. The zero location criterion in terms of the set of RCs is derived in the next section. This section associates the tested polynomial with a monic sequence of polynomials that is labeled Algorithm A. Section 2 also contains the expression for the principal minors of the Schur–Cohn matrix in terms of the RCs. Sections 3, 4, and 5 are devoted to the Type B, Type C, and Type D schemes, respectively. The presentation of each prototype is followed by a review of tests in the literature which belong to that type and is also accompanied by a simple numerical demonstration.

## 2. Reflection coefficients and zero counting

Schur [30] and Cohn [9] associated with the polynomial $p(z)$ a Hermitian *form* such that, provided all its *squares* are nonzero, its number of positive and negative squares provides the number of IUC and OUC zeros (see, e.g., [21]). Fujiwara [10] (see also [21]) showed that the Schur–Cohn conditions can be posed equivalently on the principal minors of the following $(n \times n)$-matrix:

$$
\mathbf{C} = \begin{bmatrix} p_n^* & & & 0 \\ p_{n-1}^* & & & \\ \vdots & & & \\ p_1^* & \cdots & p_{n-1}^* & p_n^* \end{bmatrix} \begin{bmatrix} p_n & p_{n-1} & \cdots & p_1 \\ & & & \vdots \\ & & & p_{n-1} \\ 0 & & & p_n \end{bmatrix} - \begin{bmatrix} p_0 & & & 0 \\ p_1 & & & \\ \vdots & & & \\ p_{n-1} & \cdots & p_1 & p_0 \end{bmatrix} \begin{bmatrix} p_0^* & p_1^* & \cdots & p_{n-1}^* \\ & & & \vdots \\ & & & p_1^* \\ 0 & & & p_0^* \end{bmatrix}. \tag{5}
$$

The Schur–Cohn zero location rule was also posed on the determinants of a sequence of $n$ matrices of sizes $2m \times 2m$, $m = 1, \ldots, n$ (called the Schur–Cohn determinants) and was also given in terms of a (not Hermitian) $(2n \times 2n)$-matrix $\Delta_{1:2n,1:2n}$ whose sequence of centrally situated submatrices $\Delta_{m:2n-m,m:2n-m}$, $m = 1, \ldots, n$, have determinants that are equal to the principal minors of $\mathbf{C}$ [15] (see also [16, Theorem 2.6], [2, Theorem 3.11]). The matrix $\mathbf{C}$ is referred to in the literature as the Schur–Cohn–Fujiwara matrix or the Hermitian Schur–Cohn matrix (see [16, Theorem 3.2] [2, Theorem 3.13], and [21, Theorem XVa]).

The term RCs stems from their interpretation in modeling lossless layered media in certain digital signal processing applications, e.g., in modeling the vocal tract in speech processing or the earth's surface in geophysical (see, e.g., [28], [23]). Another name for RCs that originates from their statistical interpretation in approximating a stationary process by an auto-regressive (AR) model is *Partial Correlation (ParCor)* coefficients. Common to both interpretations is that they involve the solution of a set of equations that may be represented in the following normal form:

$$
\mathbf{T}_n[a_{n,0}, a_{n,1}, \ldots, a_{n,n-1}, 1]^{\mathrm{t}} = [0, \ldots, 0, d_n]^{\mathrm{t}}, \tag{6}
$$

where $\mathbf{T}_n$ is a Hermitian positive definite Toeplitz matrix of size $(n+1) \times (n+1)$ and the set is to be solved for the $n+1$ unknowns $\{a_{n,i}, d_n\}$. An efficient solution to this set of equations is provided by Levinson's algorithm [29]. The Levinson algorithm is a recursive algorithm that includes (in a polynomial notation) the recursions

$$
a_m(z) = z a_{m-1}(z) - k_m a_{m-1}^{\sharp}(z) \tag{7}
$$

and a formula (that need not concern us here) to compute the $k_m$'s—the "RCs"—that brings into the solution the entries of the Toeplitz matrix. The algorithm starts with $a_0(z) = 1$ and is carried out for $m = 1, \ldots, n$. It is seen that the polynomials $a_m(z) = \sum_0^n a_{m,i} z^i$ are all monic, $a_{m,m} = 1$.

Given a polynomial $p(z)$ it is possible to determine its RCs (namely to find $k_m$'s such that (7) will produce $a_n(z) = p(z)/p_n$) by reversing the recursion (7).

**Algorithm A.** Apply the algorithm

$$
z a_{m-1}(z) = \frac{a_m(z) + k_m a_m^{\sharp}(z)}{1 - |k_m|^2}, \qquad k_m = -\frac{a_{m,0}}{a_{m,m}^*} \tag{8}
$$

with the initiation $a_n(z) = p(z)/p_n$.

**Theorem 1 (Zero location rule by RC).** *A polynomial $p(z)$ with a well-defined set of RCs $\{k_m,\ m = 1,\dots,n\}$ has $\nu$ OUC zeros (no UC zeros) and $n-\nu$ IUC zeros, where $\nu$ is given by the number of negative terms in the sequence*

$$\nu = n_-\{q_n, q_{n-1}, \dots, q_1\} \tag{9}$$

*whose members are defined by*

$$q_m := \prod_m^n (1 - |k_i|^2), \qquad m = 1,\dots,n. \tag{10}$$

We prove Theorem 1 in the Appendix. The proof is obtained by invoking the *Principle of Argument* on the recursions (7).

It is seen that the $q_m$ parameters still hold enough information to determine zero location. These parameters will play an important role in the forthcoming developments. They can be more simply obtained from the sequence of RCs by the recursion:

$$q_m = q_{m+1}(1 - |k_m|^2), \qquad q_{n+1} := 1,\ m = n, n-1, \dots, 1. \tag{11}$$

**Remark 1.** The immediate corollary of Theorem 1 that necessary and sufficient conditions for stability (i.e., $\nu = 0$) are

$$|k_m| < 1, \qquad m = 1,\dots,n,$$

is a very familiar result in digital signal processing applications, where the $k_m$'s also serve as gains in a lattice realization of the all pole filter $1/a_n(z)$ (e.g. [28]).

We define Algorithm A together with the zero location rule given by Theorem 1 as our Type A scheme in the SCMJ class. The relation of Algorithm A to the Levinson recursions also motivates the use of the term RC in the context of the SCMJ class of stability and zero location. It is clear from (8) that the sequence of the RCs of a polynomial is not affected by the possible scaling factors in (4). Consequently, the RCs in conjunction with Theorem 1 form convenient tools for properly tracking zero location rules for other algorithms in the SCMJ class.

**Definition 1 (Strong regularity).** A polynomial $p(z)$ is said to satisfy conditions of strong regularity if Algorithm A does not encounter a premature termination (and hence produces a well-defined set of $n$ RCs).

