Proc. of the 1st IFAC
Symposium on System Structure and Control
SSSC01, August 2001
Prague, Czech Republic

ZERO LOCATION OF POLYNOMIALS WITH RESPECT TO THE UNIT-CIRCLE WITHOUT NONESSENTIAL SINGULARITIES

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Abstract: The author's method to determine the distribution of the zeros of a (real or complex) polynomials with respect to the unit-circle is revisited and refined. The three-term recursion of symmetric polynomials that the method uses is generalized such that it remains regular for all polynomials whose Schur-Cohn matrix is not singular. The refined version eliminates nonessential singularities without compromising the low computational cost and the simplicity of the zero location rules of the original procedure.

Keywords: Stability criteria, Discrete-time systems, Zero sets, Polynomial methods.

1. INTRODUCTION

The unit-circle zero location problem aims to determine the number of zeros, z_i, of a polynomial inside, on and outside the unit-circle (IUC, UC and OUC zeros) $|z_i| < 1$, $|z_i| = 1$ and $|z_i| > 1$, respectively, in a finite number of operations. The current paper revises the author's solution to this problem in [Bistritz, 1984, 1986]. The revised method retains the simplicity and low computation of the original method in normal cases but refines its behavior in situations that previously were regarded as singularities and required special intervention.

The above is a key problem for the stability analysis and the design of one-dimensional and multi-dimensional discrete-time systems with impact on many related signal processing algorithm. The author's test is a relatively late contribution to this classical problem. The problem was first solved by Schur (1917) who obtained necessary and sufficient conditions for a polynomial to have only IUC zeros (to be 'stable') and it was extended to the zero location problem by Cohn [1922]. Marden [1966] and Jury [1982, 1988] considered efficient algorithms to carry out the Schur-Cohn solution. These classical ('scattering' type) algorithms and the author's new ('immittance' type) solution differ in several noticeable aspects. The classical solutions use two-term recursion and propagate polynomials of no particular form. In difference, the new test employs a three-term recursion of symmetric polynomials. The new test requires less computation than any version of the Marden-Jury and the Schur-Cohn methods (classified into four types in [Bistritz, 1996]) by a factor of two or higher (depending on version). As a new stability test, the method attracted attention of several researchers in this area. Jury and Mansour (1985) regarded the new test as the discrete-time equivalent of the Routh test (the well known stability test for continuous-time systems). Premaratne and Jury (1993) studied its relation with the principal minors of the Schur-Cohn matrix. Its computational advantage attracted efforts to generalize it also to testing stability of 2-D discrete-time systems, [Karan and Srivastava, 1986] [Premaratne, 1993] [Bistritz, 2000]. The redundancy that the new formulation exposed in testing stability of discrete-time system polynomials, led to use it to improve also other signal processing algorithms related to the Schur-Cohn problem, see Bistritz et al. [1991] and references there in.

Even though the original test in [Bistritz, 1984, 1986] was able to *always* determine the distribution of zeros of *any* polynomial, it had to cope with two type of singularities: a *first-type* singularity whose occurrence implies, and is implied by, a special pattern of a subset of the zeros of the tested polynomial (the existence of zeros whose reciprocal are also zeros of the polynomial) and a *second-type* singularity that bears no relation to constellation of the zeros (giving sense to calling them *nonessential* singularities). The revision proposed in this paper will remain regular in cases that previously implied second-type singularities. Another characterization of the difference between the original and the now revised form of the test is provided by their relations to properties of the Schur-Cohn matrix. It can be shown that while the original setting of the procedure could encounter singularities even though the tested polynomial has a nonsingular Schur-Cohn matrix (another reason for calling them *nonessential* singularities), the revised setting is guaranteed to remain regular for all polynomials whose Schur-Cohn matrix is nonsingular.

