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IMMITTANCE AND TELEPOLATION-BASED PROCEDURES TO TEST STABILITY OF CONTINUOUS-DISCRETE BIVARIATE POLYNOMIALS

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ABSTRACT

The paper presents several algebraic procedures to test whether a bivariate polynomial is continuous-discrete (C-D) stable (does not vanish in the product of the closed right half-plane times the closed exterior of the unit-circle). This problem was solved in the last ISCAS by a scattering-type tabular test based on Jury's modified stability test. Here an immittance-type counterpart for the test, that relies instead on a modified form of the author's test, is presented. The immittance tabular test has a lower cost of computation because it produces a sequence of matrices with para-conjugate column-symmetry. Telepolation-based forms for the two tabular tests are also presented. They carry out the C-D stability test by a finite number of Jury's or this author's 1-D stability tests, respectively, plus a Routh zero location procedure. As a consequence the overall complexity reduces significantly (from $O(n^6)$ to $O(n^4)$ for a bivariate polynomial of degree (n, n)).

1. INTRODUCTION

The paper considers the problem of determining whether a two-variable (2-V) polynomial of degree (n_1, n_2), say

$$Q(s,z) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} q_{ij} s^i z^j \quad , \; q_{n_1,n_2} \neq 0 \tag{1}$$

satisfies the condition

$$Q(s,z) \neq 0 \qquad \forall (s,z) \in \overline{\mathbb{R}} \times \overline{\mathbb{V}} \tag{2}$$

where, with $\mathbb{C}$ denoting the complex plane, we denote

$$\mathbb{L} = \{s : \mathcal{R}e\; s < 0\;,\; |s| < \infty\}, \;, \quad \overline{\mathbb{R}} = \mathbb{C} - \mathbb{L}\,. \tag{3}$$

and

$$\mathbb{U} = \{z : |\, z\,| < 1\}\,, \quad \overline{\mathbb{V}} = \mathbb{C} - \mathbb{U}\,. \tag{4}$$

We shall also denote the imaginary axis and the unit circle by

$$\mathbb{I} = \{s : \mathcal{R}e(s) = 0\}\,,\; \mathbb{T} = \{z : \; |z| = 1\}\,. \tag{5}$$

The problem arises in testing stability of certain linear systems that can be described by a linear differential-difference equations, where $Q(s, z)$ presents their characteristic polynomial [1]. A polynomial that satisfies the condition (2) will be called C-D (continuous-discrete) stable. The problem is also closely related to determining stability of certain differential delay equations with commensurate delay, see [2], [3].

A tabular stability test for this problem was presented in the previous ISCAS [4] based on the modified Jury test [5]. This paper will first bring an alternative tabular test that relies instead on a certain modified form of this author's test [6]. It may be regarded as the immittance-type counterpart of the scattering-type test in [4]. The current tabular test has a similar order of complexity ($O(n^6)$ for $n_1 = n_2 = n$) but the actual count of operations is lower because it produces matrices with certain symmetry instead of matrices with no symmetry that are produced by the scattering C-D stability test [4]. The paper will also consider simplification for both the scattering and the immittance tabular C-D stability tests by telepolation [7]. Using these procedures the testing of the condition (2) will be carried by a collection of $n + 1$ 1-D stability tests reducing the overall complexity from $O(n^6)$ to $O(n^4)$.

The procedures that will be presented here and the one in [4] are worthy also for testing stability of commensurate delay because C-D stability provides sufficient (though not necessary condition) for stability of such systems. However, the modification of these procedures into a comprehensive stability test (necessary and sufficient conditions) for differential systems with commensurate delay [2], [3] will be deferred to a future publication.

2. PRELIMINARIES

Our notation convention uses a same letter for both a polynomial and its matrix (or vector) of coefficients. For example $Q = (q_{i,k})$ will also denote the coefficients matrix of the polynomial (1). We use $\mathbf{z}$ to denote a vector whose entries are powers in ascending degrees of the variable, $\mathbf{z} = [1, z, \ldots, z^i, \ldots]^t$ (of length determined by context). It allows one to write $d(z) = \mathbf{z}^t d$ and $Q(s, z) = \mathbf{s}^t Q\mathbf{z}$. It is instructive for the derivation to regard $Q(s, z)$ as a 1-V polynomial in $\mathbb{C}[z]$ (the set of univariate polynomials in indeterminate z and complex coefficients) with coefficients over $\mathbb{C}[s]$. To this end, the columns of Q will be denoted $Q = [q_0\;,\; q_1\;, \ldots,\; q_{n_2}]$ which allows one to write $Q(s, z)$ as

$$Q(s,z) = \sum_{k=0}^{n_2} q_k(s) z^k = [q_0(s), q_1(s), \ldots, q_{n_2}(s)]\mathbf{z}\,. \tag{6}$$