**Remark 2.** The condition that causes the recursion to terminate prematurely is the occurrence of a $k_m$ such that $|k_m| = 1$. The offense in general is not the division by $[1 - |k_m|^2]$ apparent in Algorithm A but that a subsequent $k_{m-1} = -f_{m-1,0}/f^*_{m-1,m-1}$ is not well defined when $f_{m-1,m-1} = 0$ and an $f_{m-1,m-1} = 0$ is necessarily preceded by, and always follows, a $|k_m| = 1$. This is seen by reading from (4) that

$$f_{m-1,m-1} = \psi_m(f_{m,m} - k_m f^*_{m,0}) = \psi_m f_{m,m}(1 - |k_m|^2).$$

Several equivalent conditions for strong regularity will be given in Theorem 2 below. Strong regularity will be presumed throughout in this paper.

Fujiwara, who contributed the expression (5) for the Schur–Cohn matrix, also showed that a generating function for $\mathbf{C} = [c_{i,j}]$ is given by

$$C(z, w) = \frac{p^{\sharp}(z)p^{r}(w) - p(z)\overline{p}(w)}{1 - zw} = \sum_{i,j=0}^{n-1} c_{i,j} z^{i} w^{j}. \tag{12}$$

In more modern terminology $C(z, w)$ is sometimes referred to as a generating function for T-Bezoutian (a Bezoutian with respect to the unit circle). The T-Bezoutians and the former Bezoutians with respect to a line may be treated in a unified manner [22].

The relations between the principal minors of the Schur–Cohn matrix and the reflection coefficients will be obtained by using a couple of results known in the context of the Levinson algorithm and the inversion of Toeplitz matrices. First it may be noticed from the nested structure of the Levinson algorithm that the algorithm in fact provides solutions to all normal sets of equations of the form (6) defined by all the submatrices $\mathbf{T}_m$, $m = 0, 1, \ldots, n$. This observation implies that the Levinson algorithm creates a UDL triangular factorization for the inverse of $\mathbf{T}_n$, viz.,

$$\mathbf{T}_n^{-1} = \mathbf{A}_n \Lambda_n^{-1} \mathbf{A}_n^{\mathrm{H}}, \tag{13}$$

where

$$\mathbf{A}_n = \begin{bmatrix} 1 & a_{1,0} & \cdots & a_{n-1,0} & a_{n,0} \\ 0 & 1 & \cdots & a_{n-1,1} & a_{n,1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n,n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{13a}$$

where the superscript H denotes the conjugate transpose, and $\Lambda_n$ is the diagonal matrix

$$\Lambda_n = \mathrm{diag}[1, \lambda_1, \ldots, \lambda_n]; \qquad \lambda_m = \prod_{i=1}^{m} (1 - |k_i|^2). \tag{13b}$$

The second result that we shall use is a formula due to Gohberg and Semencul [11] that allows the expression of the inverse of a Toeplitz matrix $\mathbf{T}_n^{-1}$ in terms of just the coefficients of $a_n(z)$

$$\begin{aligned} \mathbf{T}_n^{-1} = \frac{1}{\lambda_n} & \begin{bmatrix} a_{n,n}^{*} & & & 0 \\ a_{n,n-1}^{*} & \ddots & & \\ \vdots & & \ddots & \\ a_{n,0}^{*} & \cdots & a_{n,n-1}^{*} & a_{n,n}^{*} \end{bmatrix} \begin{bmatrix} a_{n,n} & a_{n,n-1} & \cdots & a_{n,0} \\ & \ddots & & \vdots \\ & & \ddots & a_{n,n-1} \\ 0 & & & a_{n,n} \end{bmatrix} \\ - \frac{1}{\lambda_n} & \begin{bmatrix} 0 & & & 0 \\ a_{n,0} & \ddots & & \\ \vdots & & \ddots & \\ a_{n,n-1} & \cdots & a_{n,0} & 0 \end{bmatrix} \begin{bmatrix} 0 & a_{n,0}^{*} & \cdots & a_{n,n-1}^{*} \\ & \ddots & & \vdots \\ & & \ddots & a_{n,0}^{*} \\ 0 & & & 0 \end{bmatrix}. \end{aligned} \tag{14}$$

This formula indicates that a generating function for $\mathbf{R}_n := \mathbf{T}_n^{-1}$ is given by (compare to (12) and (5))

$$\mathcal{R}_n(z, w) = \frac{1}{\lambda_n} \frac{a_n^\sharp(z)a_n^r(w) - zwa_n(z)\overline{a}_n(w)}{(1 - zw)}. \tag{15}$$

The relation of the Schur–Cohn matrix for $p(z)$ and the Schur–Cohn matrix $\hat{\mathbf{C}}$ for the corresponding monic polynomial $a_n(z) = p(z)/p_n$ is seen from (5) to be

$$\mathbf{C} = |p_n|^2 \hat{\mathbf{C}} \tag{16}$$

and the generating function for $\hat{\mathbf{C}}$ is (12),

$$\hat{\mathcal{C}}(z, w) = \frac{a_n^\sharp(z)a_n^r(w) - a_n(z)\overline{a}_n(w)}{1 - zw}. \tag{17}$$

Next, the recursion (8) can be used to obtain the identity

$$\begin{aligned} a_n^\sharp(z)a_n^r(w) - a_n(z)\overline{a}_n(w) = (1 - |k_n|^2)[&a_{n-1}^\sharp(z)a_{n-1}^r(w) \\ &- zwa_{n-1}(z)\overline{a}_{n-1}(w)]. \end{aligned}$$

Set this identity in the generating function (17) for $\mathbf{C}$ and compare the result with the generating function (15) for $\mathbf{R}_{n-1} = \mathbf{T}_{n-1}^{-1}$ to find that

$$\hat{\mathcal{C}}(z, w) = \lambda_n \mathcal{R}_{n-1}(z, w).$$

Thus the next relation between the monic Schur–Cohn matrix and inverse of the $(n \times n)$-Toeplitz matrix becomes evident,

$$\frac{1}{\lambda_n} \hat{\mathbf{C}} = \mathbf{T}_{n-1}^{-1}. \tag{18}$$

We proceed to use this relation to connect the minors of the Schur–Cohn matrix with the RCs. Toward this goal, first observe from (5) that the Schur–Cohn matrix is centro-Hermitian; i.e., it has the property $\mathbf{JCJ} = \mathbf{C}$, where $\mathbf{J}$ denotes a matrix with 1's along the anti-diagonal and 0's elsewhere (the reversion matrix). The UDL triangular factorization (13) implies through (18) and the centro-Hermitian symmetry the following LDU triangular factorization for $\hat{\mathbf{C}}$:

$$\hat{\mathbf{C}} = \mathbf{B}_{n-1}\mathbf{Q}_{n-1}\mathbf{B}_{n-1}^{\mathrm{H}}, \tag{19}$$

where

$$\mathbf{B}_{n-1} = \mathbf{J}\mathbf{A}_{n-1}\mathbf{J} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ a_{n-1,n-2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1,1} & a_{n-2,1} & \cdots & 1 & 0 \\ a_{n-1,0} & a_{n-2,0} & \cdots & a_{1,0} & 1 \end{bmatrix} \tag{19a}$$

and

$$\mathbf{Q}_{n-1} = \mathrm{diag}[q_n, q_{n-1}, \ldots, q_1]; \qquad q_i = \frac{\lambda_n}{\lambda_{i-1}}, \; i = 1, \ldots, n. \tag{19b}$$

It is noticed that these $q_m$'s are indeed identical to the $q_m$'s as defined before in (10). The implicit assumption that the underlying Schur–Cohn and Toeplitz (sub)matrices are invertible follows from the strong regularity assumption. In fact we may at this point summarize several equivalent conditions for strong regularity.