The revised form is based on generalizing the form of the three-term recursion of symmetric polynomials that underlies the method. The new recursion behaves like the original recursion in normal cases however it also passes smoothly steps that previously caused second-type singularities. It retains the efficiency of the former recursion form. As a matter of fact the count of operations actually decreases because a cost-free recursion step follows each previously singular step that is now eliminated.

The capacity to overcome singularities is not essential for 1-D stability testing because the procedure may be stopped as soon as it encounters a singularity. (Singularity of any kind implies that the tested polynomial is not 'stable' [Bistritz, 1984, 1986].) However, the revised procedure obtains more elegantly full information on the distribution of the zeros of a real or complex polynomial with respect to the unit-circle and is advantageous when performing this task using a uniform recursion is important. One not so obvious but important instance is stability testing of multidimensional (m-D, m $\geq$ 2) discrete-time systems. In fact, the preparation of this paper stems from the author's recent research effort in this area. A central problem in m-D stability testing involves a decision whether a polynomial of a relatively high degree (compared to degrees in stability determination of 1-D systems) has no zeros on the unit-circle, [Gu and Lee, 1999] [Bistritz, 1999, 2000].

The next section introduces the new form of the recursion and presents the rule to obtain the zero distribution in the nonsingular case. The third section characterizes the singular case and brings the general zero location theorem. A full version of this paper, [Bistritz, 2001], will contain more details, more illustrative numerical examples and, most importantly, proofs for all the theorems and comments that were omitted in this presentation to meet space constraints.

2. THE REGULAR CASE

Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, and $\mathbb{R}[z]$ and $\mathbb{C}[z]$ the set of polynomials with coefficients in the respective sets. The reciprocal polynomial $D_n^\sharp(z)$ of a polynomial $D_n(z) \in \mathbb{C}[z]$ is defined as

$$D_n(z) = \sum_{i=0}^{n} d_i z^i \quad , \quad D_n^\sharp(z) = \sum_{i=0}^{n} \bar{d}_{n-i} z^i \,, \qquad (1)$$

where the bar denotes complex conjugate. The zeros of $D_n^\sharp(z)$ are located reciprocally, with respect to the unit-circle, to the zeros of $D_n(z)$, i.e., $D_n^\sharp(z_i) = 0$ if, and only if, $D_n(\bar{z}_i^{-1}) = 0$. A $D_n(z) \in \mathbb{C}[z]$ is called symmetric if $D_n^\sharp(z) = D_n(z)$ and anti-symmetric if $D_n^\sharp(z) = -D_n(z)$.

The method to determine the number of IUC, UC and OUC zeros of a polynomials in [Bistritz, 1984, 1986] consists of building for the tested polynomial a certain sequence of symmetric polynomials, $\{T_k(z), k = n, n-1, \ldots, 0\}$, and zero location rules posed on this sequence. The sequence is constructed using a three-term polynomial recursion that starts with two polynomials obtained from the decomposition of the tested polynomial into the sum of a its symmetric and anti-symmetric parts.

The description so far is adequate for also the revised form of the method but currently two modifications are introduced. One is that the three-term recursion takes a more general form. The second modification is that symmetric polynomials $T_k(z)$ whose exact degree is lower than their formal degree k are now allowed. In the earlier form of the test each $T_k(z)$ in the final sequence of polynomials was of exact degree k. Now, polynomials whose exact degree is lower than their formal degree are legitimate in the final sequence $\{T_k(z), k = n, n-1, \ldots, 0\}$. To account of this new allowance, a new parameter λ_k is introduced to present the deficiency between the exact and formal degree of $T_k(z) \not\equiv 0$. Thus a nonzero symmetric polynomials, $T_k(z) \not\equiv 0$, will have now the form $T_k(z) = \sum_{i=0}^{k} t_{k,i} z^i$, $t_{k,i} = 0$, $i < \lambda_k$, $0 \leq \lambda_k < (k+1)/2$ where the uppetr limit is implied by the symmetry of the polynomial

($t_{k,i} = \bar{t}_{k,k-i} \quad i = 0, \ldots, k$). It also follows that λ_k is the number of zeros of $T_k(z)$ at $z = 0$ and at $z = \infty$. The λ_k for polynomials $T_k(z) \equiv 0$ will be regarded as "not defined" (it will never be required).

