Zero Location of Polynomials With Respect to the Unit-Circle Unhampered by Nonessential Singularities

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***Abstract*—A method to determine the distribution of the zeros of a polynomial with respect to the unit-circle, proposed by this author in the past, is revisited and refined. The revised procedure remains recursive and nonsingular for polynomials whose Schur–Cohn matrix is not singular. Other nonessential singularities that previously caused interruption of the recursion are assimilated into a more general regular form of the three-term recursion of symmetric polynomials that underlies the method. The new form of the procedure does not compromise the simplicity of the rules to extract the information on the distribution that are proved using a different and more direct proof, based on the evaluation of the Cauchy index along the unit-circle. The low count of operations of the original procedure (recognized as the least cost solution for the problem) is maintained and actually gets better by the elimination of nonessential singularities. The improved features make the revised procedure a better all-around unit-circle zero location method for any real or complex polynomial. Its wider range of regularity should also benefit a variety of related signal processing and algebraic problems including some that were already affected by the original formulation.**

***Index Terms*—Discrete-time systems, immittance algorithms, polynomial methods, stability criteria, zero location.**

I. Introduction

The unit-circle zero location problem aims to determine the number of zeros, z_i, of a polynomial inside, on and outside the unit-circle (IUC, UC, and OUC zeros) $|z_i| < 1$, $|z_i| = 1$ and $|z_i| > 1$, respectively, in a finite number of arithmetic operations. The most noted application for this problem is testing stability of discrete-time systems. However, most of the available solutions to this problem are intimately related to a variety of topics in system theory and linear algebra, including efficient algorithms in digital signal processing problems, factorization of certain structured matrices, and tools for the design of one-dimensional and multi-dimensional filters.

The problem has a long history. It was first solved by [1], who obtained necessary and sufficient conditions for a polynomial to have only IUC zeros (to be "stable") and was extended to the zero location problem by [2]. Subsequent modifications to this solution were carried out by [3]–[5] and more researchers. A possible classification into four types of the forms that were published for this Schur–Cohn–Marden–Jury (SCMJ) class of methods was proposed in [6], that also cites more papers in this class, provides missing proofs or zero location rules, and gives the exact relations among the many versions.

A different approach to the solution of this problem has been proposed in [7] and [8]. This relatively late solution differs from the classical solutions in form and efficiency. The tested polynomial is associated with a sequence of symmetric polynomials, instead of polynomials of no particular form in the SCMJ class of solutions, and this sequence is obtained by a three-term recursion, instead of two-term recursions in the classical solutions. It requires less computation than the SCMJ methods by a factor of two or more, depending on the type of the SCMJ solution to which it is compared.

The current paper revisits the method in [7], [8] and proposes an improved form for it. The procedure there requires special intervention in certain circumstances called singularities that may occur and interrupt the recursion. These singularities were classified into two types; a *first-type* singularity that is related to a specific pattern in the location of a subset of the zeros of the tested polynomial (it implies and is implied by the existence of zeros whose reciprocal, with respect to the unit circle, are also zeros of the polynomial) and a *second-type* singularity that may occur and obstruct the recursion haphazardly, with no relation to any specific pattern in the location of the zeros. The disrupted recursion was resumed by replacing the offending polynomials by other polynomials that maintain the counting of the zeros. Singularities impair also the SCMJ class of methods and the importance of a full and simple solution to this zero location problem attracted much effort to overcome singular cases also for these alternative solutions, e.g., [9]–[12] and more references there in.

The new form of the procedure in this paper, offers a better way to combat singularities than in [7] and [8]. The revised procedure behaves normally like the original form but it remains recursive and nonsingular in circumstances that previously caused interruption of the recursion by second-type singularities. The only singularities that disrupt the recursive behavior of the new form correspond to first-type singularities in the former form. In other words, singularity will now occur if, and only if, the tested polynomial has UC zeros or pairs of zeros located reciprocally with respect to the unit circle. However, no separate examination or prior knowledge about such a situation is requested. The procedure detects them during its progress and handles them when they occur without wasting the already incurred computation. The improved features stem from introducing a more general form for the three-term recursion of symmetric polynomials that accommodates also symmetric polynomials with vanishing leading (and lowest power) coefficients

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and admits equally simple rules to count the zeros. The refined procedure also keeps the efficiency of the procedure (considered the method of least count of operations to solve the problem). As a matter of fact instances that correspond to elimination of previous singularities imply now further saving in arithmetic operations.

The original zero location method was recruited to improve the efficiency of methods to test the stability of also higher dimensional discrete-time systems [13]–[15]. It also led to new and more efficient forms for widely used signal processing algorithms related to the Schur–Cohn algorithm (titled "immittance" or "split" algorithms, see [16] and references there in). The now revised zero location procedure may benefit further these topics. The new recursion form is equally useful to eliminate singularities from also a modified version of the procedure [17]. (Exactly the same recursion plus very similar zero location conditions can be shown to hold for also [17]. However, for clarity only the form in [7], [8] will be considered in this paper.)

An interesting perspective on the difference in the range of regularity of the current form and the previous form of the procedure is provided by their respective relations to the Schur–Cohn matrix. The Schur–Cohn matrix (also known as the Schur–Cohn–Fujiwara matrix and the unit-circle Bezoutian) is a Hermitian matrix that can be formed for a polynomial (with number of rows and columns equal to its degree) such that its inertia is related to the distribution of the polynomial zeros with respect to the unit circle, e.g., [18]. The relations of the original zero location method to the principal minors of the Schur–Cohn matrix were studied in the context of positive-definite Schur–Cohn matrix and stable polynomials in [19] and in [16] for the modified form. Detailed relations between the new form and the rank profile of the Schur–Cohn matrix are beyond the scope of this paper. However, it can be shown that, unlike the first setting that could encounter singularities for polynomials whose Schur–Cohn matrix is not singular, the new form remains regular for all polynomials for which the Schur–Cohn matrix is nonsingular (cf. Remark 8 later on). This perspective gives further meaning to the adjective *nonessential* that has been attached now to the (former "second type") singularities that become obsolete with the current form of the procedure.

The proof of the main theorem will use Cauchy indices along the unit circle. This method of proof is different from the proof in [7], [8]. The proof there applied the argument principle to certain auxiliary not symmetric "behind the scene" polynomials. The current proof relates more directly to the three-term recursion of symmetric polynomials. It is modeled after the proof provided in [20] for the Routh test, except that it removes some unnecessary restrictions there on the Cauchy index and the Sturm sequence used to evaluate it. The current alternative proof is interesting in its own right and it strengthens the intimate relations between this discrete-time systems stability test and the Routh criterion, its continuous-time systems counterpart, noticed before also in [21] and [22].

The paper is constructed as follows. Section II presents the new form of the recursion and the rule to obtain the zero distribution in the nonsingular case and brings a numerical illustration. Section III completes the method into a general zero location theorem and brings a second numerical example. Proofs are mostly collected in the Appendix.