**Theorem 2 (Strong regularity).** *The following conditions are equivalent:*

(i) *The set of RCs are well defined (Definition 1 for strong regularity).*
(ii) *All the leading principal minors of* $\mathbf{C}$ *are not equal to zero.*
(iii) *All the leading principal minors of* $\mathbf{T}_n$ *are not equal to zero.*
(iv) *All* $|k_m| \neq 1, m = 1, \ldots, n$.

**Proof.** The equivalencies (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) are apparent from factorizations (13) and (19). The equivalence (i) $\Leftrightarrow$ (iv) was explained in Remark 1. ☐

**Remark 3.** Strong regularity implies that the degree $\eta$ of the greatest common divisor of $p(z)$ and $p^\sharp(z)$ is $\eta = 0$. In particular it implies that $p(z)$ has no UC zeros. However, although $\eta = 0$ is sufficient for no UC zeros and for nonsingularity of the matrices $\mathbf{T}_n$ and $\mathbf{C}$, it is still not a sufficient condition for strong regularity.

It is sufficient to recognize the factorization (19) as a congruency relation between $\hat{\mathbf{C}}$ and $\mathbf{Q}$ in order to conclude that the two matrices have the same number of positive and negative eigenvalues (the same *inertia*). The eigenvalues of the diagonal matrix $\mathbf{Q}$ are the $q_m$'s. The numbers of positive and negative $q_m$'s were shown in Theorem 1 to give the numbers of IUC and OUC zeros of $p(z)$. Thus proof for Theorem 1 in combination with the factorization (19) provide a new proof of the Schur–Cohn criterion.

**Theorem 3 (Schur–Cohn).** *If all the leading principal minors of the Schur–Cohn matrix for* $p(z)$, $\mathbf{C}$, *are different from zero then the numbers of OUC and IUC zeros of polynomial* $p(z)$, $\nu$ *and* $n - \nu$, *are equal to the number of negative and positive eigenvalues of* $\mathbf{C}$, *respectively.*

We proceed to examine in finer detail the factorization (19) in order to obtain an explicit expression for the principal minors of $\mathbf{C}$ in terms of the RCs. Denote the principal minors of $\mathbf{C}$ by

$$\mu_m := \det\{\mathbf{C}_{0:m-1}\}, \qquad m = 1, \ldots, n, \tag{20}$$

where the $\mathbf{C}_{0:l-1}$ represents the $l \times l$ leading submatrix of $\mathbf{C}$ (our count of rows and columns of matrices begins with $i = j = 0$).

**Theorem 4 (Minors of C in terms of $\{k_m\}$).** *Consider* $p(z)$ *and its Schur–Cohn matrix* $\mathbf{C}$, *and assume strong regularity.*

(i) *The principal minors of* **C** *are related to the set of RCs by*

$$\mu_m = |p_n|^{2m} \prod_{i=n+1-m}^{n} q_i, \qquad m = 1, \ldots, n. \tag{21}$$

(ii) *The polynomial $p(z)$ has $\nu$ OUC and $n - \nu$ IUC zeros with*

$$\nu = \mathrm{Var}\{1, \mu_1, \mu_2, \ldots, \mu_n\}, \tag{22}$$

*where* Var *denotes the number of sign variations in the indicated sequence.*

**Proof.** The lower upper *unit* triangular factorization (19) implies that the principal minors $\hat{\mu}_m$ of $\hat{\mathbf{C}}$ are given by

$$\hat{\mu}_m = \prod_{i=n+1-m}^{n} q_i, \qquad m = 1, \ldots, n. \tag{23}$$

Consequently, (21) follows from (16). Noticing that $\mu_m/\mu_{m+1} = q_{n-m}$, the rule (9) of Theorem 1 can be written as

$$\nu = n_-\left\{\mu_1, \frac{\mu_2}{\mu_1}, \ldots, \frac{\mu_n}{\mu_{n-1}}\right\}. \tag{24}$$

It remains to realize that (22) and (24) are equivalent. □

**Example 1, Part a.** Consider the polynomial

$$p(z) = 4 + 12.5z + 5z^2 + z^3,$$

which can be checked numerically to have 2 OUC zeros and 1 IUC zero. The sequence $\{a_m(z)\}$ consists of

$$a_3(z) = p(z), \qquad a_2(z) = 0.5 + 3z + z^2, \qquad a_1(z) = 2 + z, \qquad a_0(z) = 1.$$

The RCs are

$$k_1 = -2, \qquad k_2 = -0.5, \qquad k_3 = -4.$$

Thus the $q_m$ parameters (11) are as follows:

$$q_3 = -15, \qquad q_2 = -11.25, \qquad q_1 = 33.75.$$

Then according to the rule (9)

$$\#OUC = n_-\{q_3, q_2, q_1\} = 2.$$

The Schur–Cohn matrix is

$$\mathbf{C} = \begin{bmatrix} -15 & -45 & -7.5 \\ -45 & -146.25 & -45 \\ -7.5 & -45 & -15 \end{bmatrix}.$$

The principal minors of **C** are indeed $\mu_1 = -15$ $(= q_3)$, $\mu_2 = 168.75$ $(= q_3 q_2)$, and $\mu_3 = 5695.3$ $(= q_3 q_2 q_1)$. According to the rule in Theorem 4,

$$\#OUC = \mathrm{Var}\{1, -15, 168.75, 5695.3\} = 2.$$

In contrast to some concluding remarks in [31] the number of OUC zeros *is not* given by the number of sign variations of the principal minors of $\mathbf{C}$ (Var$\{-15, 168.75, 5695.3\} = 1$ in this example) nor by the principal minors of $\mathbf{T}_{n-1}$ or its inverse. In fact the signs of the principal minors of $\mathbf{C}$ and $\mathbf{T}_{n-1}^{-1}$ are affected by the appearance of $\lambda_n$ in (18) and $\lambda_n$ may be negative.

## 3. Schemes of Type B

By cautious inspection at the structure of the recursion (4), it is possible to restrict somewhat the range of relevant scalars $\psi_m$ in the search for tests of merit in the SCMJ class without the danger of missing interesting cases. For example, when testing a real polynomial there is clearly no point in allowing complex $\psi_m$ to complicate the recursion. More generally, the structure of (4) possesses the property that if $f_{n-1,n-1}$ is real then all subsequent $f_{m,m}$, $m \leq n-1$, stay real as long as the $\psi_m$'s, $m \leq n-1$, are real. It is possible to exploit this property by insisting on $\psi_n = f_{n,n}^*$,

$$zf_{n-1}(z) = f_{n,n}^*[f_n(z) + k_n f_n^\sharp(z)], \qquad f_n(z) = p(z), \tag{25}$$

so that $f_{n-1,n-1} = |f_{n,n}|^2(1 - |k_n|^2)$ is indeed real (and perceives the part of the information on $k_n$ that is acute for finding zero location) and afterwards by restricting the subsequent $\psi_m$'s to be real. As will be seen, the $f_{m,m}$ form the distinguished entries in terms of which zero location rules are given so that being real simplifies these rules. All the tests of interest in the SCMJ class featured distinguished entries that are real.