A nonzero $T_k(z)$ will be named *normal* if $\lambda_k = 0$ and *abnormal* if $\lambda_k > 0$. In the former setting, the symmetric polynomials were required to be normal for all $k < n$. An abnormal polynomial implied singularity and was replaced by a normal polynomial such that the recursion could be resumed (chosen carefully so as to not to destruct the process of collecting information on the location of zeros). In difference, currently, the sequence presented to the zero location rules may include normal, abnormal and, as will be seen soon, even identically zero polynomials.

The algorithm for the construction of the $T_k(z)$'s assumes, as before, a polynomial $D_n(z) \in \mathbb{C}[z]$ as follows.

$$D_n(z) = \sum_{i=0}^{n} d_i z^i,\ 0 \neq D_n(1) \in \mathbb{R},\ d_n \neq 0 \qquad (2)$$

This assumption means that an arbitrary polynomial $P(z)$ may need some preliminary adjustments as follows. If $P(1) = 0$ (then $P(z)$ is not stable, if its full distribution of zero is required) then zeros at $z = 1$ have to be removed (a simple task that involves only additive operations) and be added to the final report on the distribution of zeros. Assume the resulting polynomial is a polynomial of degree n, say $P_n(z)$, $P_n(1) \neq 0$. If $P_n(1) \in \mathbb{R}$ (this includes of course all $P_n(z) \in \mathbb{R}[z]$) then $D_n(z) = P_n(z)$ may be chosen. When $P_n(z) \in \mathbb{C}[z]$ and $P_n(1) \notin \mathbb{R}$, choose a $D_n(z)$ proportional to $P_n(z)$ with the property (2) e.g., $D_n(z) = \overline{P_n(1)} P_n(z)$ or $D_n(z) = P_n(z)/P_n(1)$.

The regular algorithm. Assume $D_n(z)$ as in (2) and construct for it a sequence of symmetric polynomials $\{T_k(z), k = n, \ldots, 0\}$ as follows.

$$T_n(z) = D_n(z) + D_n^\sharp(z) \qquad (3a)$$

$$T_{n-1}(z) = \frac{D_n(z) - D_n^\sharp(z)}{(z-1)} \qquad (3b)$$

For $k = n-1, \ldots, 0$:

$$\delta_{k+1} = \begin{cases} \dfrac{t_{k+1,0}}{t_{k,\lambda_k}} & \text{if } T_k(z) \not\equiv 0 \\ 0 & \text{if } t_{k+1,0} = 0 \end{cases} \qquad (4)$$

$$zT_{k-1}(z) = (\delta_{k+1} z^{-\lambda_k} + \bar{\delta}_{k+1} z^{\lambda_k+1}) T_k(z) - T_{k+1}(z)$$

The recursion (4) will be called the *regular recursion*. It is easily verified that $T_k(z) = T_k^\sharp(z)$ holds for all k. This symmetry means that it is enough to calculate only half of the coefficients of each polynomial.

The procedure for testing a $D_n(z)$ will be said to be *regular* (or *nonsingular*) if the regular algorithm succeeds to create the entire sequence till and including $T_0(0) \neq 0$ without interruption. Else, the procedure is said to be *singular*. Interruption of the regular recursion occurs when a $t_{k,0} \neq 0$ is followed by $T_{k-1}(z) \equiv 0$. In other words, the procedure is singular if, and only if, a normal polynomial is followed by an identically zero polynomials, viz.,

$$T_s(z) \text{ with } \lambda_s = 0 \Rightarrow T_{s-1} \equiv 0\,;\ 0 \leq s-1 < n \quad (5)$$

A $T_{k-1}(z) \equiv 0$ by itself does not imply singularity. This is another notable difference from the early form of this zero location method. A $T_{k-1}(z) \equiv 0$ that follows an abnormal polynomial ($\lambda_k > 0$) is now regarded as a regular step. Indeed, with $\delta_k = 0$ the next polynomial is obtained as $T_{k-2}(z) = -z^{-1} T_k(z)$ regardless of $T_{k-1}(z)$. This means that one or several (not adjacent) identically zero polynomials may legitimately appear in the final sequence of the currently revised procedure.