II. The Regular Case

Let $\mathcal{R}$ and $\mathcal{C}$ denote the set of real and complex numbers, and $\mathcal{R}[z]$ and $\mathcal{C}[z]$ the set of polynomials with coefficients in the respective sets. The reciprocal of a polynomial $D_n(z) = \sum_{i=0}^{n} d_i z^i \in \mathcal{C}[z]$ is defined by $D_n^\sharp(z) = \sum_{i=0}^{n} \overline{d}_{n-i} z^i$, where bar denotes complex conjugate. A reciprocal polynomial can also be expressed by $D_n^\sharp(z) = z^n \overline{D_n(\overline{z}^{-1})}$. It follows that the zeros of a reciprocal polynomial are the reciprocal, with respect to the unit-circle, of the zeros of $D_n(z)$, i.e., $D_n^\sharp(z_i) = 0$ if, and only if, $D_n(\overline{z}_i^{-1}) = 0$. A $D_n(z) \in \mathcal{C}[z]$ is called symmetric if $D_n^\sharp(z) = D_n(z)$ and antisymmetric if $D_n^\sharp(z) = -D_n(z)$. Symmetric polynomials play a major role in the enrolled zero location method. It is apparent that a symmetric polynomial has either UC zeros or reciprocal pair (RP) zeros (z_i & $\overline{z}_i^{-1}$).

The method in [7], [8] constructs for a polynomial

$$D_n(z) = \sum_{i=0}^{n} d_i z^i \in \mathcal{C}[z], \qquad 0 \neq D_n(1) \in \mathcal{R} \qquad d_n \neq 0 \tag{1}$$

a sequence of symmetric polynomials $\{T_k(z) = \sum_{i=0}^{k} t_{k,i} z^i, k = n, \ldots, 0\}$ (the "stability table") and then uses certain rules to extract from this sequence the distribution of the zeros of $D_n(z)$ with respect to the unit circle. The basic form of the algorithm for the construction of this sequence is as follow.

The Normal Algorithm (The Form in [7], [8]): Construct for $D_n(z)$ in (1)

$$T_n(z) = D_n(z) + D_n^\sharp(z)$$
$$T_{n-1}(z) = \frac{D_n(z) - D_n^\sharp(z)}{(z-1)}.$$

Then, for $k = n-1, \ldots, 0$, do:

$$\delta_{k+1} = \frac{T_{k+1}(0)}{T_k(0)}$$
$$zT_{k-1}(z) = \left(\delta_{k+1} + \overline{\delta}_{k+1} z\right) T_k(z) - T_{k+1}(z). \tag{2}$$

The recursion (2) has been called the *normal recursion*. A polynomial $D_n(z)$ will be called *normal* if its formal degree n is equal to its exact degree, $d_n \neq 0$, and *abnormal* otherwise. A symmetric polynomial $T_k(z)$ is normal or abnormal depending on whether $T_k(0) \neq 0$ or $T_k(0) = 0$, respectively. If the normal recursion does not encounter an abnormal polynomial it produces a sequence $\{T_n(z), \ldots, T_0(z)\}$ of normal symmetric polynomials from which the distribution of zeros of $D_n(z)$ can easily be obtained (using rules that can also be deduced from theorems later on in this paper). The normal recursion (2) is disrupted when it produces an abnormal polynomial. Such cases were called singularities and they were classified into two types. An identically zero polynomial was called first-type singularity and an abnormal polynomial that is not identically zero—a second-type singularity. The treatment of the two type of singularities differed but both cases involved the replacement of the offending polynomial and its predecessor by a certain other pair of normal polynomials such that the interrupted recursion can be resumed and such that the counting of zeros goes on seamlessly. At the end, the polynomials $T_k(z)$ submitted to the zero location rules, in order to determine the distribution of zeros, were always normal polynomials for all $k = n-1, \ldots, 0$—either naturally or because abnormal polynomials were replaced.

Currently, a more general form for the three-term recursion will be used. It can accommodate also abnormal symmetric polynomials. To account for this possibility, a new parameter λ_k is introduced to measure the deficiency between the exact and formal degree of a $T_k(z) \not\equiv 0$. In other words, λ_k counts the number of zeros of a $T_k(z) \not\equiv 0$ at $z = 0$ (and at $z = \infty$). The possible range of λ_k, for a not identically zero symmetric polynomial, is $0 \leq \lambda_k < (k+1)/2$, where the upper limit is implied by the symmetry of the polynomial $(t_{k,i} = \bar{t}_{k,k-i} \quad i = 0, \ldots, k)$. The λ_k for polynomials $T_k(z) \equiv 0$ need not be defined (it will never be required). Clearly, λ_k may also be used to say that a polynomial $T_k(z)$ is normal ($\lambda_k = 0$) or abnormal ($\lambda_k > 0$). A related difference between the previous and the current form of the procedure will be that now abnormal polynomials (and even identically zero polynomials!) will be legitimate members in the final sequence of polynomials submitted to the zero location rules.

The new procedure is initiated as before but it replaces the normal recursion (2) by a more general regular recursion shown in the next algorithm that becomes the basic form for the revised procedure. The next algorithm can be applied to a real or complex polynomial that obeys the assumptions in (1). This assumption is usually not restrictive because a polynomial can be adjusted easily to meet it. We shall return to this assumption later (in Remark 3 below) and discuss its implications and ways to relax it.

The Regular Algorithm (The New Form): Construct for the polynomial $D_n(z)$ in (1) a sequence of symmetric polynomials $\{T_k(z),\, k = n, \ldots, 0\}$ as follows:

$$T_n(z) = D_n(z) + D_n^{\sharp}(z) \tag{3a}$$

$$T_{n-1}(z) = \frac{D_n(z) - D_n^{\sharp}(z)}{(z-1)}. \tag{3b}$$

For $k = n-1, \ldots, 0$

$$\delta_{k+1} = \begin{cases} \dfrac{t_{k+1,0}}{t_{k,\lambda_k}}, & \text{if } T_k(z) \not\equiv 0 \\ 0, & \text{if } t_{k+1,0} = 0 \\ \text{not required}, & \text{if } t_{k+1,0} \neq 0 \,\&\, T_k(z) \equiv 0 \end{cases}$$

$$zT_{k-1}(z) = \left(\delta_{k+1} z^{-\lambda_k} + \bar{\delta}_{k+1} z^{\lambda_k+1}\right) T_k(z) - T_{k+1}(z). \tag{4}$$

The recursion (4) will be called the *regular recursion*. It is not difficult to show that all the polynomials that the regular algorithm produces are symmetric, $T_k(z) = T_k^{\sharp}(z)$. This symmetry can again be exploited to calculate only half of the coefficients of each polynomial.

The zero location procedure for a $D_n(z)$ will be called *regular* (as well as *nonsingular*), if the regular algorithm can create the entire sequence till and including $T_0(z) \neq 0$ without interruption. Interruption of the regular recursion occurs when (and only when) a $T_{k-1}(z) \equiv 0$ occurs after a $T_k(z)$ such that $T_k(0) \neq 0$. In such a case the zero location procedure will be said to be *singular* for $D_n(z)$. Thus, the new procedure is singular if, and only if, a normal polynomial is followed by an identically zero polynomials, *viz.*,

$$T_s(z) \text{ with } \lambda_s = 0 \quad \& \quad T_{s-1}(z) \equiv 0; 0 \leq s-1 < n. \tag{5}$$

Remark 1: A $T_0(z) = 0$ always presents a singular case and has to be treated as such (using means to be described in the next section) before the zero location rules may be applied. A helpful way not to miss the fact that termination of the procedure with a $T_0(z) = 0$ is not legitimate is to insist on regarding the termination of the construction when the sequence reaches the term $T_{-1}(z)$ (=0 by structure) rather than ending it when reaching $T_0(z)$. The assertion here is consistent with (5) because, for reasons explained in Remark 5 below, a $T_0(z) = 0$ can occur only after a normal $T_1(z)$. [Clearly, $T_1(z)$ must be either normal or identically zero.]