### *3.1 Type B: Basic form*

The prototype form corresponds to choosing $\psi_n = p_n^*$ and $\psi_m = 1$ for all $m \leq n-1$.

**Algorithm B.** *Initiation:*

$$zb_{n-1}(z) = p_n^* p(z) - p_0 p(z) \tag{B1}$$

For $m = n-1, \ldots, 1$ do:

$$zb_{m-1}(z) = b_m(z) + k_m b_m^\sharp(z), \qquad k_n = -\frac{b_{m,0}}{b_{m,m}^*} \tag{26}$$

[Alternative possible initiations are (1) to first normalize to monic the tested polynomial

$$b_n(z) = \frac{1}{p_n} p(z) \tag{B2}$$

and (2) when the leading coefficient $p_n$ is real it is also possible to start the recursions with

$$b_n(z) = \begin{cases} p(z) & \text{if } p_n > 0 \\ -p(z) & \text{if } p_n < 0 \end{cases} \tag{B3}$$

For these two alternative initiations (26) may be used, for $m = n$ already.]

**Theorem 5 (Zero location for Algorithm B).** *If Algorithm B does not terminate prematurely then $p(z)$ has $\nu$ OUC and $n - \nu$ IUC zeros, where $\nu$ is given by*

$$\nu = n_{-}\{b_{n-1,n-1}, b_{n-2,n-2}, \ldots, b_{1,1}, b_{0,0}\}. \tag{27}$$

**Proof.** From (26)

$$b_{m-1,m-1} = b_{m,m} + k_m b^*_{m,0} = b_{m,m}(1 - |k_m|^2), \qquad m \leq n - 1.$$

By comparison with (11) we obtain that

$$b_{m-1,m-1} = b_{m,m} q_m$$

in which for (B1) assume $b_{n,n} = |p_n|^2$, for (B2) $b_{n,n} = 1$, and for (B3) $b_{n,n} = |p_n|$. Thus for any of the three initiations the stated rule follows from (9) of Theorem 1. □

**Example 1 (Cont'd), Part b.** Consider again $p(z) = 4 + 12.5z + 5z^2 + z^3$. Algorithm B produces a sequence that consists of $b_3(z) = p(z)$,

$$b_2(z) = -7.5 - 45z - 15z^2, \qquad b_1(z) = -22.5 - 11.25z, \qquad b_0(z) = 33.75.$$

Indeed,

$$\#OUC = n_{-}\{-15, -11.25, 33.75\} = 2.$$

Type B schemes are special in offering the least number of operations in the SCMJ class—just $n^2 + O(n)$ operations as compared to $2n^2 + O(n)$ for tests of Type A or for the later Type D, and $3n^2 + O(n)$ for the next Type C tests.

In the rest of the paper we shall review many tests published in the literature; we define the convention that we shall follow in converting tests that have appeared in a "table" form to the more compact polynomial notation that we use.

**Remark 4 (Conversion convention for tables).** SCMJ tests have often been published in tabular forms. Usually the format is (format a:) $n + 1$ *pairs* of rows (the second row in each pair being the reversed conjugate of the first). Else, (format b:) the table consists of just $n + 1$ rows (omitting the reverted rows). We shall regard these tables as forming the coefficients of a sequence of polynomials $\{f_m(z)\}$ by the following conversion convention. We shall associate the first, third, fifth, ... in format a or the first, second, third, ... in format b with the polynomials $f_n(z)$, $f_{n-1}(z)$, $f_{n-2}(z)$ by post-multiplying each row by a vector of powers $[1, z, z^2, \ldots]^t$ of a corresponding equal length.

### 3.2 *Type B: Raible's version*

Raible proposed in [29] a test for real polynomials $p(z)$ which may be presented by the algorithm:

$$f_{m-1}(z) = f_m(z) - \xi_m f_m^r(z), \qquad \xi_m = \frac{f_{m,m}}{f_{m,0}}; \qquad f_n(z) = p^r(z). \tag{28}$$

Assuming $p_n > 0$, Raible provided the rule that the number of OUC zeros is given by

$$n_-\{f_{n-1,0}, f_{n-2,0}, \ldots, f_{0,0}\}. \tag{29}$$

The above recursion is comparable with the reversion (or the "reciprocation" see Remark 5 below) of the recursion in Algorithm B (26):

$$b_{m-1}^r(z) = b_m^r(z) + k_m \overline{b}_m(z)$$

by the identifications

$$f_m(z) = b_m^r(z), \qquad \xi_m = -k_m.$$

Consequently, as a corollary from Theorem 5 one has the following extension of the test to complex polynomials.

**Corollary 1 (Complex form of Raible's version).** *Apply to a polynomial $p(z)$ the next algorithm. Initiation:*

$$f_{n-1}(z) = p_n^* p^r(z) - p_0 \overline{p}(z) \tag{R0}$$

*For $m = n-1, \ldots, 1$ do:*

$$f_{m-1}(z) = f_m(z) - \xi_m f_m^\sharp(z), \qquad \xi_m = \frac{f_{m,m}}{f_{m,0}}. \tag{30}$$

[*Alternative valid initiations include* (1) $f_n(z) = \frac{1}{p_n} p^\sharp(z)$ *or, when $p_n$ is real, also* (2)

$$f_n(z) = \begin{cases} p^r(z) & \text{if } p_n > 0 \\ -p^r(z) & \text{if } p_n < 0 \end{cases}$$

*and for these alternatives the recursion also holds for $m = n$.*]

*Provided the algorithm does not terminate prematurely, the numbers of OUC and IUC zeros are $\nu$ and $n - \nu$, where*

$$\nu = n_-\{f_{n-1,0}, f_{n-2,0}, \ldots, f_{0,0}\}.$$

**Remark 5.** We "extrapolate" Raible's test from a real $p(z)$ to a complex $p(z)$. In any situation of this kind there is more than one way to do this. A dual extension form could identify (28) with the reciprocation of (26) such that $f_m(z) = b_m^\sharp(z)$ and $\xi_m = -k_m^*$ and consider the initiations to correspond to $f_n(z) = p^\sharp(z)$. The zero location rule for this dual form is the same as the $f_{m,0}$'s remain real.