We define a "diamond" operation for a 2-V polynomials of degree (n_1, n_2) with mixed $s - z$ indeterminates and for a matrix (of its coefficients) as follows.

$$Q^\diamond(s,z) = z^{n_2} Q^\star(-s, z^{-1}) \qquad , \qquad Q^\diamond = KQ^\star J \tag{7}$$

where J denotes the reversion matrix with 1's on the main anti-diagonal and zeros elsewhere, K denotes a diagonal matrix with diagonal elements $(-1)^k$, $k = 0, 1 \ldots$, i.e $K = diag[1, -1, 1, -1, 1, \ldots]$ of size determined by context, and $\star$ denotes complex conjugation. Note that the definitions are such that it is still possible to write $Q^\diamond(s, z) = \mathbf{s}^t Q^\diamond \mathbf{z}$.

We denote the para-conjugate operation for the 1-V polynomial $h(s) = \mathbf{s}^t h$ by

$$h^{\natural}(s) = h^{\star}(-s) \quad , \quad h^{\natural} = Kh^{\star} \tag{8}$$

so that $h^{\natural}(s) = \mathbf{s}^t h^{\natural}$. A polynomial $h(s)$ that is not a constant is called 1-C stable if "$h(s_i) = 0$ implies $s_i \in \mathbb{L}$" or equivalently if

$$h(s) \neq 0 \quad \forall s \in \overline{\mathbb{R}}\,. \tag{9}$$

For a 1-V polynomial $d(z) = \mathbf{z}^t d$ of degree n and its coefficient vector d we denote the operation of conjugate-reciprocation by

$$d^{\sharp}(z) = z^n d^{\star}(z^{-1}) \quad , \quad d^{\sharp} = Jd^{\star} \tag{10}$$

Again the reciprocated polynomial may be written as $d^{\sharp}(z) = \mathbf{z}^t d^{\sharp}$. A polynomial $d(z)$ that is not a constant is called 1-D stable if the condition "$d(z_i) = 0$ implies $z_i \in U$" or equivalently if

$$d(z) \neq 0 \quad \forall z \in \overline{\mathbb{V}}\,. \tag{11}$$

Note that the diamond operation amounts to either pre para-conjugate and post reciprocal operation or to pre para and post conjugate-reciprocal operation.

3. DISCRETE-CONTINUOUS STABILITY

We shall consider the stability condition (2) for complex coefficients $Q(s,z) \in \mathbb{C}[s,z]$ (the set of bivariate polynomials in indeterminates s,z with complex coefficients), like in [4]. (All other previous works that studied the condition (2) considered it for real polynomial only.) Polynomials $h(s) \in C[s]$ and $d(z) \in C[z]$ that satisfy (9) and (11) will too be called 1-C stable and 1-D stable, respectively.

Assume a $Q(s,z) \in \mathbb{C}[s,z]$ as in (1). The condition on the leading coefficient $q_{n_1,n_2} \neq 0$ is a readily visible assumption. If $q_{n_1,n_2} = 0$ then $Q(s,z)$ is not stable. (Note that $s = \infty$ belongs to $\overline{\mathbb{R}}$ and a zero there implies an unstable 1-C polynomial and hence also C-D instability.)

Lemma 1. *The polynomial $Q(s,z)$ (1) satisfies (2) (it is C-D stable) if, and only if, the following two conditions hold:*

(i) *$Q(s,b)$ is 1-C stable for some $b \in \overline{\mathbb{V}}$.*

(ii) $Q(s,z) \neq 0\ \forall (s,z) \in \mathbb{I} \times \overline{\mathbb{V}}$.

The above lemma appeared first in [8] with the values $b = \infty$. Then, with the value $b = 1$ in [2, Theorem 2:(2)], with $b \in \mathbb{T}$ in [9], and in its above generality in [3]. The latter two references contain more simplifying conditions for C-D stability.