Remark 1. It is also important not to overlook the fact that a $T_0(z)$ (necessarily $T_0(z) \in \mathbb{R}$) equal to zero presents a singular case (and has to be treated as such before the zero location rules may be applied). A recommended way not to overlook this point is to consider the termination of the procedure as occurring when it reaches the term $T_{-1}(z)(= 0$ by structure) rather than at one step before.

The conditions

$$\lambda_k = 0 \qquad k = n, \ldots, 0 \qquad (6)$$

are called *normal conditions*. They were so called also in [Bistritz, 1984, 1986] but written there as $T_k(0) \neq 0 \quad k = n, \ldots, 0$. In the current context normal conditions present the special case in which the exact degree of every $T_k(z)$ is equal to its formal degree.

When normal conditions hold, the regular recursion (4) simplifies to the form

$$\delta_{k+1} = \frac{T_{k+1}(0)}{T_k(0)} \qquad (7)$$

$$zT_{k-1}(z) = (\delta_{k+1} + \bar{\delta}_{k+1} z) T_k(z) - T_{k+1}(z)$$

called *normal recursion*. Normal conditions (6) provide sufficient conditions for the normal recursion to be able to complete the whole sequence till $T_0(z) \neq 0$ (but they are not necessary conditions because $\lambda_n > 0$ does not obstruct the normal recursion from completing the whole sequence).

Since formerly, the normal recursion was the only recursion form, an abnormal polynomial obstructed its flow by implying a division by zero. This situation was then called "second-type singularity". It was resolved by replacing $\{T_{k+1}(z), T_k(z)\}$ by another pair of normal polynomials that carried on seamlessly the collection of information on the distribution of zeros. In difference, the current regular recursion, as seen (4), is not obstructed by an abnormal polynomial. Hence, with the current revision of this procedure, the former "second-type" singularities become obsolete. The only singularities remaining now correspond to the previously so called "first-type" or "structured" singularity except that now they are no longer implied by any identically zero polynomial but require the condition (5). (Singularities will be described in more detail the next section).

Remark 2. In the original setting the δ_k parameters were necessarily $\delta_k \neq 0$ for all $k < n$. The regular recursion redefines the δ_k parameters such that $\delta_k = 0$ is admissible. As a matter of fact, $\delta_k = 0$ if, and only if, $\lambda_k > 0$.

Theorem 1. (**Regular case.**) *Consider $D_n(z)$ (2) and assume that the procedure is regular. Then, $D_n(z)$ has $\alpha_n = n - \nu_n$ IUC zeros, and $\gamma_n = \nu_n$ OUC zeros, where*

$$\nu_n = Var\{\sigma_n, \sigma_{n-1}, \ldots, \sigma_0\} \tag{8}$$

$\sigma_k := T_k(1)$ and $Var\{\cdot\}$ denotes the number of sign variations in the sequence.

The theorem will be proved in [Bistritz, 2001] using Sturm's method to calculate the Cauchy index along the unit-circle. This proof is close to the proof given in Gantmacher [1959] to the Routh test and is more compact than the proof presented in [Bistritz, 1984, 1986] that was based on studying increments in the distribution of zeros of successive polynomials in an auxiliary sequence of not symmetric polynomials, $D_k(z) = T_k(z) + (z-1)T_{k-1}(z)$ $k = n, \ldots, 0$.