## 4. Schemes of Type C

The goal in this section is to obtain an algorithm in the SCMJ class that produces a sequence whose leading coefficients are the principal minors of the Schur–Cohn matrix. For the derivation process it is convenient to define one more set of auxiliary parameters that is related to the RCs through the $q_m$'s (10) by

$$e_m := \prod_{i=m}^{n} q_i; \qquad e_m = e_{m+1} q_m, \qquad e_{n+1} := 1,\ m = n, \ldots, 1. \tag{31}$$

The sequence $\{e_m\}$ may be seen from (23) to represent the principal minors of $\hat{\mathbf{C}}$ in reversed order,

$$\hat{\mu}_m = e_{n+1-m}. \tag{32}$$

Assume that there exists a recursible set of scalars $\psi_m$ for which (4) or (25) produces a sequence $\{c_m(z)\}$ with the property $c_m(z) = e_{m+1} a_m(z)$, where $\{a_m(z)\}$ is the monic sequence. Assume, momentarily, that the algorithm starts with $c_n(z) = a_n(z)$. Then $c_{n-1,n-1} = \psi_n(1 - |k_n|^2)$, and for the choice $\psi_n = 1$ we get $c_{n-1,n-1} = e_n$. At the next step, $c_{n-2,n-2} = \psi_{n-1}(1 - |k_n|^2)(1 - |k_{n-1}|^2)$ and $c_{n-2,n-2} = e_{n-1}$ is again possible for the choice $\psi_{n-1} = (1 - |k_n|^2) = c_{n-1,n-1}$. Seemingly, the pattern is that the choice

$$\psi_{n+1-i} = c_{n+1-i,n+1-i}/c_{n+2-1,n+2-i} \quad \text{yields} \quad c_{n-i,n-i} = e_{n+1-i}. \tag{33}$$

We verify this pattern by an induction step. Suppose the assertion in (33) holds for $c_{n-i,n-i}$'s until $i = 1, \ldots, l$. Then at the next step,

$$c_{n-l-1,n-l-1} = \psi_{n-l} q_{n-l} c_{n-l,n-l},$$

indeed reduces, for the choice $\psi_{n-l} = c_{n-l,n-l}/c_{n+1-l,n+1-l}$, to $c_{n-l-1,n-l-1} = e_{n-l}$.

This completes the proof that the choosing

$$\psi_n = 1; \qquad \psi_{n-1} = c_{n-1,n-1}; \qquad \psi_m = \frac{c_{m,m}}{c_{m+1,m+1}};\ m \le n-2, \tag{34}$$

yields, for the initiation $c_n(z) = p(z)/p_n$, a sequence of polynomials $\{c_m(z)\}$ such that

$$c_m(z) = e_{m+1} a_m(z) \quad \text{and} \quad c_{m,m} = e_{m+1} = \hat{\mu}_{n-m}. \tag{35}$$

The sought algorithm emerges after removing the restriction to monic initiation to be as follows.

### *4.1 Type C: Basic form*

The prototype algorithm for Type C schemes is as follows.

**Algorithm C.** *Initiation:*

$$zc_{n-1}(z) = p_n^* p(z) - p_0 p^\sharp(z) \tag{C1}$$

For $m = n-1, \ldots, 1$ do:

$$zc_{m-1}(z) = \phi_m \{c_{m,m} c_m(z) - c_{m,0} c_m^\sharp(z)\} \tag{36}$$

with $\phi_m$ given by

$$\phi_{n-1} = 1; \qquad \phi_m = \frac{1}{c_{m+1,m+1}}, \quad m = n-2, n-3, \ldots, 1.$$

**Theorem 6 (Zero location for Algorithm C).** *If Algorithm C does not terminate prematurely then $p(z)$ has $\nu$ OUC and $n-\nu$ IUC zeros, where $\nu$ is given by*

$$\nu = \mathrm{Var}\{1, c_{n-1,n-1}, c_{n-2,n-2}, \ldots, c_{0,0}\}. \tag{37}$$

**Proof.** For the initiation with a monic polynomial we obtained the relation (35). The definition of the $e_m$'s in combination with the rule (9) implies that

$$\nu = \mathrm{Var}\{1, e_n, e_{n-1}, \ldots, e_1\}. \tag{38}$$

For the more general initiation (C1) the relation (35) starts instead with $c_{n-1}(z) = |p_n|^2 a_{n-1}(z)$ and subsequently

$$c_{m-1,m-1} = (|p_n|^2)^{n+1-m} e_m, \qquad m = 1, \ldots, n. \tag{39}$$

Thus (38) implies the stated rule. □

Algorithm C also achieves the goal of producing the principal minors of the Schur–Cohn matrix. (Initiations other then (C1) similar to those proposed before for Algorithm B will also lead to the rule (37) but will miss the identification of $c_{m,m}$ with the minors of **C** stated below.) The property is summarized as follows.

**Theorem 7 (Algorithm C and the minors of the matrix C).** *The leading coefficients of the sequence of polynomials produced by Algorithm C (with initiation (C1)) form the principal minors of the Schur–Cohn matrix* **C** *of $p(z)$ as follows*

$$c_{m,m} = \mu_{n-m} \quad \textit{where} \quad \mu_{m+1} := \det\{\mathbf{C}_{0:m}\}, \qquad m = 0, \ldots, n-1. \tag{40}$$

**Proof.** Immediate from (31), (39), and (21). □

**Example 1 (Cont'd), Part c.** Consider again $p(z) = 4 + 12.5z + 5z^2 + z^3$. The algorithm constructs

$$c_3(z) = p(z); \qquad c_2(z) = -7.5 - 45z - 15z^2;$$

$$c_1(z) = -33.75 + 168.75z, \quad (\phi_2 := 1); \qquad c_0(z) = 5695.3, \quad (\phi_1 = -1/15).$$

Evaluate (37) with the requested values

$$\#OUC = \mathrm{Var}\{1, -15, 168.75, 5695.3\} = 2.$$

Theorem 6 implies for $p(z)$ 2 OUC zeros and 1 IUC zero. The principal minors of the Schur–Cohn matrix were computed in Part a of the example. The comparison shows that the principal minors are indeed given by $\mu_1 = c_{2,2} = -15$, $\mu_2 = c_{1,1} = 168.75$, and $\mu_3 = c_{0,0} = 5695.3$, as claimed in Theorem 7.

### 4.2 *Type C: Jury's versions*

Schemes of Type C were advanced by Jury on several occasions. They are considered to be advantageous in algorithms to test the stability of two-dimensional systems [18] and to offer better accuracy in finite arithmetic precision [1]. Two versions are available, an earlier version [13, p. 104], [12], which considers only real polynomials, and a modified version [14], [15], [18], [19], which in [18] also treats complex polynomials.