In [4], the above lemma was used, in conjunction with Jury's modified 1-D stability test for complex polynomials [5], to obtain a tabular test for C-D stability that assigns to $Q(s,z)$ a sequence of n_2 2-V polynomials $\{C_m(s,z) = \sum_{k=0}^{m} c_{[m]k}(s) z^k,\ m = n_2 - 1, \ldots, 0\}$ where C_m are matrices with no particular symmetry.

Here, a tabular test that associate $Q(s,z)$ with a sequence of n_2 2-V polynomials $\{E_m(s,z) = \sum_{k=0}^{m} c_{[m]k}(s) z^k,\ m = n_2 - 1, \ldots, 0\}$ where $E_m(s,z) = E^{\diamondsuit}(s,z)$. This para-conjugate column symmetry admits the computation of only half of the entries of the coefficient matrices E_m (say only the left half columns).

The new procedure is derived by using the author's 1-D stability test for complex polynomials [6] to examine the condition (ii) of Lemma 1. Assume that $Q(s,1) \not\equiv 0$ (implied if $Q(s,1)$ is 1-C stable a necessary condition for C-D stability that will be tested before any further step is followed). Then define

$$M(s,z) := Q^{\natural}(s,1) Q(s,z) = \mathbf{s}^t M \mathbf{z} \tag{12}$$

where $Q^{\natural}(s,1) = Q^{\star}(-s,1)$. Note that condition (ii) of Lemma 1 holds if, and only if, $M(s,z) \neq 0\ \forall (s,z) \in \mathbb{I} \times \overline{\mathbb{V}}$. Then testing 1-D stability of $p_s(z) = Q(s,z)$ assuming $s \in \mathbb{I}$ with the author's 1-D stability test leads eventually to the next algorithm and theorem.

Algorithm 1. Immittance C-D stability table.
Construct for $D(z_1,z_2)$ a sequence of polynomials $\{E_m(\tilde{s},z) = \sum_{k=0}^{n-m} e_{[m]\,k}(\tilde{s}) z^k,\ m = 0,1,\ldots,n(=n_2)\}$, where $E_m = E_m^{\sharp}$ $\forall m$, as follows.

(i) Initiation. $M(s,z) = Q^{\star}(-s,1) Q(s,z)$

$$E_0(s,z) = M(s,z) + M^{\diamondsuit}(s,z))\,,\ E_1(s,z) = \frac{M(s,z) - M^{\diamondsuit}(s,z)}{z-1}$$

$$q_0(s) = E_0(s,1) \tag{13}$$

(ii) Recursion. For $m = 1, \ldots, n-1$ obtain $E_{m-1}(s,z)$ by:

$$g_m(s) = e_{[m-1]\,0}(s) e^{\natural}_{[m]\,0}(s)\,,\ q_m(s) = e_{[m]\,0}(s) e^{\natural}_{[m]\,0}(s) \tag{14}$$

$$zE_{m+1}(s,z) = \frac{g_m(s)E_m(s,z) + g_m^{\natural}(s) z E_m(s,z) - q_m(s) E_{m-1}(s,z)}{q_{m-1}(s)}$$

The polynomials $q_m(s)$ in the denominators are always factors of the numerator so that all $E_m(s,z)$ are indeed polynomials. The degree of $E_0(s,z)$ is $(2n_1,n_2)$ and the degrees of $E_m(s,z)$ for $m = 1, \ldots n_2$ are $(2mn_1, n_2 - m)$. The produced polynomials posses the symmetry $E_m^{\diamondsuit}(s,z) = E_m(s,z)$ for all m. For real / complex $Q(s,z)$ the coefficient matrices E_m are real / complex, respectively.

Theorem 1 [Stability conditions for Algorithm 1]. Assume algorithm 1 is applied to $Q(s,z)$ (1) and let

$$\epsilon(s) := E_n(s,z) \tag{15}$$

denote last polynomial that it produces. $Q(s,z)$ is stable if, and only if, the following three conditions hold.
(i) $Q(s,b)$ is 1-C stable for some $b \in \overline{\mathbb{V}}$.
(ii) $Q(a,z)$ is 1-D stable for some $a \in \mathbb{I}$.
(iii) $\epsilon(s) \neq 0 \quad \forall s \in \mathbb{I}$.