The parameters σ_k for Theorem 1 can be obtained using less computation than involved in summing the coefficients of each $T_k(z)$. Setting $z = 1$ in the nonsingular recursion (4) proves the recurrence relation,

$$\sigma_{k-2} = 2\mathcal{R}e\{\delta_k\}\sigma_{k-1} - \sigma_k\,. \tag{9}$$

This recursion may be used to obtain all the σ_k's for Theorem 1 in just n arithmetic operations using one of the following ways: (i) Running (9) in descending order (possibly in parallel with the nonsingular recursion steps) starting with $\sigma_n = T_n(1)$ and $\sigma_{n-1} = T_{n-1}(1)$. (ii) Running it in ascending order, after the table is completed, starting with $\sigma_{-1} := 0$ and $\sigma_0 = T_0(z)$. (iii) Running the recursion in ascending order as in (ii) but initiating it with $\hat{\sigma}_{-1} := 0$ and $\hat{\sigma}_0 := 1$. The latter case amounts to forming a normalized sequence $\{\hat{\sigma}_k\}_{n:0}$, where $\hat{\sigma}_k := \sigma_k/\sigma_0$ that clearly has the same number of sign variations.

It is seen that each of the two sets of parameters $\{\delta_k\}_{1:n}$ and $\{\sigma_k\}_{0:n}$ contains all the information on the zero distribution for the regular case. Similar relations between the σ_k and the δ_k and the recursion (9) were available before only for the more limited set of polynomials that obey normal conditions.

Theorem 1 infers in particular that a polynomial that obeys regular conditions has no UC zeros. Indeed according to the characterization of singular cases in the next section UC zeros form sufficient (but not necessary) conditions for singularity.

Remark 3. Note that a $\lambda_k > 0$ implies $T_{k-2}(z) = -z^{-1}T_k(z)$, i.e. a cost-free next polynomial (that involves just shift and sign change). In other words, not only that the new form conveniently circumvents situations that previously were regarded as singular and involved irregular intervention (whose cost was higher than the number of operations involved in a normal), but now there is even a gain of a free step per each such occasion. Later it will be seen that the general zero location rule remains intact. Thus the new advantages do not compromise any of the previous attractive features of the procedure.

Remark 4. Some of the $\sigma_k = T_k(1)$ may be zero. However, no two consecutive σ_k's can be zero because common zeros of adjacent $T_k(z)$ and $T_{k-1}(z)$ at $z = 1$ can be shown to imply a zero there of $D_n(z)$, in contradiction to the assumption (2). It can be shown [Bistritz, 2001] that vanishing σ_k never poses ambiguity on the sign variation rule; it can not appear at the beginning or the end of the sign variation rule and when it appears in an intermediate segment $\{\sigma_{k+1}, 0, \sigma_{k-1}\}$ its neighbors are such that $\sigma_{k+1} = -\sigma_{k-1} \neq 0$.

Example 1. *Consider the third example in [Bistritz, 1984] that was brought there to illustrate a second-type singularity.*

$$D_4(z) = [2, 7, 8, 5, 6]\mathbf{z}$$

where here and on $\mathbf{z} = [1, z, z^2, \ldots]$ (of proper length determined by context). Following (3),the algorithm is initiated with

$$T_4(z) = [8, 12, 16, 12, 8]\mathbf{z}$$

$$T_3(z) = [4, 2, 2, 4]\mathbf{z}$$

Here, $\lambda_3 = 0$ so the first recursion step is normal; $\delta_4 = \frac{t_{40}}{t_{30}} = 2$, and

$$T_2(z) = z^{-1}[\delta_4(1+z)T_3(z) - T_4(z)] = [0, -8, 0]\mathbf{z}$$

Next, $\lambda_2 = 1$ therefore an abnormal step follows; $\delta_3 = \frac{t_{30}}{t_{21}} = -0.5$ and