*4.2.1. Jury's modified test.* The construction rules for the table in [18] may be described in polynomial notation, following the conversion convention of Remark 4, by the algorithm

$$f_{m-1}(z) = \frac{1}{\eta_m}\{f_{m,0}\overline{f}_m - f^*_{m,m}f^r_m(z)\}, \qquad f_n(z) = p^r(z), \tag{41}$$

where

$$\eta_n = \eta_{n-1} = 1, \qquad \eta_m = f_{m+1,0} \quad \text{for} \quad m = n-2, n-3, \ldots .$$

A careful step-by-step comparison of this recursion with Algorithm C (36) reveals the following relations with the Type C polynomials:

$$f_{n-2i}(z) = c^r_{n-2i}(z), \qquad f_{n-2i+1}(z) = c^{\sharp}_{n-2i+1}(z), \; i = 0, 1, 2, \ldots .$$

In the above use is made of the fact that $f_{m,0} = c_{m,m}$ are real for $m \le n-1$. The distinguished entries that in [18], [19] are denoted by $\Delta_m$ correspond in the current notation to $f_{n-m,0}$, and therefore

$$\Delta_{n-m} = f_{m,0} = c_{m,m} = \det\{\mathbf{C}_{0:m-1}\}, \qquad m = 1, 2, \ldots, n, \tag{42}$$

where the last equality follows from Theorem 7. This provides a proof for the statement in [18] that the $\Delta_m$ form the principal minors of the Schur–Cohn matrix.

Furthermore, Theorem 6 provides a zero location rule for the algorithm (41).

**Corollary 2 (Jury's modified version).** *Consider the algorithm* (41). *Provided* (41) *does not terminate prematurely, the numbers of OUC and IUC zeros are* $\nu$ *and* $n-\nu$, *where*

$$\nu = \text{Var}\{1, f_{n-1,0}, f_{n-2,0}, \ldots, f_{0,0}\}.$$

References [18] and [19] state without a proof only the stability conditions (that all $\Delta_m = f_{m,0} > 0$) for this table form. The current proof is different from a proof in an unpublished report [17] available from the author. Also the zero location rule we deduce for this table here is in disagreement with the rule proposed in [17] (Part c of Example 1 is a ready counterexample that the number of OUC zeros is not $\nu = n_{-}\{f_{n-1,0}, f_{n-2,0}, \dots, f_{0,0}\}$).

*4.2.2. The earlier version.* The modified version is an enhancement to an earlier version of this table in [13, p. 104] and [12]. Because these versions consider only real polynomials, the extension to complex polynomials is not unique (recall the earlier Remark 5). A possible extension to complex polynomials is as follows:

$$f_{m-1}(z) = \frac{1}{\eta_m}\{f_{m,0}\overline{f}_m - f^*_{m,m}f^r_m(z)\}, \qquad f_n(z) = p(z). \tag{43}$$

The seemingly minute change of the initiation by comparison with (41) has a remarkable impact on the accompanying stability and zero location rules and on the relations of the distinguished entries to the Schur–Cohn determinants $\mu_m$. Currently, the sequence $\{f_m(z)\}$ can be shown to relate to the Algorithm C prototype sequence (36) as follows:

$$f_{n-2i}(z) = c^{\sharp}_{n-2i}(z), \qquad f_{n-2i+1}(z) = -c^r_{n-2i+1}(z),\ i = 1, 2, \dots . \tag{44}$$

Consequently, the relations with the Schur–Cohn determinants are now

$$f_{n-m,0} = (-1)^m \det\{\mathbf{C}_{0:m-1}\}, \qquad m = 1, \dots, n.$$

The next corollary follows at once.

**Corollary 3 (Jury's earlier version).** *Consider the algorithm* (43). *Provided it does not terminate prematurely, the numbers of OUC and IUC zeros are* $\nu$ *and* $n - \nu$, *where*

$$\nu = \mathrm{Var}\{1, -f_{n-1,0}, f_{n-2,0}, -f_{n-3,0}, \dots, (-1)^n f_{0,0}\}.$$

These results support the stability conditions

$$f_{n-2i,0} > 0, \qquad f_{n-2i+1,0} < 0,\ i = 1, 2, \dots$$

and the relations to the Schur–Cohn determinant provided by Jury [13, p. 105]. The enhancement of the modified version of Section 4.2.1 over this earlier version is in simpler forms for the stability conditions and the relations to the principal minors of **C**.

## 5. Schemes of Type D

In the first paragraph of Section 3 we explained that it suffices to search for interesting tests that obey the recursion (4) with $\psi_n = f^*_{n,n}$ and real $\psi_m$, $m < n$. It is in particular interesting in this subclass to examine the association of the zero location rule to a recursion like (35),

$$zf_{m-1}(z) = \phi_m(f_{m,m}f_m(z) - f_{m,0}f_m^{\sharp}(z)\},$$

in which all $\phi_m = 1$. The specialty of this scheme is that it avoids the operation of division.

### *5.1 Type D: Basic form*

The prototype algorithm for Type D schemes is as follows.

**Algorithm D.** For $m = n-1, \ldots, 1$ do:

$$zd_{m-1}(z) = d_{m,m}^{*}d_m(z) - d_{m,0}d_m^{\sharp}(z), \qquad d_n(z) = p(z) \tag{45}$$

($d_{m,m}$ are real for $m \leq n-1$).

We try to obtain a zero location rule for this algorithm. Comparing the highest power coefficients gives

$$d_{m-1,m-1} = |d_{m,m}|^2(1-|k_m|^2), \qquad m = n, n-1, \ldots, 1. \tag{46}$$

Therefore, the leading coefficients $\{d_{m,m}\}$ are real for all $m \leq n-1$. Furthermore, a step-by-step comparison of the $d_m(z)$ with $a_m(z)$ yields

$$\begin{aligned} d_{n-1}(z) &= |d_{n,n}|^2(1-|k_n|^2)a_{n-1}(z), \\ d_{n-2}(z) &= |d_{n-1,n-1}|^2(1-|k_{n-1}|^2)a_{n-2}(z), \text{ etc.} \end{aligned}$$

The following relations become apparent:

$$\begin{aligned} q_n &= (1-|k_n|^2) = \frac{d_{n-1,n-1}}{|d_{n,n}|^2} \\ q_{n-1} &= (1-|k_n|^2)(1-|k_{n-1}|^2) = \frac{d_{n-1,n-1}d_{n-2,n-2}}{|d_{n,n}|^2 d_{n-1,n-1}^2} = \frac{d_{n-2,n-2}}{|d_{n,n}|^2 d_{n-1,n-1}} \\ q_{n-2} &= q_{n-1}(1-|k_{n-2}|^2) = \frac{d_{n-3,n-3}}{d_{n-2,n-2}^2}\,\frac{d_{n-2,n-2}}{|d_{n,n}|^2 d_{n-1,n-1}} \\ &= \frac{d_{n-3,n-3}}{|d_{n,n}|^2 d_{n-1,n-1}d_{n-2,n-2}} \\ &\;\vdots \\ q_{n-i} &= \frac{d_{n-i-1,n-i-1}}{|d_{n,n}|^2\prod_{l=1}^{i} d_{n-l,n-l}}. \end{aligned}$$

The above relations provide the key to posing the zero location rule for Algorithm D in terms of its leading coefficients.