Remark 2: In the original setting the δ_k parameters were necessarily $\delta_k \neq 0$ for all $k < n$. Now, the regular recursion redefines the δ_k parameters differently. Consequently $\delta_k = 0$ is now admissible. In fact, $\delta_k = 0$ if, and only if, $\lambda_k > 0$.

The following conditions

$$\lambda_k = 0, \qquad k = n, \ldots, 0 \tag{6}$$

have been called *normal conditions* in [7] and [8] [expressed there by $T_k(0) \neq 0, \quad k = n, \ldots, 0$]. In the current context, normal conditions present the special case in which the exact degree of every $T_k(z)$ is equal to its formal degree. When normal conditions hold, the regular recursion (4) simplifies to the original normal recursion form (2). Normal conditions (6) are sufficient conditions for the normal algorithm to produce the entire sequence till $T_0(z) \neq 0$ (but are not necessary conditions simply because $\lambda_n > 0$ does not obstruct the normal algorithm).

Theorem 1 (The Regular Case): Consider $D_n(z)$ (1) and assume that the procedure is regular. Then, $D_n(z)$ has $\alpha_n = n - \nu_n$ IUC zeros, and $\gamma_n = \nu_n$ OUC zeros, where

$$\nu_n = \mathrm{Var}\{\sigma_n, \sigma_{n-1}, \ldots, \sigma_0\} \tag{7}$$

$\sigma_k := T_k(1)$, and $\mathrm{Var}\{\cdot\}$ denotes the number of sign variations in the sequence.

This theorem is proved in the Appendix. The current proof is different from the proof presented in [7], [8] where increments in the distribution of zeros of successive polynomials in an auxiliary sequence of not symmetric polynomials, $D_k(z) = T_k(z) + (z-1)T_{k-1}(z)\; k = n, \ldots, 0$ were used to prove a similar sign variation rule. It is modeled after the proof provided in [20] for the Routh test (another evidence to the Routh-like form of this procedure) but it employs a more liberal definition of the Sturm sequence to evaluate more directly the Cauchy index along the unit-circle.

It is apparent that the regular algorithm is not obstructed by abnormal polynomials. Abnormal polynomials were previously responsible for "second-type" singularities. The only singularity that is possible now occurs in circumstances described in (5) and it will be the subject of the next section. Since "second-type" singularities have now been eliminated, it might be expected that the remaining singularity coincides with the previously called "first-type" singularity. The reply to this expectation is divided between appearance and characterization as follows. Formerly, first-type singularity was shown to imply and be implied by $D_n(z)$ having UC or reciprocal pair of zeros. Singularities in the new procedure will be shown to relate similarly to this special pattern of zeros. (A hint is already provided by Theorem 1 that infers that a polynomial that obeys regular conditions has no UC zeros.) On the other hand, formerly a $T_{k-1}(z) \equiv 0$ always presented a first-type singularity. In difference, now a $T_{k-1}(z) \equiv 0$, by itself, does not present a singularity. Currently, a $T_{k-1}(z) \equiv 0$ that follows an abnormal polynomial ($\lambda_k > 0$) does not interrupt the recursion. It implies $\delta_k = 0$ and consequently $T_{k-2}(z) = -z^{-1}T_k(z)$ is obtained regardless of $T_{k-1}(z)$. This means that one or several (not adjacent) identically zero polynomials may legitimately appear in the final sequence of the currently revised procedure. The exact characterization of singularities is brought in the forthcoming Theorem 2 and Example 2 will illustrate the new situation.

The parameters σ_k for Theorem 1 can be obtained also in a different manner. Setting $z = 1$ in the regular recursion (4) proves the recurrence relation

$$\sigma_{k-2} = 2\mathcal{R}e\{\delta_k\}\sigma_{k-1} - \sigma_k. \tag{8}$$

This recursion may be used to obtain adequate parameters for Theorem 1 in several ways: i) Running (8) in descending order (possibly in parallel with the regular recursion steps) starting with $\sigma_n = T_n(1)$ and $\sigma_{n-1} = T_{n-1}(1)$. ii) Running it in ascending order, after the table has been completed (so that all the δ_k are available), starting with $\sigma_{-1} := 0$ and $\sigma_0 = T_0(z)$. iii) Running the recursion in ascending order as in ii) but initiating it with $\hat{\sigma}_{-1} := 0$ and $\hat{\sigma}_0 := 1$. The latter case amounts to forming a normalized sequence $\{\hat{\sigma}_k\}_{n:0}$, where $\hat{\sigma}_k := \sigma_k/\sigma_0$ that clearly has the same number of sign variations. One use of this approach is to obtain the parameters for the zero location rules in just n arithmetic operations [less computation than summing the coefficients of each $T_k(z)$] but it has other potential applications as well (one is hinted in Remark 3.

It follows that any of the two sets of parameters $\{\delta_k\}_{1:n}$ and $\{\sigma_k\}_{0:n}$ contains all the information on the zero distribution for the regular case. Similar relations between the σ_k and the δ_k and the recursion (8) were available also previously but formerly they were limited to polynomials that obey normal conditions.

Remark 3: In this remark we shall dwell in some depth on the assumption in (1). An arbitrary polynomial $P(z)$ can be made to meet it by a sequence of preliminary adjustments as follows. If $P(1) = 0$ [enough to declare $P(z)$ as not stable] then zeros at $z = 1$ have to be removed till a polynomial of lower degree, say $P_n(z)$ of degree n, such that $P_n(1) \neq 0$ is reached. (In such a case zeros at $z = 1$ should be remembered for the final report on the distribution of zeros). If $P_n(1) \in \mathcal{R}$ (which holds of course for all $P_n(z) \in \mathcal{R}[z]$) then $D_n(z) = P_n(z)$ may be chosen. Else when $P_n(z) \in \mathcal{C}[z]$ and $P_n(1) \notin \mathcal{R}$, $D_n(z)$ can be chosen by scaling $P_n(z)$ to acquire it with the property (1) e.g., $D_n(z) = \overline{P_n(1)}P_n(z)$ or $D_n(z) = P_n(z)/P_n(1)$. Next, a few words on what purpose this assumption is set to serve (and how it can be relaxed). The requirement $D_n(1) \in \mathcal{R}$ admits the division in (3b), i.e., is necessary for $T_{n-1}(z)$ to be a polynomial. It can then be shown that the value at $z = 1$ of $T_{n-1}(z)$ is $T_{n-1}(1) = 2\mathcal{R}e\{D_n'(1)\} - n\overline{D_n(1)}$ [where $D_n'(z) = dD_n(z)/dz$]. It follows that all following $T_k(1)$ produced by the regular recursion are real as well [cf. (8)] so that the sign variation rule makes sense. The additional requirement, $D_n(1) \neq 0$, is posed in order to have $T_n(1) = 2D_n(1) \neq 0$ because else a $T_n(1) = 0$ that is followed by a $T_{n-1}(1) = 0$ implies that all subsequent $T_k(1)$ produced by the regular recursion vanish. This would pose an obstacle on the sign variation rule [cf. (7)]. If however, (for some more remote applications than a stand alone zero location problem), the assumption (1) is restrictive or not desirable, several fixes are possible. Assuming regularity, it is possible to replace in the zero location rule the $T_k(1)$ by parameters $\hat{\sigma}_k$ explained next to (8). This way, it is possible to drop the requirement $D_n(1) \neq 0$ and require only that $D_n(1)$ is real. Furthermore, it can be shown that the revised zero location method is applicable also for a modified form of the zero location procedure [17] that differs in its initiation and, consequently, admits complex values for $D_n(1)$ posing instead the requirement $\mathcal{R}e\{D_n(1)\} \neq 0$. The latter requirement can further be relaxed to also just $\mathcal{C} \ni D_n(1) \neq 0$, see [23, Remark 2].