$$T_1(z)=z^{-1}[\delta_3(z^{-1}+z^2)T_2(z)-T_3(z)] = [-2,-2]\mathbf{z}$$

Next $\lambda_1 = 0$ implies again a normal step; $\delta_2 = \frac{t_{20}}{t_{10}} = 0$ and

$$T_0(z) = z^{-1}\{\delta_2(1+z)T_1(z) - T_2(z)\} = 8$$

Since $T_0(z)$ is normal ($\neq 0$), the information required for the zero location rule (9) is already available at this point. However, the algorithm is defined as terminating after one more step that creates $\delta_1 = \frac{t_{10}}{t_{00}} = -0.25$ (and the term $T_{-1} = 0$). Substituting the values of the polynomials at $z = 1$ into (9) gives

$$\nu_4 = Var\{56, 12, -8, -4, 8\} = 2 \quad .$$

Therefore according to Theorem 1, $D_4(z)$ has no UC zeros (the procedure is not singular), $\alpha = n - \nu_n = 4 - 2 = 2$ IUC zeros and $\gamma = \nu_n = 2$ OUC zeros. Alternatively, the σ_k parameters may also be obtained from $\{\delta_k\}_{4:1} = \{2, -0.5, 0, -0.25\}$ using (9) in one of the three ways mentioned there. To illustrate Remarks 2 and 3, notice that $\delta_2 = 0$ and therefore $T_0(z) = -z^{-1}T_2(z)$ may be deduced with no further computation.

Remark 5. The necessary and sufficient conditions for stability (all IUC zeros) is $\nu = 0$. It is often more convenient to express the stability condition as $\sigma_k > 0$, $\forall k$. This is possible by arranging $\sigma_n = 2D_n(1) > 0$. This arrangements is typically taken care of when a $P_n(z) \in \mathbb{C}[z]$ is adjusted to meet the requirement (2) by $D_n(z) = P_n(1)^\star P_n(z)$ or $D_n(z) = P_n(z)/P_n(1)$.

Remark 6. It is possible to adapt the approach shown in [Bistritz, 1986] for the normal recursions and similarly replace the regular recursion of complex polynomial by a pair of interlacing recursion of real symmetric and anti-symmetric polynomials.

3. THE GENERAL CASE

In order to determine the zero location of an arbitrary $D_n(z)$, it remains to show how the method deals with singular cases. The only singularity possible currently corresponds to first-type singularity in the earlier form of the method. It is associated with the existence of zeros z_o of $D_n(z)$ such that their reciprocal with respect to the unit-circle, $\bar{z}_o^{-1}$, is also a zero of $D_n(z)$. Among zeros that fall in this category, distinction is made between unit-circle ("UC zeros") and zeros not on the unit-circle that appear in reciprocal pairs ("RP zeros"), $(z_r, \bar{z}_r^{-1})$ $(|z_r| \neq 1)$. The singular case is characterized in the next theorem.

Theorem 2. *If the regular recursion is interrupted by $T_{s-1}(z) \equiv 0$ that follows a normal $T_s(z)$ (i.e. with $\lambda_s = 0$) then, $T_s(z)$ contains all the UC and RP of zeros of $D_n(z)$. Conversely, if the total number of UC and RP of zeros of $D_n(z)$ is s, then the regular recursion is interrupted by a $T_{s-1}(z) \equiv 0$ that follows a normal $T_s(z)$.*

The proof of this theorem uses the fact that the regular recursion acts as a greatest common divisor (gcd) algorithm for $T_n(z)$ and $T_{n-1}(z)$ hence it also determines the gcd of $D_n(z)$ and $D_n^\sharp(z)$ [Bistritz, 2001]. The above characterization states that singularity occurs (and right after a normal polynomial of degree s, $T_s(z)$) if, and only if, $D_n(z)$ and $D_n^\sharp(z)$ have a gcd of degree s given (up to a constant value) by $T_s(z)$. Note that the above theorem covers also the regular procedure as a special case. A regular procedure fits in Theorem 2 to the case $s = 0$ - no common zeros. In this case $0 \neq T_0(z) \in R$ is the highest degree gcd of $D_n(z)$ and $D_n^\sharp(z)$). The above characterization fits this special case even further because if the recursion is carried out one step further it produces the formally defined "$T_{-1}(z) = 0$" term.