**Theorem 8 (Zero location for Algorithm D).** *If Algorithm D does not terminate prematurely then $p(z)$ has $\nu$ OUC and $n-\nu$ IUC zeros, where $\nu$ is given by*

$$\nu = n_{-}\{g_n, g_{n-1}, \ldots, g_1\}, \qquad g_m := \prod_{i=m}^{n} d_{i-1,i-1}, \tag{47}$$

*where the auxiliary parameter $g_m$'s may also be calculated recursively by*

$$g_m = d_{m-1,m-1} g_{m+1}, \qquad g_{n+1} := 1,\ m = n, n-1, \ldots .$$

**Proof.** The rule follows from the basic rule (9) after developing further the expressions obtained for $d_{m,m}$ in terms of $g_m$'s into the following relations:

$$g_n = |d_{n,n}|^2 q_n, \qquad g_{n-1} = [|d_{n,n}|^2]^2 q_n^2 q_{n-1}, \text{ etc.}$$

that is seen to lead to

$$g_{n-i} = [|d_{n,n}|^2]^{2^i} q_n^{2^i} q_{n-1}^{2^{i-1}} \cdots q_{n-i+1}^2 q_{n-i}.$$

It is possible to obtain from here either a direct expression for $g_m$'s in terms of $q_m$'s

$$g_m = q_m q_{m+1}^2 q_{m+2}^{2^2} \cdots q_n^{2^{n-m}} [|d_{n,n}|^2]^{2^{n-m}}, \qquad m = 1, \ldots, n, \tag{48}$$

or, alternatively, a more compact but recursive relation between these parameters, viz.,

$$g_m = g_{m+1}^2 q_m.$$

Either of these relations makes it clear that $\operatorname{sgn}\{g_m\} = \operatorname{sgn}\{q_m\}$. □

**Example 1 (Cont'd), Part d.** The sequence $\{d_m(z)\}$ produced for $p(z) = 4 + 12.5z + 5z^2 + z^3$ consists of $d_3(z) = p(z)$ and

$$d_2(z) = -7.5 - 45z - 15z^2, \qquad d_1(z) = -33.75 + 168.75z, \qquad d_0(z) = -85430.$$

Therefore

$$g_3 = d_{2,2} = -15, \quad g_2 = d_{2,2} d_{1,1} = -2531.2, \quad g_1 = d_{2,2} d_{1,1} d_{0,0} = 2.1624 \cdot 10^8.$$

Then

$$\#OUC = n_{-}\{-15, -2531.2, 2.1624 \cdot 10^8\} = 2.$$

### 5.2 *Type D: Marden's version*

Marden's original test is a Type D scheme devoid of division [25] (see also [26, Theorem (42.1)] and [16, Theorem 5.6]).

Marden considers the algorithm

$$f_{m-1}(z) = f_{m,0}^* f_m(z) - f_{m,m} f_m^\sharp(z), \qquad f_n(z) = p(z), \tag{49}$$

which, in a step-by-step identification procedure of Marden's sequence $\{f_m(z)\}$ with the sequence $\{d_m(z)\}$ of Algorithm D shows that

$$f_n(z) = d_n(z); \qquad f_{n-1}(z) = -d^{\sharp}_{n-1}(z);$$

$$f_m(z) = d^{\sharp}_m(z) \quad \text{for} \quad m = n-2, n-3, \ldots, 0.$$

Marden then defines the auxiliary parameters

$$p_m = \prod_{i=1}^{m} f_{n-i,n-i}, \qquad m = 1, \ldots, n, \tag{50}$$

which, in our notation (47), corresponds to

$$p_m = -\prod_{i=1}^{m} d_{n-i,n-i} = -g_{n+1-m}, \qquad m = 1, \ldots, n.$$

Consequently, Marden's theorem is reproducible from Theorem 8.

**Corollary 4 (Marden).** *Consider for a polynomial $p(z)$ the algorithm* (51) *and define the parameters* (50). *If all the products $p_k$ are nonzero then $p(z)$ has $\pi$ IUC and $n - \pi$ OUC zeros, where*

$$\pi = n_-\{p_1, p_2, \ldots, p_n\}. \tag{51}$$

### 5.3 *Type D: The Maria–Fahmy version*

Maria and Fahmy proposed in [27] a table for complex polynomials that relied on the test for real polynomials that was proposed by Jury and Blanchard in [20] (see same in [13, p. 98]). The table in [29] converts by our standard convention of Remark 4 to the algorithm

$$f_{m-1}(z) = f_{m,0}\overline{f}_m(z) - f^*_{m,m} f^r_m(z), \qquad f_n(z) = p(z). \tag{52}$$

The following relation to the polynomials in the main D-type algorithm may be detected:

$$f_n(z) = d_n(z); \qquad f_{n-1}(z) = -d^r_{n-1}(z);$$

$$f_{n-2i}(z) = d^{\sharp}_{n-2i}(z); \qquad f_{n-(2i+1)} = d^r_{n-(2i+1)}(z) \qquad \text{for } i = 1, 2, \ldots .$$

For these relations Theorem 8 can be invoked to extend the stability conditions in [27],

$$f_{n-1,0} < 0, \qquad f_{m,m} > 0, \; m = n-2, n-3, \ldots, 0$$

into the next zero location rule.

**Corollary 5 (Maria and Fahmy's version).** *Consider algorithm* (52). *If it does not terminate prematurely define*

$$p_m = \prod_{i=1}^{m} f_{n-i,n-i}, \qquad m = 1, \ldots, n.$$

*Then* $p(z)$ *has* $\pi$ *IUC and* $n - \pi$ *OUC zeros, where*

$$\pi = n_-\{p_1, p_2, \ldots, p_n\}. \tag{53}$$

The fact that the generalization to complex polynomials that Maria and Fahmy obtained using the real tests in [20] is not identical to Marden's test although [20] was derived from Marden's test is again a demonstration of the mentioned nonunique way in which a real test procedure can be extended to complex polynomials.

### *5.2 Type D: Chen's version*

The table of Chen and Chan in [7] and Chen and Shiao in [8] translates into the following polynomial recursion:

$$zf_{m-1}(z) = f_{m,m} f_m(z) - f_{m,0} f_m^{\sharp}(z), \qquad f_n(z) = p(z)/p_n.$$

They considered only real polynomials and assumed the tested polynomial is first scaled to be monic. With these constraints the recursion coincides with Algorithm D. They provide the right stability conditions but their rule for the number of OUC zeros

$$\nu = n_-\{f_{n-1,n-1}, f_{n-2,n-2}, \ldots, f_{0,0}\}$$

is wrong, as the next counterexample may illustrate.