Note that $\epsilon(s)$ is truly a 1-V polynomial (of s only) of degree $2n_1n_2$. The property $E_m^{\diamondsuit}(s,z) = E_m(s,z)$ implies that ϵ exhibits para-conjugate symmetry $\epsilon(s) = \epsilon^{\star}(-s)$. Thus, in the real case (Q is real) it is an even polynomial with vanishing odd powers of s coefficients while in the complex case (Q is complex) even/odd powers of s coefficients are purely real/imaginary.

A rigorous derivation of Algorithm 1 and a proof of Theorem 1 will not be given currently. They can be deduced starting with the test in [6] and following a path analogous to the way the main theorem in [10] was reached. The fact that Theorem 1 looks like the main theorem in [4] is not coincidental. In fact, an alternative proof for the above theorem can be reached after showing that both the algorithm used here and in [4] terminate with an identical $\epsilon(s)$ that is the greatest common divisor of $Q(s,z)$ and $Q^{\diamondsuit}(s,z)$ with respect to the variable z. (Other equivalent terms are the resultant or the determinant of the unit-circle Bezoutian of $Q(s,z)$ and $Q^{\diamondsuit}(s,z)$ with respect to the variable z.)

A general algebraic approach to examine condition (iii) in Theorem 1 is using the extension of the Routh test into a zero location procedure for complex 1-V polynomials that also handle "singular" situations (because singularities are not inconsistent with condition iii being true) [14]. The testing of this condition can be carried out by approximately n^4 (for $n_1 = n_2 = n$) flops. The testing of condition (i) can be carried out by any of the tests in the Schur-Cohn Marden and Jury class of stability tests (e.g. [5]) or more efficiently with the author's test [6] in $O(n^2)$. Condition (ii) is a 1-C stability test that like condition (iii) can be handled by the Routh test and too has $O(n^2)$ of complexity. Two convenient choices for b in Theorem 1 are either $b = 1$ (check 1-C stability for a polynomial whose coefficients are the sum of columns of Q) or $b = \infty$ (the last column of Q). Two convenient choices for a are $a = 0$ - the first row of Q and $a = \infty$ - the last row of Q).

4. ILLUSTRATION

We recall that the test works for both real and complex coefficient polynomials. Here we shall illustrate the test with the following real polynomial of degree (3,3) that we used also in [4].

$$Q(s,z) = [1\ s\ s^2\ s^3] \begin{bmatrix} 6 & 6 & -10 & 15 \\ 5 & 8 & -15 & 25 \\ 2 & 2 & -4 & 7 \\ 1 & 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{bmatrix}$$

To test whether or not it is C-D stable we carry out the test that consists of Algorithm 1 and Theorem 1 through the following four steps.

Step 1: Test condition (i) in Theorem 1 for say $Q(s,1) = [17\,,\ 23\,,\ 7\,,\ 3]\mathbf{s}$ and realize that it is 1-C stable.

Step 2: Test condition (ii) in Theorem 1 for say $Q(0,z) = [6\,,\ 6\,,\ -10\,,\ 15]\mathbf{z}$ and realize that it is 1-D stable.

Step 3: Apply algorithm 1. Form $Q^\diamond$ and then obtain $M(s,z)$ (12)

$$Q^\diamond = \begin{bmatrix} 15 & -10 & 6 & 6 \\ -25 & 15 & -8 & -5 \\ 7 & -4 & 2 & 2 \\ -3 & 2 & -1 & -1 \end{bmatrix}, M = \begin{bmatrix} 102 & 102 & -170 & 255 \\ -53 & -2 & -25 & 80 \\ -24 & -33 & 63 & -95 \\ 1 & 1 & -2 & 0 \\ -3 & -3 & 6 & -9 \end{bmatrix}$$

Proceed with Algorithm 1 to obtain the sequence of 2-V polynomials $\{\mathbf{s}^t E_m \mathbf{z}, m = 0, \ldots, 3\}$ with coefficient matrices as follows.