Remark 4: Note that a $\lambda_k > 0$ implies $T_{k-2}(z) = -z^{-1}T_k(z)$, i.e., a cost-free next polynomial (just shift and sign change). Thus, not only that the new form of the procedure conveniently circumvents previous singularities (that required irregular intervention with a cost that exceeds the cost of a normal step) viz. $\lambda_k > 0$ rewards the count of operations by λ_k free recursion steps.

Remark 5: Some of the $\sigma_k = T_k(1)$ may be zero. However, no two consecutive σ_ks can be zero because common zeros of adjacent $T_k(z)$ and $T_{k-1}(z)$ at $z = 1$ would imply a zero of $D_n(z)$ at $z = 1$ (using an inherent property of the recursion, see also the proof for the forthcoming Theorem 2), in contradiction to the assumption (1). A vanishing σ_k will never pose ambiguity on the sign variation for the following reasons. The sequence starts with $\sigma_n = 2D_n(1) \neq 0$ by assumption. If at some intermediate step $k > 0$, a $\sigma_k = 0$ occurs, then (8) implies that $\sigma_{k+1} = -\sigma_{k-1}$. Thus the segment $\{\sigma_{k+1}, 0, \sigma_{k-1}\}$ contributes a definite sign variation. Finally, the last element σ_0 can never be zero, cf. Remark 1.

Example 1: Consider the third example in [7] that was brought there to illustrate a second-type singularity

$$D_4(z) = [2, 7, 8, 5, 6]\mathbf{z}$$

where here and on $\mathbf{z} = [1, z, z^2, \ldots]^t$ (of proper length determined by context). Following (3), the algorithm is initiated with

$$T_4(z) = [8, 12, 16, 12, 8]\mathbf{z}$$
$$T_3(z) = [4, 2, 2, 4]\mathbf{z}.$$

Here, $\lambda_3 = 0$ so the first recursion step is normal; $\delta_4 = t_{40}/t_{30} = 2$, and

$$T_2(z) = z^{-1}[\delta_4(1+z)T_3(z) - T_4(z)] = [0, -8, 0]\mathbf{z}$$

Next, $\lambda_2 = 1$ so this is an abnormal step; $\delta_3 = t_{30}/t_{21} = -0.5$ and

$$T_1(z) = z^{-1}\left[\delta_3\left(z^{-1}+z^2\right)T_2(z) - T_3(z)\right] = [-2,\,-2]\mathbf{z}.$$

Next $\lambda_1 = 0$ presents a normal step; $\delta_2 = t_{20}/t_{10} = 0$ and

$$T_0(z) = z^{-1}\{\delta_2(1+z)T_1(z) - T_2(z)\} = 8.$$

Since $T_0(z)$ is normal ($\neq 0$), all the information required for the zero location rule (7) has already reached. However, it is noted that the algorithm is defined as terminating after one more step (and Remark 1 explains why it is better not to overlook this matter). The next step produces $\delta_1 = t_{10}/t_{00} = -0.25$ and the term $T_{-1}(z) = 0$. Substituting values into (7) gives

$$\nu_4 = \mathrm{Var}\{56,\,12,\,-8,\,-4,\,8\} = 2.$$

Therefore according to Theorem 1, $D_4(z)$ has no UC zeros (the procedure is not singular), $\alpha = 4 - \nu_4 = 2$ IUC zeros and $\gamma = \nu_4 = 2$ OUC zeros.

The example can also be used to realize the alternative ways to obtain σ_k parameters via (8) in parallel to the main recursion or by recovering $\{\hat{\sigma}_k\}_{n:0} = \{7,\,1.5,\,-1,\,-0.5,\,1\}$ from $\{\delta_k\}_{4:1} = \{2,\,-0.5,\,0,\,-0.25\}$. As illustration for Remark 3, notice that $\delta_2 = 0$ implies that $T_0(z) = -z^{-1}T_2(z)$ that becomes available for no arithmetic cost. Finally, it is also possible to arrange the algorithm in a tabular form as done in [7]. The rows of the table are the coefficients of the symmetric polynomials and its entries can be obtain by an adequate translation of the effect of the algorithm on them. The table for this example is

$$\begin{array}{ccccccccc|l}
8 & & 12 & & 16 & & 12 & & 8 & \lambda_4 = 0 \\
& 4 & & 2 & & 2 & & 4 & & \lambda_3 = 0,\ \delta_4 = 2 \\
& & 0 & & -8 & & 0 & & & \lambda_2 = 1,\ \delta_3 = -0.5 \\
& & & -2 & & -2 & & & & \lambda_1 = 0,\ \delta_2 = 0 \\
& & & & 8 & & & & & \lambda_0 = 0,\ \delta_1 = -0.25.
\end{array}$$

Some annotation on the relevant parameters has been added at a right hand side column. The symmetry of the coefficient vectors (the rows of the table) can be used to drop the right hand side (say) of the table. The second example in this paper will be presented only in this brief tabular form.

A stability criterion according to this method corresponds to the special case $\nu = 0$ in Theorem 1. As a matter of fact and as will become apparent immediately, the new procedure offers nothing new to this case compared to the former form. Nevertheless, this special case is important enough to be characterized separately as the next corollary.

Corollary 1 (Stability): (**a**) $D_n(z)$ (1) has n IUC zeros (is stable) if, and only if, all $\sigma_k = T_k(1)$ (are non zero and) have the same sign. This condition may be equally presented by e.g.,

$$\frac{\sigma_k}{\sigma_n} > 0, \qquad k = n-1,\,\ldots,\,0. \tag{9}$$

(**b**) The following are necessary conditions for stability: i) $\mathcal{R}e\{\delta_k\} > 0$, $k = n,\,\ldots,\,1$. ii) The normal conditions, defined in (6).

Proof: Part (a) follows as the special case $\nu = 0$ in Theorem 1, where the expression (9) is chosen to stress *definite* signs, i.e., that all $\sigma_k \neq 0$ are necessary condition for stability. It was shown in Remark 5 that a $\sigma_k = 0$ implies a full sign variation hence OUC zeros. For part (b): If all σ_k have same sign then $\mathcal{R}e\{\delta_k\} > 0$, $\forall\, k$ is seen from (8). Condition i) implies condition ii) because a $\lambda_k > 0$ implies $\delta_k = 0$. ■

Note that, since normal conditions form necessary conditions for stability, the appearance of an abnormal polynomial is enough to declare the tested polynomial as not stable and the stability test need not be continued. Since for normal conditions the regular recursion coincides with the original normal recursion, the revised procedure offers no added value for using the method only as a stability criterion.

Remarks 6: It is possible to express the necessary and sufficient conditions for stability by $\sigma_k > 0$, $\forall\, k$ by requiring $D_n(1) > 1$ [instead of the form assumed in (1)] because $\sigma_n = 2D_n(1)$. In the complex case, this nicety will typically be granted after arranging a $P_n(z) \in \mathcal{C}[z]$ to meet the requirement (1) by $D_n(z) = P_n(1)^{\star}P_n(z)$ or $D_n(z) = P_n(z)/P_n(1)$.