**Example 2.** The polynomial $p(z) = 0.5 + 9z + 12z^2 + z^3$ has 1 OUC and 2 IUC zeros. It is real and monic so that both Algorithm D and Chen's test associate to it the same sequence of polynomials: $d_3 = p(z)$ and

$$d_2(z) = 3{+}7.5z{+}0.75z^2, \qquad d_1(z) = -16.875{-}8.4375z, \qquad d_0(z) = -213.57.$$

Using (47)

$$g_3 = d_{2,2} = 0.75, \qquad g_2 = d_{2,2}d_{1,1} = -6.3281, \qquad g_1 = d_{2,2}d_{1,1}d_{0,0} = 1351.5.$$

Theorem 8 obtains the correct number of OUC zeros

$$\nu = n_-\{g_3, g_2, g_1\} = n_-\{0.75, -6.3281, 1351.5\} = 1.$$

Instead, Chen's rule suggests that the number of OUC zeros is

$$n_-\{d_{2,2}, d_{1,1}, d_{0,0}\} = n_-\{0.75, -8.4375, -213.57\} = 2.$$

## 6. Conclusions

The paper used the reflection coefficient to parametrize the tests in the Schur–Cohn and Marden–Jury tests class of methods for determining the zero location of a polynomial with respect to the unit circle. The SCMJ class was classified into four useful types of recursions. Although the polynomials in the four prototype sequences that were defined differ only in scaling factors from each other, this difference has at times quite an intricate effect on expressing the zero location rule in terms of the "native" polynomial coefficients. The invariance of the set of reflection coefficients for all tests in the SCMJ class facilitated the derivation of zero location rules for Types B, C, and D in terms of the leading coefficients of the polynomials in the sequence. The current systematic approach makes it possible to classify essentially any test published in this class into one of the defined types in spite of the masking effect of operations such as difference in initiation, reversion, conjugation, or reciprocation and sign variation of polynomials in corresponding sequences. The classification process led at times to corrections to wrongly-stated zero location rules, to the support of results stated with no accessible proof, or to generalization to some extent of the reviewed tests.

In the course of pursuing the zero location rule for the C-type scheme we in fact obtained yet another proof to the relation between zero location of a polynomial and the principal minors (hence the inertia) of the Schur–Cohn matrix generated for that polynomial.

## Appendix: Proof of Theorem 1

Let $(\pi_m, \nu_m)$ denote the number of (OUC, IUC) zeros of $a_m(z)$ (strong regularity implies no UC zeros). Obtain from the recursion (4) and its reciprocation the two equations

$$\frac{za_{m-1}(z)}{a_m(z)} = 1 + k_m \frac{a_m^\sharp(z)}{a_m(z)}, \qquad \frac{a_{m-1}^\sharp(z)}{k_m^* a_m(z)} = 1 + \frac{1}{k_m^*}\frac{a_m^\sharp(z)}{a_m(z)}. \tag{a.1, a.2}$$

Observe that $\left|\frac{a_m^\sharp(z)}{a_m(z)}\right| = 1$ for $|z| = 1$. Applying the principle of argument on (a.1) for $|k_m| < 1$ and on (a.2) for $|k_m| > 1$ proves, respectively, that

$$\begin{array}{lll} \text{if} & |k_m| < 1 & (\pi_m, \nu_m) = (\pi_{m-1} + 1, \nu_{m-1}) \\ \text{if} & |k_m| > 1 & (\pi_m, \nu_m) = (\nu_{m-1} + 1, \pi_{m-1}). \end{array} \tag{a.3}$$

Define for each fixed $m$, $m \leq n$, its own sequence (10) of $q_l$s, $\{q_l^{(m)}, l = 1, \ldots\ldots, m\}$ by

$$q_l^{(m)} = \prod_{i=l}^{m}(1 - |k_i|^2), \qquad l = 1, \ldots, m, \quad q_{m+1}^{(m)} := 1.$$

Note that in this new notation the original sequence of $q_m$'s in (10) corresponds to $\{q_m = q_m^{(n)}, m = 1, \ldots, n\}$.

Normalize each such sequence $\{q_i^{(m)}\}$ by dividing each of its entries by its minimal indexed member $q_1^{(m)}$

$$\frac{q_{l+1}^{(m)}}{q_1^{(m)}} = \prod_{i=1}^{l} \frac{1}{(1-|k_i|^2)} := \hat{q}_l, \qquad l = 1, \ldots, n.$$

The resulting new parameters $\hat{q}_l$ are independent of the index $m$ (therefore superscripts $(m)$ were dropped) and they can be computed recursively as follows.

$$\hat{q}_l = \hat{q}_{l-1}\frac{1}{(1-|k_l|^2)}, \qquad \hat{q}_0 := 1,\ l = 1, \ldots, n. \tag{a.4}$$

Thus the sequence in (9) is given by $\{q_1, q_2, \ldots, q_n\} = \{[\hat{q}_1^{(n)}, \hat{q}_2^{(n)}, \ldots, \hat{q}_n^{(n)}]q_1^{(n)}\}$, where the used notational convention is defined by

$$\{[\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_n]q_1\} := \{\hat{q}_1 q_1 \hat{q}_2 q_1, \ldots, \hat{q}_n q_1\}.$$

**Lemma 1.** *The number of (IUC, OUC) zeros of the mth degree polynomial $a_m(z)$, $(m - \nu_m, \nu_m)$, is given by*

$$\nu_m = n_-\{q_1^{(m)}, q_2^{(m)}, \ldots, q_m^{(m)}\} = n_-\{[\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_m]q_1^{(m)}\}. \tag{a.5}$$

**Proof.** The proof of the lemma is by induction over $m = 1, 2, \ldots, n$. The $m = 1$ is clear from (a.3). Assume the assertion (a.5) holds until $m = l - 1$ so that the numbers of IUC and OUC zeros of $a_{l-1}(z)$ are given, respectively, by

$$\nu_{l-1} = n_-\{[\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{l-1}]q_1^{(l-1)}\} \quad \text{and} \quad \pi_{l-1} = n_+\{[\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{l-1}]q_1^{(l-1)}\}.$$

We have to show that the above implies that (a.5) gives the correct numbers of IUC and OUC zeros also for $a_l(z)$.

There are two possibilities; either $|k_l| < 1$ or $|k_l| > 1$. For the case $|k_l| < 1$ (a.3) implies that $\nu_l = \nu_{l-1}$ (and $\pi_l = \pi_{l-1} + 1$). It is needed to check whether (a.5) provides the same result. For $|k_l| < 1$ (a.4) shows that $\text{sgn}(q_1^{(l)}) = \text{sgn}(q_1^{(l-1)})$. Therefore

$$\begin{aligned}\nu_l &= n_-\{[\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{l-1}, \hat{q}_l]q_1^{(l)}\} = n_-\{[\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{l-1}]q_1^{(l)}\} \\ &= n_-\{[\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{l-1}]q_l^{(l-1)}\} = \nu_{l-1}.\end{aligned}$$

In the above we dropped a by definition positive term $q_1^{(l)}\hat{q}_l = 1 > 0$ from the $n_-$ count. In the case $|k_l| > 1$ (a.3) implies that $\nu_l = \pi_{l-1}$ and again it has to be verified that the rule (a.5) is consistent with this case too. For $|k_l| > 1$ (a.4) indicates that $\text{sgn}(q_1^{(l)}) = -\text{sgn}(q_1^{(l-1)})$. Therefore

$$\begin{aligned}\nu_l &= n_-\{[\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{l-1}, \hat{q}_l]q_1^{(l)}\} = n_-\{[\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{l-1}]q_l^{l}\} \\ &= n_+\{[\hat{q}_1, \hat{q}, \ldots, \hat{q}_{l-1}]q_l^{(l-1)}\} = \pi_{l-1}.\end{aligned}$$ □

Theorem 1 corresponds to $m = n$ in this lemma.

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