$$E_0 = \begin{bmatrix} 357 & -68 & -68 & 357 \\ -133 & 23 & -23 & 133 \\ -390 & 99 & 99 & -390 \\ -32 & 26 & -26 & 32 \\ 1 & 3 & -3 & -1 \\ -12 & 3 & 3 & -12 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 153 & -119 & 153 \\ -27 & 0 & 27 \\ -312 & 3 & -312 \\ -8 & 0 & 8 \\ -71 & 25 & -71 \\ -1 & 0 & 1 \\ -6 & 3 & -6 \end{bmatrix}, E_2 = \begin{bmatrix} 5967 & 5967 \\ 1557 & -1557 \\ -49293 & -49293 \\ -7704 & 7704 \\ 71015 & 71015 \\ 283 & -283 \\ 30943 & 30943 \\ 1353 & -1353 \\ 5520 & 5520 \\ 268 & -268 \\ 470 & 470 \\ 19 & -19 \\ 18 & 18 \end{bmatrix}$$

$E_3^t = [646425, 0, -8915057, 0, 35480226, 0, -27528155, 0, -22357775, 0, -6569912, 0, -1050718, 0, -99997, 0, -5414, 0, -135]$

Step 4: Examine whether $\epsilon(s) = \mathbf{s}^t E_3 \neq 0$ for $s \in \mathbb{I}$. The condition can be shown to hold. Thus the tested polynomial is C-D stable.

Algorithm 1 can be shown to require $O(n^6)$ operations assuming for simplicity $n_1 = n_2 = n$. It is the dominant part in the overall cost of the procedure. The testing of the condition (i) and (ii), require $O(n^2)$ flops with actual number that depends on the 1-C and 1-D stability test that is employed. The testing of condition (iii) by the Routh zero location procedure requires $O(n^4)$ operations. Using a more accurate evaluation of the cost of computation would show that the computational cost of this tabular test compares to the cost of the immittance 2-D stability test in [10]. The cost of the tabular test in [4] has too $O(n^6)$ with a cost figure that compares to the cost for the scattering 2-D tabular test [13]. Thus the current tabular test here is more efficient than the test in [4] by a factor similar to the computational advantage of the 2-D tabular stability test in [10] over the 2-D tabular test in [13]. This advantage stems from the "diamond" symmetry (para-conjugate column) symmetry of the matrices E_m that admits the computation of (roughly) only half of the entries of the matrices.

5. TELEPOLATION

In this section we shall briefly show how the above tabular test, as well as the test in [4], can be simplified by telepolation. This approach has been introduced in [7] to simplify the immittance 2-D tabular test in [12]. It was also used in [11] to simplify the test in [10] and to simplify the the Hu-Jury 2-D stability test in [13]. In all these cases interpolation was applied to an "efficient" tabular test (i.e. one with $O(n^6)$ complexity and a single "positivity" condition) to replace the construction of the table by a finite collection of certain 1-D stability tests. This procedure also reduced the overall complexity to $O(n^4)$.

The immittance-type tabular test presented here, as well as the scattering-type test in [4], qualify for a similar treatment. The idea is to circumvent the algorithm for the construction of the table and instead obtain the polynomial $\epsilon(s)$ that celebrates in Theorem 1 by interpolation.

As a polynomial of degree $2M$, $M := n_1 n_2$, $\epsilon(s)$ can be determined from any $2M+1$ sample values $b_i = \epsilon(s_i)$ at different sampling points $s_i, i = 0, \pm 1 \ldots \pm M$ by solving the set of equations

$$[1, s_i, s_i^2 \ldots, s_i^{2M}]\epsilon = b_i \quad i = 0, \pm 1 \ldots \pm M \qquad (16)$$

This is a Vandermonde set of equations for which efficient $O(N^2)$ solutions (here $N = 2M$) are known [15].

The key requirement is to obtain sample values of $\epsilon(s)$ from which this polynomial can be interpolated. For $s_i \in \mathbb{I}$ sample values $\epsilon(s_i)$ can be obtained by applying the next algorithm to $p_{s_i}(z) := M(s_i, z)$.