When testing a complex polynomial, it is possible to replace the regular recursion by a pair of interlacing recursion of real symmetric and anti-symmetric polynomials, extending the approach shown in [8] to the regular recursions. Let the symmetric $T_k(z) \in C[z]$ be written as $T_k(z) = S_k(z) + jA_k(z)$, where $S_k(z),\,A_k(z) \in R[z]$, $S_k^{\sim}(z) = S_k(z)$ and $A_k^{\sim}(z) = -A_k(z)$. Then, the regular recursion (4) can be carried out by the following coupled three-term recursion of real polynomials

$$\begin{aligned} zS_{k-1}(z) = {} & \delta_{k+1}^r\left(z^{-\lambda_k} + z^{\lambda_k+1}\right)S_k(z) \\ & - \delta_{k+1}^i\left(z^{-\lambda_k} - z^{\lambda_k+1}\right)A_k(z) - S_{k+1}(z) \end{aligned} \tag{10a}$$

$$\begin{aligned} zA_{k-1}(z) = {} & \delta_{k+1}^r\left(z^{-\lambda_k} + z^{\lambda_k+1}\right)A_k(z) \\ & + \delta_{k+1}^i\left(z^{-\lambda_k} - z^{\lambda_k+1}\right)S_k(z) - A_{k+1}(z) \end{aligned} \tag{10b}$$

where $\delta_k = \delta_k^r + j\delta_k^i$. The initiation, $S_k(z)$ and $A_k(z)$ for $k = n,\,n-1$, are obtained from (3a), (3b). Notice that σ_ks for the zero location rules are given by $\sigma_k = S_k(1)$. They may also be obtained from the set of δ_k^rs using (8).

III. The General Case

In order to determine the zero location of an arbitrary polynomial, it remains to deal with singular cases. Singularity occurs when an identically zero polynomial follows a normal polynomial as stated in (5). It is associated with the existence of zeros z_o of $D_n(z)$ such that their reciprocal with respect to the unit-circle, $\overline{z}_o^{-1}$, is also a zero of $D_n(z)$. In this category distinction will be made between zeros on the unit-circle ("UC zeros") and zeros not on the unit-circle that appear in reciprocal pairs ("RP zeros"), $(z_r,\,\overline{z}_r^{-1})$, $|z_r| \neq 1$.

Theorem 2: (**a**) If the regular recursion is interrupted by $T_{s-1}(z) \equiv 0$ that follows a normal $T_s(z)$ (i.e., $\lambda_s = 0$) then, $T_s(z)$ contains all the UC and RP zeros of $D_n(z)$. Conversely, if the total number of UC and RP zeros of $D_n(z)$ is s, then the regular recursion is interrupted by a $T_{s-1}(z) \equiv 0$ that follows a normal $T_s(z)$.

(**b**) If the regular recursion produces a $T_{k-1}(z) \equiv 0$ that follows an abnormal $T_k(z)$ (i.e., $\lambda_k > 0$), then the situation described in part (a) will occur for $T_s(z)$ of degree $s = k - 2\lambda_k$ given by $T_s(z) = (-1)^{\lambda_k} T_k(z)/z^{\lambda_k}$.

The proof of this theorem is brought in the Appendix. It uses the fact that the regular recursion acts as a greatest common divisor (g.c.d.) algorithm for $T_n(z)$ and $T_{n-1}(z)$ hence it also determines the g.c.d. of $D_n(z)$ and $D_n^{\sim}(z)$.

Remark 7: Part (a) of Theorem 2 can be rephrased as saying that singularity occurs [and if so then right after $T_s(z)$] if, and only if, $D_n(z)$ and $D_n^{\sim}(z)$ have a g.c.d. of degree s given (up to a constant value) by $T_s(z)$. Note that the theorem covers correctly also the regular case. The regular case is represented in Theorem 2 by the value $s = 0$. For $s = 0$ the highest degree g.c.d. of $D_n(z)$ and $D_n^{\sim}(z)$ is the real and constant $T_0(z) \neq 0$. Taking the recursion one step beyond $s = 0$ produces the term "$T_{-1}(z) = 0$" (cf. Remark 1) which is still in agreement with the description in Theorem 2. Next part (b) of the theorem enlightens further the complementary case of an identical zero polynomials that follows an abnormal polynomial. It does not present an immediate singularity (the regular recursion bypasses it without interruption) but it implies a singularity at $s = k - 2\lambda_k$ if (and only if) $s > 0$. It is even more acutely stated that such a situation implies *always* (whether or not later the procedure turns to be singular) that the degree $s = k - 2\lambda_k$ polynomial $T_k(z)/z^{\lambda_k}$ presents the common finite zeros of $D_n(z)$ and $D_n^{\sim}(z)$ (if any). Bearing in mind that $k - 2\lambda_k \geq 0$, cases where an identically zero polynomial occurs without implying a subsequent singularity are described by $k = 2\lambda_k$. In explicit words, a $T_{k-1}(z) \equiv 0$ does not predict a later singularity if, and only if, it has odd degree and it is preceded by an abnormal polynomial $T_k(z)$ that has only a single nonzero coefficient (at the center of its coefficient vector, necessarily).

Remark 8: Our introductory comment that characterized the advantage of the current procedure as remaining nonsingular for all polynomials that have a nonsingular Schur–Cohn matrix becomes now apparent. It follows from the links that Theorem 2 describes between singularity in the current form of zero location procedure to g.c.d of $D_n(z)$ and $D_n^{\sim}(z)$ of degree $s > 0$ that are also well known as necessary and sufficient conditions for the Schur–Cohn Bezoutian to be singular (e.g., [18]).

Overcoming Singularities: Whenever a $T_s(z)$ with $\lambda_s = 0$ is followed by a $T_{s-1}(z) \equiv 0$ for a $s > 0$ proceed as follows:

i) Differentiate $T_s(z)$ and denote the derivative by $P_{s-1}(z) := T_s'(z)$. Form

$$D_{s-1}(z) = K P_{s-1}^{\sim}(z) \tag{11a}$$

where K is any scaling number that fulfills the requirement that $D_{s-1}(1)$ is real and has a sign opposite to the sign of $T_s(1)$. Some adequate choices for K are

$$K_1 = -\frac{\mathcal{R}e\{P_{s-1}(1)\}}{\overline{P_{s-1}(1)}}$$

$$K_2 = -\frac{T_s(1)}{\overline{P_{s-1}(1)}}, \quad K_3 = -T_s(1)P_{s-1}(1). \tag{11b}$$

ii) Resume the regular recursion with the two polynomials

$$T_{s-1}(z) = D_{s-1}(z) + D_{s-1}^{\sim}(z) \tag{12a}$$

$$T_{s-2}(z) = \left[D_{s-1}(z) - D_{s-1}^{\sim}(z)\right] \Big/ (z-1). \tag{12b}$$

The above method for overcoming a singularity is similar to the treatment of first-type singularity in the former form of the procedure. The choice K_1 was proposed in [8] and it reduces to the convenient value $K = -1$ for the real case [7]. More substantiation for this method of overcoming singularities appears in the Appendix.

Remark 9: A singular situation (5) will occur more than once if (and only if) $D_n(z)$ has UC or RP zeros of multiplicity higher than one. Any time $T_{k-1}(z) \equiv 0$, occurs after a normal $T_k(z)$ it has to be treated again by (10) and (11). Singular steps will occur in a number of times equal to the highest multiplicity of UC or RP zeros of $D_n(z)$ (because differentiation lowers multiplicities each time by one).

The complete rules to obtain the distribution of zeros by the current unit-circle zero location method are presented in the next theorem.