Overcoming singularities. When a $T_s(z)$ with $\lambda_s = 0$ is followed by a $T_{s-1}(z) \equiv 0$ proceed as follows:

(i) Differentiate $T_s(z)$ to obtain $P_{s-1}(z) = T_s^{'}(z)$. Then form

$$D_{s-1}(z) = KP_{s-1}^\sharp(z)\,,\; K = -\frac{\mathcal{R}e\{P_{s-1}(1)\}}{\overline{P_{s-1}(1)}} \qquad (10)$$

where K may be also any other scaling number chosen such that $D_{s-1}(1)$ is real and of sign opposite to the sign of $T_s(1)$ [Bistritz, 2001]. The above choice was offered in Bistritz [1986] and it reduces to the simple value $K = -1$ for the real case Bistritz [1984].

(ii) Resume the nonsingular recursion with the two polynomials

$$T_{s-1}(z) = D_{s-1}(z) + D_{s-1}^\sharp(z) \qquad (11a)$$

$$T_{s-2}(z) = [D_{s-1}(z) - D_{s-1}^\sharp(z)]/(z-1) \qquad (11b)$$

Remark 7. A singular situation (5) will occur more than once if (and only if) $D_n(z)$ has UC or RP of zeros with multiplicity higher than one. Subsequent $T_k(z) \equiv 0$, $k < s-1$ should be treated again by (10). Singular steps will occur a number of times equal to the highest multiplicity of a UC or RP of zeros of $D_n(z)$ (because differentiation lowers the multiplicities each time by one).

The next theorem summarizes the zero location rule in the general case.

Theorem 3. (General case.) *Assume the proposed algorithm is applied to $D_n(z)$ (2) and that, possibly after encountering singular steps treated each time by (10), it finally produces a sequence $\{T_k(z), k = n, \ldots, 0\}$ (with $T_0(z) \neq 0$). Let s denote the degree after which a singularity occurred for the first time (with $s = 0$ corresponding to a procedure with no singularity). Let*

$$\nu_n = Var\{\sigma_n, \sigma_{n-1}, \ldots, \sigma_1, \sigma_0\} \tag{12}$$

and

$$\nu_s = Var\{\sigma_s, \sigma_{s-1}, \ldots, \sigma_1, \sigma_0\}\,. \tag{13}$$

Then, the number of IUC zeros of $D_n(z)$ is $\alpha_n = n - \nu_n$, its number of UC zeros is $\beta_n = 2\nu_s - s$ and its number of OUC zeros is $\gamma_n = n - \alpha_n - \beta_n$ (and there are $s - \nu_s$ pairs of reciprocal zeros).

A second example in Bistritz [2001] will illustrates a singularity case, as well as the fact that an identically zero polynomial is no longer sufficient condition for singularity.

4. CONCLUSION

The paper presented a refined form for the method to determine the distribution of zeros of a polynomial with respect to the unit circle in Bistritz [1984, 1986]. The revised procedure remains regular in situations that previously caused "second-type" singularities. Its efficiency compares favorably with the efficiency of the original procedure. The modifications also retains the originally simple zero location rules.

The revised method reduces to the original form in normal conditions (this includes the use of the method as a stability criterion), but it handles in a more uniform and elegant manner the more general problem of determining the location of zeros of a real or complex polynomial with respect to the unit circle. These advantages are expected to benefit also other applications, like the stability testing of multidimensional discrete-time systems and additional discrete-time system and digital signal processing algorithm related to the unit-circle zero location problem.

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