Algorithm 2. Assume $p(z)$ is a polynomial of degree n with complex coefficients and that $p(1) \neq 0$. Form $\hat{p}(z) = p(1)^\star p(z)$ and construct a sequence $e_m(z) = \sum_{i=0}^{n-m} e_{m,i} z^i$, $m = 0, 1, \ldots, n$ of symmetric polynomials as follows.

$$e_0(z) = \hat{p}(z) + \hat{p}^\sharp(z)\ ,\ q_0 = e_0(1)$$
$$e_1(z) = \frac{\hat{p}(z) - \hat{p}^\sharp(z)}{(z-1)} \qquad (17)$$

For $m = 1, \ldots, n-1$ obtain $e_{m+1}(z)$:

$$g_m = e_{m-1,0} e_{m,0}^\star \quad , \quad q_m = |e_{m,0}|^2$$

$$ze_{m+1} = \frac{(g_m + g_m^\star z)e_m(z) - q_m e_{m-1}(z)}{q_{m-1}} \quad (18)$$

It is apparent that Algorithm 2 presents the projection of Algorithm 1 for s fixed to $s = s_i$. Thus the last entry $e_{n,0}$ that Algorithm 1 produces is equal to $\epsilon(s_i)$, i.e. $b_i = e_{n,0}$.

Algorithm 2 is the companion 1-D stability test that was used also in [11] to telepolate the 2-D tabular stability test in [10]. The next theorem is proved in [11].

Theorem 2. *Assume Algorithm 2 is applied to a complex polynomial $p(z)$ of degree n such that $p(1) \neq 0$. Then $p(z)$ is stable if, and only if,*

$$e_m(1) > 0 \quad , \quad m = 0, 1, \ldots, n. \quad (19)$$

Furthermore, $\mathcal{R}e\{e_{m,0}\} > 0$ for all m, are necessary conditions for stability of $p(z)$.

Algorithm 2 and Theorem 2 can be used as the accompanying 1-D stability test for the telepolation of the current tabular C-D stability test. During the use of the accompanying stability test to acquire a sample value $\epsilon(s_i)$, one should watch for any violation of a necessary condition for 1-D stability of $p_{s_i}(z) = M(s_i, z)$ (according to Theorem 2) because in any such case $Q(s, z)$ can be declared at once as not C-D stable and the telepolation-based stability testing procedure reaches an earlier termination.

In the complex case (Q is complex) the number of required 1-D stability tests is $2M+1$. In the real case (Q is real), only $M+1$ 1-D stability tests are needed because it is enough to obtain sample values for $b_i = \epsilon(s_i)$, $s_i = 0, 1, \ldots M$. Then, using the fact that $\epsilon(s_i) = \epsilon(-s_i)$, it is possible to choose the remaining M sample points as $s_{-i} = -s_i$ for which the already obtained sample values $b_i = \epsilon(s_{-i})$ can be used.

It is possible to simplify similarly also the scattering tabular C-D test presented in [4]. This time the accompanying 1-D stability test that has to be used is the modified Jury test depicted as Algorithm 1 and Theorem 1 in [4]. Namely, it can be shown that applying this 1-D stability test to $p_{s_i}(z)$ ($s_i \in \mathbb{I}$) produces at its end $b_i = c_{n,n}$.

Note that each 1-D stability test requires $O(n^2)$ flops to test a polynomial of degree n. Assume for simplicity $n_1 = n_2 = n$ then $M = n^2$ tests require $O(n^4)$ flops. So sample values are acquired in $O(n^4)$ flops. The recovery of $\epsilon(s)$ (solution of the interpolation set of equation) requires too $O(n^4)$ flops. Thus the telepolation-based C-D test can be completed in $O(n^4)$ flops. The actual number of flops will be lower for the immittance-type telepolation-based C-D test than for its scattering-type counterpart because sample values are obtained by the immittance 1-D stability test (i.e. by the above Algorithm 2) in less arithmetic operations than by the modified Jury test.

6. CONCLUSION

The paper presented several procedures to determine whether a bivariate polynomial stable in the continuous-discrete sense (does not vanish in the closed right half-plane times the closed exterior of the unit-circle. First a tabular test that forms the immittance counterpart of the scattering-type tabular test that was presented for this problem in [4]. Both tabular tests have $O(n^6)$ complexity (for a bivariate polynomial of degree (n, n)) but the immittance test has a lower cost because it produces matrices with para-conjugate column symmetry. Next, the paper showed simplification for both the tabular tests by telepolation (interpolation). The telepolation-based procedures test C-D stability by a finite collection of (modified Jury or Bistritz) 1-D stability tests plus a Routh test for no zeros on the imaginary axis. Both telepolation-based procedures reduce the computational cost from $O(n^6)$ to $O(n^4)$ where the immittance version requires again less computation than the scattering version.

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