Theorem 3 (General Case): Assume the proposed algorithm is applied to $D_n(z)$ (1) and that, possibly after encountering singular steps treated each time by the procedure stated above, it finally produces a sequence $\{T_k(z),\, k = n, \ldots, 0\}$. Let s denote the degree after which a singularity occurred for the first time (with $s = 0$ presenting a nonsingular procedure). Let

$$\nu_n = \mathrm{Var}\{\sigma_n, \sigma_{n-1}, \ldots, \sigma_1, \sigma_0\} \tag{13}$$

and

$$\nu_s = \mathrm{Var}\{\sigma_s, \sigma_{s-1}, \ldots, \sigma_1, \sigma_0\}. \tag{14}$$

Then, the number of IUC zeros of $D_n(z)$ is $\alpha_n = n - \nu_n$, its number of UC zeros is $\beta_n = 2\nu_s - s$ and its number of OUC zeros is $\gamma_n = n - \alpha_n - \beta_n$ (and there are $s - \nu_s$ pairs of reciprocal zeros).

Example 2: To illustrate a singular case and other identically zero polynomial consider $D_9(z) = [-2, 7, -3, -16, 16, 10, -1, -3, -8, 4]\mathbf{z}$. For brevity this time we shall present the procedure only by the resulting table (with commentary on the participating parameters at an extra column at the right end, similar to the tabular presentation of Example 1), as shown at the bottom of the next page, $T_6(z)$ and $T_4(z)$ are identically zero but they do not present a singularity because they are preceded by abnormal polynomials. As illustration for Remark 4, notice that $\delta_k = 0 \Rightarrow T_{k-2}(z) = -z^{-1}T_k(z)$ for $k = 7, 6, 5$. The procedure becomes singular at $s = 3$ when the normal polynomial $T_3(z)$ is proceeded by an identically zero polynomial. The procedure is resumed with $T_2(z)$ and $T_1(z)$ created from $T_3(z)$ using (11) (with $K = -1$) and (12). The distribution of zeros can finally be determined using Theorem 3. Here $n = 9$, $s = 3$, setting values into (13) and (14) reveals that

$$\nu_n = \mathrm{Var}\{8, -12, -16, 0, 16, 0, -16, 48, 144, -144\} = 5$$

and that $\nu_s = 2$. Accordingly, there are $\alpha_n = n - \nu_n = 9 - 5 = 4$ IUC zeros; the number of UC zeros is $\beta_n = 2\nu_s - s = 2{\cdot}2 - 3 = 1$ and the number of OUC zeros is $\gamma_n = n - \alpha_n - \beta_n = 9 - 4 - 1 = 4$. (In addition $s - \nu_s = 1$ means one OUC and IUC zeros that form a reciprocal pair.) As illustration for Theorem 2, it can be checked that the zeros of $T_3(z) = 16(z+1)(z-1/2)(z-2)$ are zeros of $D_9(z)$. Namely, the above counted 1 UC zero of $D_9(z)$ is at $z = -1$ and the 1 reciprocal pair of zeros are at $z = 1/2,\ 2$. Furthermore, part b) of Theorem 2 is illustrated by noting that $s = k - 2\lambda_k$ for $k = 7,\ 5$. In other words, the two early identical zero polynomials predict the forthcoming singularity. It is also apparent they reveal the g.c.d. zeros that cause the singularity. Namely, the finite nonzero zeros of $T_7(z)$ and $T_5(z)$ are equal to the zeros of $T_3(z)$.

IV. Concluding Remarks

The paper has presented a revised form for the method to determine the distribution of zeros of a polynomial with respect to the unit circle in [7], [8]. The method now uses a three-term recursion of symmetric polynomials of a more general form that assimilates situations that previously were regarded as singularities and disrupted the procedure for polynomials whose Schur–Cohn matrix is not singular. The new form of the procedure does not compromise neither the simplicity of the rules to extract the information on the distribution of the zeros, nor the efficiency of original form. As a matter of fact, each instance a previous singularity is circumvented by the regular recursion, implies now one or several cost-free recursion steps. Consequently, the cost of the revised procedure is less or equal to the cost of the original procedure (already recognized as the method of least arithmetic operations for the task).

The importance of a complete solution for the unit-circle zero location problem motivated considerable effort to overcome singularities that occur also in the SCMJ class of methods (see [9]–[12] and references there in). The many publications dealing with singularities in these longer known solutions, may symptomize a genuine difficulty to reach a satisfactory general solution in this classical ("scattering") framework. By this token, the elegant way the current procedure overcomes singularities conveys the impression that the newer formulation (the "immittance" approach) is an inherently more suitable environment to deal with singularities in this problem.

The new form of the procedure may be valuable for various application on which the original procedure already had an impact. Other applications may also benefit from the wider range of regularity of the new three-term recursion. For example, the writing of this paper at this time stems from realizing the importance of a uniform and recursive unit-circle zero location algorithm for multidimensional stability tests in schemes like [23]. Finally, but not least importantly, the new shape of the procedure makes it a more pleasant general method to determine the distribution of the zeros of any real or complex coefficient polynomial with respect to the unit-circle.

Appendix

Part 1—Proof for Theorem 1: Write the unit-circle as

$$T = \left\{ z | z = e^{j2\phi},\ \phi \in [0,\ \pi] \right\}.$$

Define for a polynomial $P_n(z) \in \mathcal{C}[z]$ of degree n a "balanced polynomial" by $\hat{P}_n(z) := z^{-n/2} P_n(z)$. $\hat{P}_n(z)$ is real for $z \in T$, if (and only if) $P_n(z)$ is symmetric, [8, Theorem 3]. Consider from here and on the sequence of symmetric polynomials $\{T_n(z),\ \ldots,\ T_0(z)\}$ produced completely by the regular algorithm. Therefore each $\hat{T}_k(z)$ is real for $z \in T$. Multiplying the two sides of the regular recursion (4) by $z^{-k/2}$ gives the next recursion for the corresponding balanced polynomials,

$$\hat{T}_{k+1}(z) = \left(\delta_{k+1} z^{-\lambda_k - 1/2} + \overline{\delta}_{k+1} z^{\lambda_k + 1/2} \right) \hat{T}_k(z) - \hat{T}_{k-1}(z). \tag{A.1}$$

Consider the function that results when a balanced symmetric polynomials takes values on T, $\tilde{T}_k(\phi) := \hat{T}_k(e^{j2\phi})$ and call it a trigonometric polynomial [it can be expressed as a polynomial in $\cos(\phi)$ and $\sin(\phi)$]. Thus $\tilde{T}_k(\phi)$ is real valued for $\phi \in [0,\ \pi]$. The recursion for the trigonometric polynomials is obtained by substitution of $z = e^{j2\phi}$ into (A.1)

$$\tilde{T}_{k+1}(\phi) = \left[2\delta^r_{k+1} \cos(2\lambda_k + 1)\phi + 2\delta^i_{k+1} \sin(2\lambda_k + 1)\phi \right] \cdot \tilde{T}_k(\phi) - \tilde{T}_{k-1}(\phi) \tag{A.2}$$

where $\delta^r_k,\ \delta^i_k$ denote the real and imaginary parts of $\delta_k = \delta^r_k + j\delta^i_k$.

2	−1	−6	−17	−26	−26	−17	−6	−1	2	$\lambda_9 = 0$
6	−9	−9	6	0	6	−9	−9	6		$\lambda_8 = 0,\ \delta_9 = 2/6$
−0	−0	−16	−24	−24	16	0	0			$\lambda_7 = 2,\ \delta_8 = 6/16$
0	0	0	0	0	0	0				$\lambda_6 n/a,\ \delta_7 = 0$
0	−16	24	24	−16	0					$\lambda_5 = 1 \delta_6 = 0$
0	0	0	0	0						$\lambda_4 n/a,\ \delta_5 = 0$
16	−24	−24	16							$\lambda_3 = 0,\ \delta_4 = 0$
0	0	0								$s = 3$ (singularity)
−24	96	−24								$\lambda_2 = 0$
72	72									$\lambda_1 = 0,\ \delta_2 = -24/72$
−144										$\lambda_0 = 0,\ \delta_1 = -72/144$

Proposition: The sequence of trigonometric polynomials $\{\tilde{T}_k(\phi), k = n, \ldots, 0\}$, related via (A.2) to a sequence $\{T_k(z), k = n, \ldots, 0\}$ produced by the regular procedure has the next properties.

Property 1) For every $\phi \in (0, \pi)$, if $\tilde{T}_k(\phi) = 0$ then $\tilde{T}_{k+1}(\phi)\tilde{T}_{k-1}(\phi) < 0$.
Property 2) $\tilde{T}_0(\phi) \neq 0 \,\forall \phi \in (0, \pi)$.
Property 3) $\tilde{T}_\ell(\pi) = (-1)^\ell \tilde{T}_\ell(0), \forall \ell = 0, \ldots, n$.

Proof: To prove Property 1), assume $\tilde{T}_k(\phi_o) = 0$. Then, it is apparent from (A.2) that $\tilde{T}_{k+1}(\phi_o) = -\tilde{T}_{k-1}(\phi_o)$. The possibility that they are both equal to zero is prohibited because it implies singularity (as explained in the proof for Theorem 2). Property 2) is satisfied too because by the nonsingularity assumption, $\tilde{T}_0(\phi) = T_0(z) = \sigma_0 \neq 0$. The proof of Property 3) is by induction. For $\ell = 0$, $\tilde{T}_0(\pi) = \tilde{T}_0(0)$ [$\tilde{T}_0(z)$ is a nonzero constant]. Since $T_{-1}(z) = 0$ it follows that $\tilde{T}_1(0) = 2\delta_1^r \tilde{T}_0(0)$ and $\tilde{T}_1(\pi) = -2\delta_1^r \tilde{T}_0(\pi) = -2\delta_1^r \tilde{T}_0(0) = (-1)\tilde{T}(0)$. Therefore the property holds for also $\ell = 1$. Assume it holds for $i = 0, \ldots, \ell - 1$, then $\tilde{T}_\ell(\pi) = -2\delta_\ell^r \tilde{T}_{\ell-1}(\pi) - \tilde{T}_{\ell-2}(\pi) = -2\delta_\ell^r (-1)^{\ell-1}\tilde{T}_{\ell-1}(0) - (-1)^{\ell-2}\tilde{T}_{\ell-2}(0) = (-1)^\ell[2\delta_\ell^r \tilde{T}_{\ell-1}(0) - \tilde{T}_{\ell-2}(0)] = (-1)^\ell \tilde{T}_\ell(0)$. This completes the proof of Property 3). ■

Assume that $D_n(z)$ has k IUC zeros. Then, by the argument principle, the argument of $D_n(z)$, $\arg D_n(z)$, changes as z encircles T by the amount $\Delta_T \arg D_n(z) = 2\pi k$. Therefore,

$$\begin{aligned}\Delta_T \arg \hat{D}_n(z) &= \Delta_T z^{-n/2} + \Delta_T \arg D_n(z)\\ &= -\pi n + 2\pi k.\end{aligned} \tag{A.3}$$

From (3), $2D_n(z) = T_n(z) + (z-1)T_{n-1}(z)$ and therefore,

$$2\hat{D}_n(e^{j2\phi}) = \tilde{T}_n(\phi) + j2\sin(\phi)\tilde{T}_{n-1}(\phi). \tag{A.4}$$

Since all $\tilde{T}_k(\phi)$ are real valued for $\phi \in [0, \pi]$,

$$\arg \hat{D}_n(e^{j2\phi}) = \arctan\left(\frac{2\sin(\phi)\tilde{T}_{n-1}(\phi)}{\tilde{T}_n(\phi)}\right). \tag{A.5}$$

The Cauchy index is defined in [20] for a rational real function $\rho(x)$ and for x in a real interval $[a, b]$ as $\mathcal{I}_a^b \rho(x) :=$ (number of jumps of $\rho(z)$ from $-\infty$ to $+\infty$) $-$ (number of jumps of $\rho(z)$ from $+\infty$ to $-\infty$).

This index can be calculated using a Sturm sequence of real polynomials from boundary values of these polynomials and it was used in this manner to prove the Routh test in [20]. A sequence of polynomials over the real interval $[-1, 1]$ was used in [24] to obtain a proof for the original setting of this unit-circle zero location method. The proof there assumed the case of real polynomials and strictly adhered to the above definition of the Cauchy index and to the definition of a Sturm sequence of polynomials in [20]. Next, a more direct proof for the case of complex polynomials and the newer setting general is enrolled using the properties prepared in the above proposition to evaluate Cauchy index directly along the unit circle as follows.

A jump from $-\infty$ to $+\infty$ and from $+\infty$ to $-\infty$ correspond, respectively, to a $+\pi$ and $-\pi$ change in the value of the $\arctan$ function. Therefore, one obtains from (A.5) that the change in the argument of $\hat{D}_n(z)$ as z traverses T is given by

$$\Delta_T \arg \hat{D}_n(e^{j2\phi}) = -\pi \mathcal{I}_0^\pi \frac{2\sin(\phi)\tilde{T}_{n-1}(\phi)}{\tilde{T}_n(\phi)}. \tag{A.6}$$

After dropping the factor $2\sin(\phi)$ that has a constant sign on $(0, \pi)$, and combining the last expression with (A.3), one obtains,

$$n - 2k = \mathcal{I}_0^\pi \frac{\tilde{T}_{n-1}(\phi)}{\tilde{T}_n(\phi)}. \tag{A.7}$$

Using the first two properties of the above proposition, it becomes apparent that the trigonometric sequence meets all the properties (even though they are not simple polynomials) that were truly used in [20, p. 175] to evaluate the Cauchy index by a Sturm sequence over the interval $[0, \pi]$. Therefore Sturm's theorem can be applied to calculate the values of the above Cauchy index from boundary values of this sequence, *viz.*,

$$\begin{aligned}\mathcal{I}_0^\pi \frac{\tilde{T}_{n-1}(\phi)}{\tilde{T}_n(\phi)} &= \mathrm{Var}\left\{\tilde{T}_n(0), \ldots, \tilde{T}_0(0)\right\}\\ &\quad - \mathrm{Var}\left\{\tilde{T}_n(\pi), \ldots, \tilde{T}_0(\pi)\right\}.\end{aligned} \tag{A.8}$$

Using Property 3) of the proposition, it follows that

$$\begin{aligned}&\mathrm{Var}\left\{\tilde{T}_n(\pi), \ldots, \tilde{T}_\ell(\pi), \ldots, \tilde{T}_0(\pi)\right\}\\ &= \mathrm{Var}\left\{(-1)^n\tilde{T}_n(0), \ldots, (-1)^\ell\tilde{T}_\ell(0), \ldots, \tilde{T}_0(0)\right\}\\ &= n - \mathrm{Var}\left\{\tilde{T}_n(0), \ldots, \tilde{T}_\ell(0), \ldots, \tilde{T}_0(0)\right\}.\end{aligned} \tag{A.9}$$

Next, substitute (A.9) into (A.8), use $\tilde{T}_\ell(0) = \hat{T}_\ell(e^{j0}) = T_\ell(1)$ and combine the result with (A.7) to obtain

$$n - k = \mathrm{Var}\{T_n(1), \ldots, T_0(1)\}. \tag{A.10}$$

Recall that k denotes the number of IUC zeros of $D_n(z)$, and since nonsingularity implies no UC zeros (see Theorem 2), it follows that $D_n(z)$ has $n - k$ OUC zeros given by the above count of sign variations. This completes the proof of Theorem 1.

Part 2—Proof for Theorem 2: First, recall that the zeros of a symmetric polynomial $T_k(z)$, as well as the common zeros of $D_n(z)$ and $D_n^\sharp(z)$, are either UC zeros or RP zeros. It is also important to remember that zeros of $D_n(z)$ at $z = 1$ and at $z = \infty$ were prohibited by assumption (1). The proof uses the property of the regular three-term recursion that if adjacent polynomials $T_k(z)$, $T_{k+1}(z)$ vanish at a common z_o then $T_{k+2}(z_o) = 0$ and $T_{k-1}(z_o) = 0$ there as well.

Assume the greatest common divisor (g.c.d.) of $D_n(z)$ and $D_n^\sharp(z)$ has degree s. Then these s zeros are also common to $T_n(z)$ and $T_{n-1}(z)$. Therefore, by the mentioned property of the recursion, these are zeros of $T_m(z)$ for also $m = n-2, \ldots$ till $T_s(z)$. This $T_s(z)$ must be normal (zeros at 0 and ∞ can not be g.c.d. zeros) and must be followed by $T_{s-1}(z) \equiv 0$ (because its degree is too low to accommodate differently s zeros).

To prove the converse, assume $T_s(z)$ is normal ($\lambda_s = 0$) and is followed by $T_{s-1}(z) \equiv 0$. Then,

$$0 = \left(\delta_{s+1} + \overline{\delta}_{s+1} z\right) T_s(z) - T_{s+1}(z). \tag{A.11}$$

If $\delta_{s+1} \neq 0$ then the above equation asserts that the zeros of $T_{s+1}(z)$ consist of the s zeros of $T_s(z)$ plus the zero of the factor $(\delta_{s+1} + \overline{\delta}_{s+1} z)$ and therefore all higher degree polynomials also vanish at the zeros of $T_s(z)$. When $\delta_{s+1} = 0$ then $T_{s+1}(z) \equiv 0$ and the zeros $T_s(z)$ still propagate to upward polynomials, in fact the next higher degree polynomial will be $T_{s+2}(z) = -zT_s(z)$. Thus, either way, $T_m(z)$ vanish at the s zeros of $T_s(z)$ for all $m > s$. Therefore, the zeros of $T_s(z)$ are zeros of also $D_n(z) = (T_n(z) + (z-1)T_{n-1}(z))/2$ and of $D_n^{-}(z) = (T_n(z) - (z-1)T_{n-1}(z))/2$. This completes the proof of part (a).

To prove part (b), assume $T_k(z)$ with $\lambda_k > 0$ is followed by a $T_{k-1}(z) \equiv 0$. In this case,

$$0 = \left(\delta_{k+1} z^{-\lambda_k} + \overline{\delta}_{k+1} z^{\lambda_k + 1}\right) T_k(z) - T_{k+1}(z) \tag{A.12}$$

implies that the $k+1$ zeros of $T_{k+1}(z)$ consist of the $k - 2\lambda_k$ finite zeros of $T_k(z)/z^{\lambda_k}$ plus $2\lambda_k + 1$ zeros of the factor $(\delta_{k+1} + \overline{\delta}_k z^{2\lambda_k+1})$. The claim in part 2 follows at once by repeating the argument above on the propagation of the common $k - 2\lambda_k$ zeros of the two adjacent polynomials $T_k(z)$ and $T_{k+1}(z)$ down and up the recursion.

Part 3—On the Scheme to Overcome Singularity: The treatment of singularity here is similar to the treatment of first type singularities in [7], [8]. It uses the property that $P_{s-1}(z) = T_s'(z)$ has as many OUC zeros as $T_s(z)$ [2]. Therefore, $D_{s-1}^{-}(z)$ has as many IUC zeros as $T_s(z)$. As shown in [7], $\mathrm{Sgn} T_s(1) = -\mathrm{Sgn} T_{s-1}(1)$ is required to seam correctly the zero location rule based on the concatenation of the two partial sequences $\{T_n(1), \ldots, T_s(1)\}$ and $\{T_{s-1}(1), T_{s-2}(1), \ldots\}$. It remains to show that the proposed options for K satisfy this requirement. Differentiation of $T_s(z) = z^s \overline{T}_s(z^{-1})$ and setting into the result $z = 1$ shows that $2\mathcal{R}e\{P_{s-1}(1)\} = sT_s(1)$. It follows that K_1 satisfies the sign requirement $\mathrm{Sgn} T_s(1) = -\mathrm{Sgn} T_{s-1}(1)$. This sign requirement is obviously satisfied by also K_2 and K_3.

Part 4—Proof for Theorem 3: If $T_s(z)$ is normal ($\lambda_s = 0$) and is followed by an identically zero polynomial, then according to Theorem 2, it divides all the previous polynomials $T_k(z) k = s, \ldots, n$. Therefore this common factor cancels out in the ratio of trigonometric polynomials used in the proof to Theorem 1. Consequently the proof for Theorem 1 may be regarded as evaluation of the Cauchy index for the trigonometric sequence that corresponds to the reduced degree polynomials $T_k(z)/T_s(z)$, $k = n, \ldots, s$. It then shows that $D_n(z)/T_s(z)$ has $n - s - \mathrm{Var}\{T_n(1), T_{n-1}(1), \ldots, T_s(1)\}$ IUC zeros. Next, Theorem 1 may be applied to test the polynomial $D_{s-1}(z)$ assuming the procedure is nonsingular (else, when a singularity recurs, the current reasoning has to be repeated on a subsequent subset, or subsets, of the sequence). Consequently, $\nu_s = \mathrm{Var}\{T_{s-1}(1) T_{s-2}(1), \ldots, T_0(1)\}$ provides information on the zero distribution of $D_{s-1}(z)$. However, the number of IUC zeros of $D_{s-1}(z)$, $s - \nu_s$ is equal to the number of IUC zeros of $T_s(z)$. $T_s(z)$, being a symmetric polynomial, has then also $s - \nu_s$ OUC zeros (located reciprocally to the IUC zeros) and the remaining $s - 2(s - \nu_s)$ zeros are UC zeros. These are also the UC zeros (and *all* the UC zeros) of $D_n(z)$. Therefore $\beta_n = 2\nu_s - s$. The remaining assertions in Theorem 3 follow at once. If singularities recur they have to be resolved each time in a similar manner. In fact singularities must occur in a number of times equal to the highest multiplicity of a UC zero or RP zeros of $D_n(z)$ (because each time by treatment by differentiation reduces the multiplicity of zeros by one). Note that determining the distribution triple $\alpha_n, \beta_n, \gamma_n$ depends only the degree s associated with the *first* occurring singularity. However, the location of subsequent singularities can be used to obtain more subtle information on the multiplicity of UC and RP zeros of $D_n(z)$ by nesting and superposing conclusions from the above analysis.

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